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A COMPARISON OF DISCRETE FIXED POINT THEOREMS VIA A BIMATRIX GAME

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Dedicated to Professor Wataru Takahashi on his 70th birthday

Abstract. There are three types of discrete fixed point theorems: type M and type C deal with monotone mappings and contraction mappings, respectively. Type B is based on Brouwer's fixed point theorem. The main aim of this paper is to compare type B with type M by applying them to a bimatrix game. For this purpose we characterize the direction preserving condition that is used in type B in terms of the best response mappings of the bimatrix game. Further we extend the characterization to a non-cooperative n-person game.

1. Introduction

There are three types of discrete fixed point theorems. Type M deals with monotone mapping such as Tarski's fixed point theorem [8]. Topkis [9] applied Tarski's fixed point theorem to a non-cooperative n-person game to show the existence of the pure-strategy Nash equilibrium, see also Sato-Kawasaki [6]. Type C deals with contraction mappings. Robert [5] showed that any contraction mapping from the Boolean algebra $\{0,1\}^n$ into itself has a unique fixed point. Shih-Dong [7] presented a marvelous result that any locally contractive mapping from $\{0,1\}^n$ into itself also has a unique fixed point. Richard [4] extended Shih-Dong's result to integer intervals. Further, Kawasaki-Kira-Kira [3] obtained an extension of [5] by way of [4]. Type B is based on Brouwer's fixed point theorem. Iimura [1] introduced an important assumption that guarantees a discrete fixed point. Iimura-Murota-Tamura [2] corrected the main theorem of [1]. The basic idea of type B is as follows. Let $X \subset \mathbb{Z}^n$ be a finite set and $f : X \to X$ a mapping.

(1) Give a simplicial decomposition of the convex hull co$X$ of X.
(2) Extend f to a piecewise linear mapping, say $\hat{f}$, by using the simplicial decomposition.
(3) Apply Brouwer's theorem to $\hat{f}$ on co$X$, and obtain a fixed point, say y, of $\hat{f}$.
(4) Impose an assumption for a vertex of the simplex including y be a fixed point of f.

The assumption introduced in [1] is called the direction preserving condition (1.1). We say two points x, $x' \in X$ to be *cell-connected* if they belong to a same simplex of the simplicial decomposition, and denote the binary relation by $x \sim x'$. A mapping

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$f = (f_1, \ldots, f_n) : X \to X$ is said to be *direction preserving* if

$$x \sim x' \Rightarrow (f_i(x) - x_i)(f_i(x') - x'_i) \geq 0 \quad (i = 1, \ldots, n). \tag{1.1}$$

Here we note that the original definition is slightly different from (1.1), see Remark 2.4 below. Further, Yang [10] weakened the assumption as (1.2), which is called the *locally gross direction preserving condition.*

$$x \sim x' \Rightarrow \sum_{i=1}^{n} (f_i(x) - x_i)(f_i(x') - x'_i) \geq 0. \tag{1.2}$$

For a set-valued mapping $F(x)$, we call a mapping f such that $f(x) \in F(x)$ $(\forall x)$ a *selection* of F. The following theorem was given by Yang [10].

Theorem 1.1. *Let F be a set-valued mapping from X into itself and a simplicial decomposition of the convex hull* co$X$ *be given. If a selection f of F satisfies (1.2), then F has a fixed point $\bar{x}$, that is, $\bar{x} \in F(\bar{x})$.*

An important application of fixed point theorems is a bimatrix game. A bimatrix game consists of two players and $m \times n$ payoff matrices $A = (a_{ij})$ and $B = (b_{ij})$. Players 1 and 2 maximize $x^T Ay$ and $x^T By$, respectively, where $x \in P_m$ and $y \in P_n$ are probability vectors. A pair of probability vectors $(\bar{x}, \bar{y})$ is called a *Nash equilibrium* if

$$x^T A\bar{y} \leq \bar{x}^T A\bar{y}, \ \bar{x}^T By \leq \bar{x}^T B\bar{y} \quad \forall x \in P_m, \forall y \in P_n.$$

In particular, when $\bar{x}$ and $\bar{y}$ are standard unit vectors e_i and e_j, respectively, $(\bar{x}, \bar{y})$ is called a *pure-strategy Nash equilibrium.* The set of best responses is defined as follows:

$$F_1(j) = \{i \mid a_{ij} \geq a_{i'j} \ \forall i'\}, \quad F_2(i) = \{j \mid b_{ij} \geq b_{ij'} \ \forall j'\}.$$

Then a pure-strategy Nash equilibrium (e_i, e_j) is characterized by $(i, j) \in F(i, j) := F_1(j) \times F_2(i)$.

In Section 2, we characterize the direction preserving condition for the best response mappings of a bimatrix game, and give a sufficient condition that the bimatrix game has a pure-strategy Nash equilibrium (Theorem 2.5). In Section 3, we define a generalized Freudenthal decomposition in $\mathbb{R}^n$, and characterize the direction preserving condition in a non-cooperative n-person game. In Section 4, we briefly review type M to make a comparative review of types B and M.

2. Direction preserving condition in a bimatrix game

In this section we characterize the direction preserving condition in a bimatrix game. We show that the simplicial decomposition of the rectangular grid (Figure 1-left) is essential for the characterization. We deal with the Freudenthal decomposition in $\mathbb{R}^2$ (Figure 1-right), its rotation, and a general simplicial decomposition of the rectangular grid. Then we get a sufficient condition for the bimatrix game to have a pure-strategy Nash equilibrium.

Before going any further, we remark that the column (raw) number of matrices begins with not 1 but 0 in this paper. That is convenient because we define a simplicial decomposition of a grid in $\mathbb{R}^n$ by shifting a simplicial decomposition of the hypercube $[0, 1]^n$.

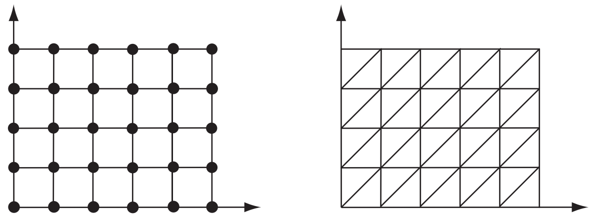

FIGURE 1. For the rectangular grid (left), right is the Freudenthal decomposition.

First, it is clear that, for any selection $f = (f_1, f_2)$ of F, (1.1) reduces to

$$\begin{matrix} (f_1(j) - i)(f_1(j') - i') \geq 0 \\ (f_2(i) - j)(f_2(i') - j') \geq 0 \end{matrix} \quad \forall (i,j) \sim (i',j'). \tag{2.1}$$

Theorem 2.1. *When we take the Freudenthal decomposition of the rectangular grid, a selection $f = (f_1, f_2)$ of the best response F is direction preserving if and only if*

$$\begin{matrix} f_1(j) \leq f_1(j+1) \leq f_1(j) + 1 \\ f_2(i) \leq f_2(i+1) \leq f_2(i) + 1 \end{matrix} \quad \forall (i,j). \tag{2.2}$$

Proof. Taking $i = f_1(j) + 1$ in the first inequality of (2.1), we have

$$f_1(j') \leq i' \quad \forall (i',j') \sim (f_1(j) + 1, j).$$

Since $(f_1(j) + 1, j + 1) \sim (f_1(j) + 1, j)$, we get $f_1(j + 1) \leq f_1(j) + 1$. Taking $i = f_1(j) - 1$ in the first inequality of (2.1), we have

$$f_1(j') \geq i' \quad \forall (i',j') \sim (f_1(j) - 1, j).$$

Since $(f_1(j), j + 1) \sim (f_1(j) - 1, j)$, we get $f_1(j + 1) \geq f_1(j)$. Similarly, we obtain $f_2(i) \leq f_2(i + 1) \leq f_2(i) + 1$ from the second inequality of (2.1). Conversely, let $(i, j) \sim (i', j')$. Then, we may assume that $(i', j') = (i, j) + (d_1, d_2)$ for some $(d_1, d_2) \in \{0, 1\}^2$. By (2.2), there exists $\delta_i \in \{0, 1\}$ such that $f_1(j') = f_1(j) + \delta_1$, so that

$$(f_1(j) - i)(f_1(j') - i') = (f_1(j) - i)(f_1(j) + \delta_1 - i - d_1). \tag{2.3}$$

If $f_1(j)-i > 0$, then $f_1(j)+\delta_1-i-d_1 \geq 0$. If $f_1(j)-i < 0$, then $f_1(j)+\delta_1-i-d_1 \leq 0$. In both cases, RHS of (2.3) is nonnegative. Similarly, we have $(f_2(i)-j)(f_2(i')-j') \geq 0$. □

The following theorem is similarly proved as Theorem 2.1.

Theorem 2.2. *When we take the simplicial decomposition in Figure 3-left, a selection f of the best response F is direction preserving if and only if*

$$\begin{matrix} f_1(j) - 1 \leq f_1(j+1) \leq f_1(j) \\ f_2(i) - 1 \leq f_2(i+1) \leq f_2(i) \end{matrix} \quad \forall (i,j). \tag{2.4}$$

Actually, for any simplicial decomposition of the rectangular grid, we can characterize the direction preserving condition. The given simplicial decomposition can be regarded as an undirected graph, say G (Figure 4-left). Let G_V (G_H) be the

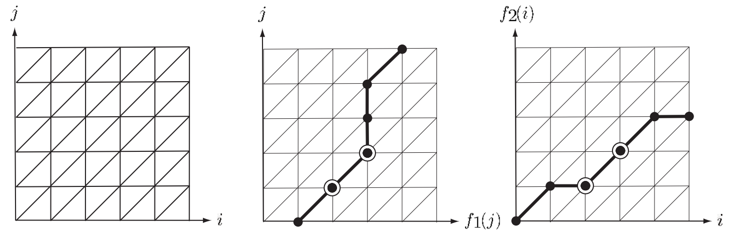

FIGURE 2. When we take the Freudenthal decomposition, any direction preserving best response is monotone with at most 1 increment. Double circles indicate pure-strategy Nash equilibria.

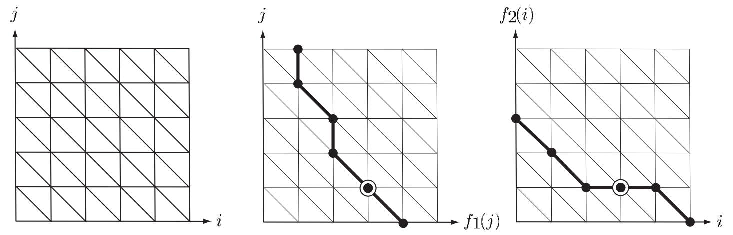

FIGURE 3. When we take a rotation of the Freudenthal decomposition, any direction preserving best response is monotone with at most 1 decrement. The double circle indicates a pure-strategy Nash equilibrium.

graph obtained by deleting the horizontal (vertical) edges from G, see Figure 4-center (right). The direction preserving condition is characterized in terms of G_V and G_H.

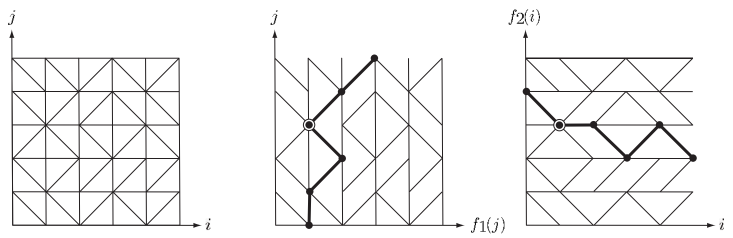

FIGURE 4. For any simplicial decomposition (left), center and right denote G_V and G_H, respectively. Double circle indicates a pure-strategy Nash equilibrium.

Theorem 2.3. *When we take an arbitrary simplicial decomposition of the rectangular grid, a selection* $f = (f_1, f_2)$ *of the best response* F *is direction preserving if and only if polygonal line* $(f_1(0), 0)$, $(f_1(1), 1)$, $\ldots$, $(f_1(n), n)$ *is a subgraph of* G_V *and polygonal line* $(0, f_2(0))$, $(1, f_2(1))$, $\ldots$, $(m, f_2(m))$ *is a subgraph of* G_H.

Proof. Necessity: (By induction on j) Assume that the polygonal line $(f_1(0), 0)$, $\ldots$, $(f_1(j), j)$ is a subgraph of G_V. When $0 < f_1(j) < m$, we see from (1.1)

$$f_1(j') \leq i' \qquad \forall (i', j') \sim (f_1(j) + 1, j), \tag{2.5}$$

$$f_1(j') \geq i' \qquad \forall (i', j') \sim (f_1(j) - 1, j). \tag{2.6}$$

Since $(f_1(j)+1, j+1) \sim (f_1(j)+1, j)$ and $(f_1(j)-1, j+1) \sim (f_1(j)-1, j)$, we get from (2.5) and (2.6) that $f_1(j) - 1 \leq f_1(j+1) \leq f_1(j) + 1$.

Case 1: When $(f_1(j), j+1) \sim (f_1(j)+1, j)$, we see from (2.5) that $f_1(j+1) \leq f_1(j)$. Case 2: When $(f_1(j), j+1) \not\sim (f_1(j)+1, j)$, $(f_1(j), j)$ must be cell-connected to $(f_1(j)+1, j+1)$. Case 3: When $(f_1(j), j+1) \sim (f_1(j)-1, j)$, we see from (2.6) that $f_1(j+1) \geq f_1(j)$. Case 4: When $(f_1(j), j+1) \not\sim (f_1(j)-1, j)$, $(f_1(j), j)$ must be cell-connected to $(f_1(j)-1, j+1)$. Since $f_1(j+1) \in \mathbb{Z}$,

$$f_1(j+1) = \begin{cases} f_1(j) & \text{Case 1 and Case 3,} \\ f_1(j) \text{ or } f_1(j) & \text{Case 1 and Case 4,} \\ f_1(j) \text{ or } f_1(j)+1 & \text{Case 2 and Case 3,} \\ f_1(j)-1,\ f_1(j), \text{ or } f_1(j)+1 & \text{Case 2 and Case 4.} \end{cases} \tag{2.7}$$

Figure 5 indicates four patterns of simplicial decompositions around $(f_1(j), j)$. In any case, $\{(f_1(j), j), (f_1(j+1), j+1)\}$ is an edge of G_V.

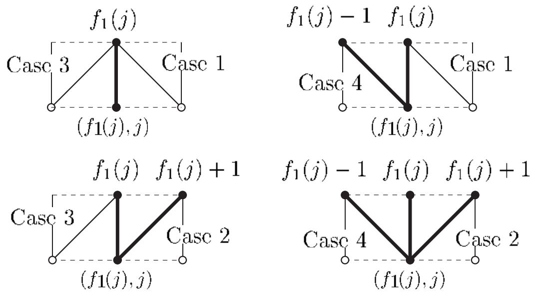

FIGURE 5. Simplicial decompositions around $(f_1(j), j)$.

When $f_1(j) = 0$ or m, it is similarly prove that the edge joining $(f_1(j), j)$ and $(f_1(j+1), j+1)$ belongs to G_V. Hence polygonal line $(f_1(0), 0)$, $\ldots$, $(f_1(j), j)$, $(f_1(j+1), j+1)$ is a subgraph of G_V. Similarly, polygonal line $(0, f_2(0))$, $(1, f_2(1))$, $\ldots$, $(m, f_2(m))$ is also a subgraph of G_H.

Sufficiency: We have to show that

$$(f_1(j) - i)(f_1(j') - i') \geq 0 \quad \forall (i', j') \sim (i, j). \tag{2.8}$$

Case A: when $j' = j+1$, since the edge joining $(f_1(j), j)$ and $(f_1(j+1), j+1)$ is an edge of G_V, we have $|f_1(j+1) - f_1(j)| \leq 1$. Since $|i' - i| \leq 1$, (2.8) trivially holds when $|f_1(j) - i| \geq 2$. Case A1: When $f_1(j) - i = 1$, (2.8) reduces to $f_1(j+1) \geq i'$. Since $(i, j) = (f_1(j) - 1, j)$ is the lower-left vertex of each pattern in Figure 5, any $(i', j') = (i', j+1) \sim (i, j)$ satisfies $f_1(j+1) \geq i'$. Case A2: when $f_1(j) - i = -1$, (2.8) reduces to $f_1(j+1) \leq i'$. Since $(i, j) = (f_1(j) + 1, j)$ is the lower-right vertex of each pattern in Figure 5, any $(i', j') = (i', j+1) \sim (i, j)$ satisfies $f_1(j+1) \leq i'$.

Hence (2.8) holds in Case A. Case B: when $j' = j - 1$, (2.8) is similarly proved. Case C: when $j' = j$, (2.8) trivially holds.

It is also proved that $(f_2(i) - j)(f_2(i') - j') \geq 0$ for any $(i', j') \sim (i, j)$ as well as (2.8). □

Remark 2.4. The original definition of the direction preserving condition in [1] adopted $||x' - x||_\infty \leq 1$ instead of $x \sim x'$ in (1.1). In that case, the original direction preserving condition is characterized as follows.

$$f_1(j+1) = f_1(j),\ f_2(i+1) = f_2(i) \quad \forall i,\ j,$$

which is too strict, see Figure 6.

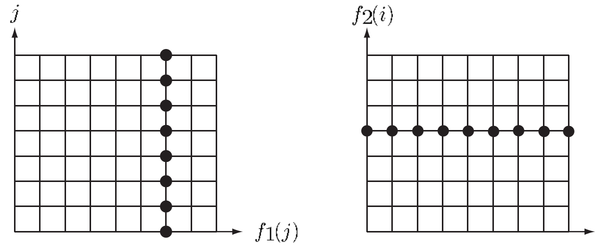

FIGURE 6. If we adopt $||x'-x||_\infty \leq 1$ instead of $x \sim x'$, any direction preserving best response must be constant.

Theorem 2.3 is restated in terms of Nash equilibrium as below.

Theorem 2.5. *If there exists a simplicial decomposition of the rectangular grid and a selection $f = (f_1, f_2)$ of the best response F such that polygonal line $(f_1(0), 0)$, $(f_1(1), 1)$, ..., $(f_1(n), n)$ is a subgraph of G_V and polygonal line $(0, f_2(0))$, $(1, f_2(1))$, ..., $(m, f_2(m))$ is a subgraph of G_H, then there exists a pure-strategy Nash equilibrium.*

3. Direction preserving condition in an n-person game

In this section, we consider the direction preserving condition for best responses in non-cooperative n-person games. Let $X_i = \{0, 1, \ldots, m_i\}$ be the set of pure strategies of player i, $X := \prod_{i=1}^n X_i$, and $X_{-i} := \prod_{j \neq i}^n X_j$. Any element of X_{-i} is denoted as x_{-i}. So $x \in X$ is expressed as $x = (x_i, x_{-i})$. Let $r_i(x)$ be the reward function of player i for $x \in X$,

$$F_i(x_{-i}) := \{x_i \in X_i \mid r(x_i, x_{-i}) \geq r(y_i, x_{-i})\ \forall y_i \in X_i\},$$

$$F(x) := \prod_{i=1}^n F_i(x_{-i}).$$

Then $x \in X$ is a pure-strategy Nash equilibrium if and only if $x \in F(x)$. Let $f = (f_1, \ldots, f_n)$ be a selection of F, that is, $f_i(x_{-i}) \in F_i(x_{-i})$ for any x and i. Then the direction preserving condition (1.1) reduces to

$$(f_i(x_{-i}) - x_i)(f_i(x'_{-i}) - x'_i) \geq 0 \quad \forall x \sim x',\ \forall i. \tag{3.1}$$

The Freudenthal decomposition of the grid of $\prod_{i=1}^{n}\{0,1,\ldots,m_i\}$ is defined as follows. For any permutation $\pi \in \mathfrak{S}_n$ on $\{1,\ldots,n\}$, put

$$\sigma_\pi = \mathrm{co}\{0, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, \ldots, e_{\pi(1)} + e_{\pi(2)} + \cdots + e_{\pi(n)}\}.$$

Then $\{\sigma_\pi \mid \pi \in \mathfrak{S}_n\}$ gives a simplicial decomposition of the hypercube $[0,1]^n$. We shift this decomposition to the whole grid to obtain *the Freudenthal decomposition* in $\mathbb{R}^n$.

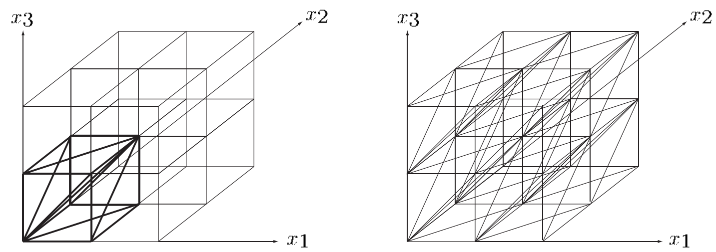

FIGURE 7. Left is the Freudenthal decomposition of $[0,1]^3$. Right is the Freudenthal decomposition of the whole grid.

Theorem 3.1. *When we take the Freudenthal decomposition in $\mathbb{R}^n$, a selection f of the best response F is direction preserving if and only if*

$$f_i(x_{-i}) \le f_i(x_{-i} + d_{-i}) \le f_i(x_{-i}) + 1 \quad \forall x \in X,\ \forall d \in \{0,1\}^n,\ \forall i. \tag{3.2}$$

Proof. Theorem 3.1 is a special case of Theorem 3.3 below. □

As well as Theorem 2.2, we can take a rotation of the Freudenthal decomposition and obtain an extension of Theorem 3.1. Namely, let e'_i be either e_i or $-e_i$. For any permutation $\pi \in \mathfrak{S}_n$, put

$$\sigma'_\pi = \mathrm{co}\{0, e'_{\pi(1)}, e'_{\pi(1)} + e'_{\pi(2)}, \ldots, e'_{\pi(1)} + e'_{\pi(2)} + \cdots + e'_{\pi(n)}\}.$$

Then $\{\sigma'_\pi \mid \pi \in \mathfrak{S}_n\}$ gives a simplicial decomposition of a hypercube $\mathrm{co}\{\sum_{j\in J} e'_j \mid J \subset \{1,\ldots,n\}\}$, so that $\{\sigma'_\pi + \sum_{e'_j=-e_j} e_j \mid \pi \in \mathfrak{S}_n\}$ gives a simplicial decomposition of $[0,1]^n$. We define the *generalized Freudenthal decomposition* by shifting the latter decomposition to the whole grid.

Next, we equip the integer lattice $\{\sum_{j\in J} e'_j \mid J \subset N := \{1,\ldots,n\}\}$ with a partial order $\preceq$ by

$$\sum_{j\in I} e'_j \preceq \sum_{j\in J} e'_j \iff I \subset J,$$

and extend it to $\mathbb{Z}^n$ by parallel translation.

Lemma 3.2. For any generalized Freudenthal decomposition, $x \sim x'$ if and only if they are comparable ($x \preceq x'$ or $x \succeq x'$) and $||x - x'||_\infty \le 1$.

Proof. When $x \sim x'$, they are vertices of a same n-simplex. Hence it is clear that $||x - x'||_\infty \le 1$. Since the simplex is expressed as $\sigma = \sigma'_\pi + \sum_{e'_j=-e_j} e_j + z$ for some

$\pi \in \mathfrak{S}_n$ and $z \in \mathbb{Z}^n$, we have

$$x = e'_{\pi(1)} + \cdots + e'_{\pi(k)} + \sum_{e'_j = -e_j} e_j + z,\ x' = e'_{\pi(1)} + \cdots + e'_{\pi(l)} + \sum_{e'_j = -e_j} e_j + z.$$

for some k and l. If $k \le l$, then $x \preceq x'$. If $k \ge l$, then $x \succeq x'$.

Conversely, when $x \preceq x'$ and $||x - x'||_\infty \le 1$, there exists some $z \in \mathbb{Z}^n$ such that $x, x' \in [0,1]^n + z$ and $x - z \preceq x' - z$. Hence there exist $I \subset I' \subset N$ such that $x - z = \sum_{j \in I} e'_j$ and $x' - z = \sum_{j \in I'} e'_j$. Taking a permutation π satisfying

$$I = \{\pi(1), \dots, \pi(|I|)\}, \quad I' = \{\pi(1), \dots, \pi(|I|), \dots, \pi(|I'|)\},$$

we see that $x - z,\ x' - z \in \sigma'_\pi$, so that $x \sim x'$. □

Theorem 3.3. *When we take a generalized Freudenthal decomposition in $\mathbb{R}^n$, a selection f of the best response F is direction preserving if and only if (3.3) holds for any $x \in X$ and $d \in \{\sum_{j \in J} e'_j \mid J \subset N\}$*

$$\begin{cases} f_i(x_{-i}) \le f_i(x_{-i} + d_{-i}) \le f_i(x_{-i}) + 1 & \text{if } e'_i = e_i, \\ f_i(x_{-i}) \ge f_i(x_{-i} + d_{-i}) \ge f_i(x_{-i}) - 1 & \text{if } e'_i = -e_i. \end{cases} \tag{3.3}$$

So, if f satisfies (3.3), then there exists $\bar{x} \in X$ such that $f_i(\bar{x}_{-i}) = \bar{x}_i$ for any $i = 1, \dots, n$. Namely, $\bar{x}$ is a pure-strategy Nash equilibrium.

Proof. Necessity: For any $x \in X$ and $d := \sum_{j \in I} e'_j$ $(I \subset N)$, we see from Lemma 3.2 that $y := (f_i(x_{-i}) + 1, x_{-i}) \sim y' := (f_i(x_{-i}) + 1, x_{-i} + d_{-i})$. Since $y_{-i} = x_{-i}$ and $y'_{-i} = x_{-i} + d_{-i}$, we have by (3.1)

$$0 \le (f_i(y_{-i}) - y_i)(f_i(y'_{-i}) - y'_i) = f_i(x_{-i}) + 1 - f_i(x_{-i} + d_{-i}).$$

Since $y := (f_i(x_{-i}) - 1, x_{-i}) \sim y' := y + \sum_{j(\ne i) \in I} e'_j = (f_i(x_{-i}) - 1, x_{-i} + d_{-i})$, we have by (3.1)

$$0 \le (f_i(y_{-i}) - y_i)(f_i(y'_{-i}) - y'_i) = f_i(x_{-i} + d_{-i}) - f_i(x_{-i}) + 1.$$

In the case of $e'_i = e_i$, we have, by Lemma 3.2 and (3.1),

$$z := (f_i(x_{-i}) - 1, x_{-i}) \sim z' := z + d = (f_i(x_{-i}), x_{-i} + d_{-i}),$$

$$0 \le (f_i(z_{-i}) - z_i)(f_i(z'_{-i}) - z'_i) = f_i(x_{-i} + d_{-i}) - f_i(x_{-i}).$$

In the case of $e'_i = -e_i$, we have, by Lemma 3.2 and (3.1),

$$z := (f_i(x_{-i}) + 1, x_{-i}) \sim z' := z + d = (f_i(x_{-i}), x_{-i} + d_{-i}),$$

$$0 \le (f_i(z_{-i}) - z_i)(f_i(z'_{-i}) - z'_i) = f_i(x_{-i}) - f_i(x_{-i} + d_{-i}).$$

Sufficiency: Assume that $x \sim x'$. Then it follows from Lemma 3.2 that $x' = x + d$ or $x = x' + d$ for some $d = \sum_{j \in I} e'_j$ $(I \subset N)$. It suffices to consider the first case. Case 1: When $e'_i = e_i$, we have $d_i \in \{0, 1\}$. It follows from the first half of (3.3) that $f_i(x'_{-i}) = f_i(x_{-i} + d_{-i}) = f_i(x_{-i}) + \delta_i$ for some $\delta_i \in \{0, 1\}$. So

$$(f_i(x_{-i}) - x_i)(f_i(x'_{-i}) - x'_i) = (f_i(x_{-i}) - x_i)(f_i(x_{-i}) + \delta_i - x_i - d_i). \tag{3.4}$$

If $f_i(x_{-i}) > x_i$, then $f_i(x_{-i}) + \delta_i - x_i - d_i \ge 0$. If $f_i(x_{-i}) < x_i$, then $f_i(x_{-i}) + \delta_i - x_i - d_i \le 0$. RHS of (3.4) is nonnegative in either case.

Case 2: When $e'_i = -e_i$, we have $d_i \in \{0, -1\}$. It follows from (3.3) that $f_i(x'_{-i}) = f_i(x_{-i}) - \delta_i$ for some $\delta_i \in \{0, 1\}$. So

$$(f_i(x_{-i}) - x_i)(f_i(x'_{-i}) - x'_i) = (f_i(x_{-i}) - x_i)(f_i(x_{-i}) - \delta_i - x_i - d_i). \tag{3.5}$$

If $f_i(x_{-i}) > x_i$, then $f_i(x_{-i}) - \delta_i - x_i - d_i \geq 0$. If $f_i(x_{-i}) < x_i$, then $f_i(x_{-i}) - \delta_i - x_i - d_i \leq 0$. RHS of (3.5) is nonnegative in either case. □

4. Discrete fixed point theorem for monotone mappings

Topkis [9] derived a discrete fixed point theorem (Theorem 4.1) for monotone mappings from Tarski's fixed point theorem [8]. In this section, we apply Theorem 4.1 to a bimatrix game, and make a comparative review of type B (Theorems 2.1, 2.2) and type M (Theorem 4.1).

Let X_i, X, X_{-i}, F, and f be same with those in Section 3. We assume that each X_i is equipped with an order $0 \preceq 1 \preceq \cdots \preceq m_i$ or $m_i \preceq m_i - 1 \preceq \cdots \preceq 0$. They induce a component-wise partial order $\preceq$ in X and X_{-i}.

Theorem 4.1 ([9])**.** *If mappings $f_i : X_{-i} \to X_i$ $(i = 1, \ldots, n)$ satisfy*

$$x_{-i} \preceq x'_{-i} \quad \Rightarrow \quad f_i(x_{-i}) \preceq f_i(x'_{-i}), \tag{4.1}$$

then there exists $\bar{x} \in X$ such that $f_i(\bar{x}_{-i}) = \bar{x}_i$ for any $i = 1, \ldots, n$. Namely, $\bar{x}$ is a pure-strategy Nash equilibrium.

In the following we consider a bimatrix game. When we define $(x_1, x_2) \preceq (x'_1, x'_2)$ by $x_1 \leq x'_1$ and $x_2 \leq x'_2$, (4.1) reduces to (4.2) below, which implies that f_1 and f_2 are nondecreasing, see Figure 8.

$$x_2 \leq x'_2 \;\Rightarrow\; f_1(x_2) \leq f_1(x'_2), \quad x_1 \leq x'_1 \;\Rightarrow\; f_2(x_1) \leq f_2(x'_1). \tag{4.2}$$

When we define $(x_1, x_2) \preceq (x'_1, x'_2)$ by $x_1 \leq x'_1$ and $x_2 \geq x'_2$, then (4.1) reduces to (4.3) below, which implies that f_1 and f_2 are nonincreasing.

$$x_2 \geq x'_2 \;\Rightarrow\; f_1(x_2) \leq f_1(x'_2), \quad x_1 \leq x'_1 \;\Rightarrow\; f_2(x_1) \geq f_2(x'_1). \tag{4.3}$$

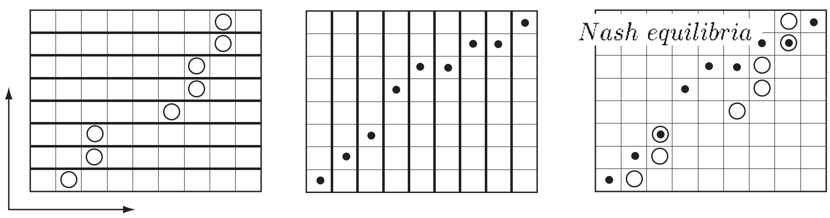

Figure 8. If best response mappings f_1 and f_2 are nondecreasing, there exists a pure-strategy Nash equilibrium.

As we have seen in Sections 2, 3 and 4, if a best response $f = (f_1, f_2)$ satisfies one of the following conditions for a bimatrix game, there exists a pure-strategy Nash equilibrium.

(a) Both f_1 and f_2 are nondecreasing.
(b) Both f_1 and f_2 are nonincreasing.
(c) Both f_1 and f_2 are monotone with at most 1 increment.

(d) Both f_1 and f_2 are monotone with at most 1 decrement.
(e) The polygonal line connecting $(f_1(j), j)$ $(j = 0, \ldots, n)$ is a subgraph of G_V, and the polygonal line connecting $(i, f_2(i))$ $(i = 0, \ldots, m)$ is a subgraph of G_H.

It should be noted that (e) does not require any monotonicity. Type M (Theorem 4.1) can deal with (a)-(d), and Type B (Theorems 2.1, 2.2, 3.1, and 3.3) can deal with (c)-(e).

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