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MINIMAL POINTS, VARIATIONAL PRINCIPLES, AND VARIABLE PREFERENCES IN SET OPTIMIZATION

TRUONG Q. BAO, BORIS S. MORDUKHOVICH*, AND ANTONE SOUBEYRAN

Dedicated to Wataru Takahashi in honor of his 70th birthday

Abstract. The paper is devoted to variational analysis of set-valued mappings acting from quasimetric spaces into topological spaces with variable ordering structures. Besides the mathematical novelty, our motivation comes from applications to adaptive dynamical models of behavioral sciences. We develop a unified dynamical approach to variational principles in such settings based on the new minimal point theorem for product sets with general ordering. This approach allows us, in particular, to establish enhanced versions of the Ekeland variational principle for set-valued mappings ordered by variable preference.

1. Introduction and motivations

Basic principles, methods, and results of variational analysis are crucial in many areas of nonlinear analysis, optimization, and their applications; see, e.g., [8, 36, 40] and the references therein. The importance of variational analysis has been proven for problems of multiobjective optimization with single-valued vectorial and then set-valued objectives. The reader can find more information on these and related topics in the books [9, 19, 36] and a number of papers; in particular, those cited below. Besides the growing interest to optimization problems with set-valued objectives motivated by applications to economics, finance, etc. (as, e.g., in [3, 24, 25]), quite recently [4, 5, 6, 10, 13, 14] a serious attention has been drawn to multiobjective optimization problems with variable ordering structures.

Our interest to study such problems has been mainly motivated by applications to adaptive dynamical models of behavioral sciences. The mathematical realization of the variational rationality approach by Soubeyran [43, 44] to behavioral models requires a *simultaneous* consideration of *set-valued* mapping defined on *quasimetric* spaces with values ordered by *variable* preferences. This has been partly demonstrated in [5, 6] on the basis of variational analysis and fixed point theory with applications to particular models related to goal systems in psychology [32] and capability theory of wellbeing [41]. Further mathematical developments are needed for their own interest and subsequent applications.

This paper is focused on deriving basic variational principles and results concerning set-valued mappings acting in the aforementioned settings. First we establish a

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general *parametric minimal point* theorem for product sets endowed with abstract (pre)ordering relationships that interrelate with the imposed quasimetric structure of the decision space. The proof of this theorem offers a *dynamical process*, which is on one hand at the mainstream of variational analysis, while on the other hand corresponds to the very nature of the variational rationality approach in applications to behavioral models. From this result we derive, in particular, new set-valued versions of the Ekeland variational principle for mappings from quasimetric into ordering spaces with variable preferences. Furthermore, we apply the parametric minimal point theorem to obtaining another type of variational principles with set-valued quasidistance perturbations.

The rest of the paper is organized as follows. Section 2 presents basic definitions and preliminary material on quasimetric spaces and ordering relationships widely used for the formulations and proofs of the main results below. Section 3 is devoted to the parametric minimal point theorem, its various consequences, modifications, and illustrations.

Section 4 contains applications of the parametric minimal point theorem and the approach developed in its proof to deriving new variational principles of the Ekeland type for the case of set-valued mappings discussed above. We given a short overview of the major extensions of the Ekeland principle and their proofs in vector and multiobjective settings and then show that basically all the known results of this type follow from our theorems. Section 5 applies the parametric minimal point theorem to deriving variational principles with set-valued quasidistance perturbations. In the final Section 6 we discuss further applications of the obtained results to some models of behavioral sciences.

2. Basic definitions and preliminaries

First we present and discuss the definitions of quasimetric spaces and the corresponding notions of closedness, compactness, and completeness in such spaces; see, e.g., [11].

Recall that (X, d) is a metric space if the distance function $d : X \times X \to I\!R$ satisfies the condition: (1) $d(x, y) \geq 0$; (2) $d(x, y) = 0$ if and only if $x = y$; (3) $d(x, y) = d(y, x)$; (4) $d(x, z) \leq d(x, y) + d(y, z)$ on X. In this paper we deal with a broader class of *quasimetric spaces* (X, q), where the quasimetric $q\colon X \times X \to I\!R$ satisfies conditions (1), (4) and (2′) $d(x, x) = 0$ for all $x \in X$, but *not* the symmetry condition (3).

Note that considering quasimetrics is beneficial even of finite-dimensional spaces. Furthermore, quasimetrics are common in real life. Let, e.g., X be the set of mountain villages and $q(x, y)$ stand for the walking times between elements of X. It is a quasimetric because travel uphill takes longer than travel downhill. Another case is a taxicab net with one-way streets, where a path from A to B comprises a different set of streets than a path from B to A. A simple and remarkable example is the *Sorgenfrey quasimetric* on $I\!R$ defined by

$$q(x, y) := \begin{cases} x - y & \text{if } x \geq y, \\ 1 & \text{otherwise} \end{cases} \tag{2.1}$$

describing the filing down a metal stick: it is easy to reduce its size, but not to grow it.

Classical topology tells us that every metric space (X, d) can be viewed as a topological space on which the topology is constructed by taking as a base of the neighborhood filter of $x \in X$ given by the balls $B(x, \varepsilon) := \{y \in \mid d(x, y) < \varepsilon\}$. We can similarly proceed with quasimetric spaces (X, q), while the absence of symmetry for q requires considering two different topologies corresponding to the left and right balls as

$$I\!B_l(x, \varepsilon) := \big\{y \in X \big| \, q(x, y) < \varepsilon\big\}, \quad I\!B_r(x, \varepsilon) := \big\{y \in X \big| \, q(y, x) < \varepsilon\big\}.$$

Recall some notions from quasimetric spaces used in what follows:

• We say that the sequence $\{x_n\} \subset X$ (*left-sequentially*) *converges* to $\bar{x} \in X$ and denote it by $x_n \to \bar{x}$ if $q(x_n, x_*) \to 0$ as $k \to \infty$.

• $\{x_n\} \subset X$ is (*left-sequential*) *Cauchy* if for each $k \in I\!N$ there is $N_k \in I\!N$ such that

$$q(x_n, x_m) < 1/k \text{ for all } m \geq n \geq N_k.$$

• A quasimetric space (X, q) is (*left-sequentially*) *complete* if each left-sequential Cauchy sequence is convergent and its limit belongs to X.

• (X, q) is *Hausdorff topological* if we have the implication

$$\Big[\lim_{n\to\infty} q(x_n, \bar{x}) = 0, \lim_{n\to\infty} q(x_n, \bar{u}) = 0\Big] \Longrightarrow \bar{x} = \bar{u}. \tag{2.2}$$

• A quasimetric space (X, q) ordered by a preorder $\preceq$ (see below) satisfies the *Hausdorff decreasing condition* if for every decreasing sequence $\{x_k\} \subset X$ and $\bar{x}, \bar{u} \in X$ with $\bar{x} \preceq \bar{u}$ the implication in (2.2) holds.

• A nonempty subset $\Omega \subset X$ of the quasimetric space (X, q) is (*left-sequentially*) *closed* if for any convergent sequence $\{x_n\} \subset \Omega$ with the limit $\bar{x}$ we have $\bar{x} \in \Omega$.

For brevity we *drop* mentioning "left-sequential" in what follows.

It is easy to see that the Sorgenfrey line $(I\!R, q)$ in (2.1) is Hausdorff topological. Observe that there are simple quasimetric spaces with $X = I\!R$ for which Hausdorff condition (2.2) fails. Consider, e.g., $(I\!R, q')$ with

$$q'(x, y) := \begin{cases} x - y & \text{if } x \geq y, \\ e^{x-y} & \text{otherwise.} \end{cases} \tag{2.3}$$

To verify that (2.2) does not hold, it is sufficient to show that the sequence $\{x_k\}$ with $x_k := -k$ has more than one limit. Fix $a \in I\!R$ and observe that $q(x_k, a) = q(-k, a) = e^{(-k-a)}$ for all $k > -a$, i.e., the numbers x_k left-sequentially converge to a. Since a was chosen arbitrarily, this sequence has infinitely many limits.

Next we recall the definitions of binary relations and preferences taken from [45, Definition 1.4]; cf. also [19, 28, 35, 36] and the references therein.

Given a nonempty set Z, a *binary relation* $\mathcal{R}$ on Z is a collection of ordered pairs of elements in Z, i.e., it is defined by a subset $Q \subset Z \times Z$ as follows: $u\mathcal{R}z$ if and only if $(u, z) \in Q$. Let us identify properties of $\mathcal{R}$ on Z. It is said to be:

• *reflexive* if $z\mathcal{R}z$ for all $z \in Z$;

• *antisymmetric* if $[z\mathcal{R}u, \ u\,\mathcal{R}z] \Longrightarrow u = z$ for all $z, u \in Z$;

• *transitive* if $[z\mathcal{R}u,\ u\mathcal{R}w] \Longrightarrow z\mathcal{R}w$ for all $z, u, w \in Z$.

Now we define the major ordering relations used in this paper.

Definition 2.1 (preorders and partial orders)**.** Consider a binary relation $\preceq$ on a nonempty set Z. We say that it is:

(i) a PREORDER if it is reflexive and transitive;
(ii) a PARTIAL ORDER if it is an antisymmetric preorder.

A nonempty set equipped with a preorder (respectively, a partial order) is called a *preordered* (respectively, an *ordered*) set.

Definition 2.2 (minimal points to sets)**.** Let Ξ be a preordered set with the preorder $\preceq$. We say that $\bar z \in \Xi$ is a MINIMAL POINT of Ξ with respect to $\preceq$ if there is no $z \in \Xi \setminus \{\bar z\}$ such that $z \preceq \bar z$. The collection of these minimal points is denoted by $\operatorname{Min}(\Xi; \preceq)$.

Given a preorder $\preceq$, we associate with it the *level-set mapping* $\mathcal{L} : Z \rightrightarrows Z$ defined by

$$\mathcal{L}(z) := \big\{u \in Z \big|\ u \preceq z\big\}$$

and observe that following descriptions of the above properties via $\mathcal{L}$:

• $z \in \mathcal{L}(z)$ for all $z \in Z$ if and only if the preorder $\preceq$ is reflexive.

• The preorder $\preceq$ is transitive if and only if $[u \preceq z \Longrightarrow u - \mathcal{L}(u) + z - \mathcal{L}(z) \subset z - \mathcal{L}(z)]$.

• $\bar z \in \operatorname{Min}(\Xi; \preceq)$ if and only if $\mathcal{L}(\bar z) = \{\bar z\}$.

Next we recall the concept of *Pareto efficiency*, which can be formulated in terms of a preorder as follows. Given a real topological space Z and a nonempty convex cone $\Theta \subset Z$, denote by $\preceq_\Theta$ the Pareto preference relation:

$$u \preceq_\Theta z \ \text{ if and only if } \ u \in z - \Theta. \tag{2.4}$$

Then $\bar z \in \Xi$ is a (*Pareto*) *minimal point* of Ξ with respect to Θ if $\bar z \in \operatorname{Min}(\Xi; \preceq_\Theta)$, i.e.,

$$\Xi \cap (\bar z - \Theta) = \{\bar z\}. \tag{2.5}$$

As usual in vector optimization, in this case we write $\operatorname{Min}(\Xi; \Theta)$ instead of $\operatorname{Min}(\Xi; \preceq_\Theta)$.

As mentioned in Section 1, a serious attention in the literature has been recently paid to multiobjective optimization problems with *variable ordering structures*, the notion that goes back to Yu [50, 51]. In the abstract sense, a variable ordering structure is given by a set-valued mapping $K : Z \rightrightarrows Z$, which defines a binary relation that may not be transitive or even compatible with positive scalar multiplication. In this paper we are interested in some special classes of variable ordering structures that will be specified later.

In contrast to the classical vector optimization with the Pareto ordering relation $\preceq_\Theta$ induced by a convex ordering cone $\Theta \subset Z$, we define now two "less" ordering relations for each variable ordering structure K. This is due to the fact that for any $u, z \in Z$ there are two different ordering cones $K[u]$ and $K[z]$.

Definition 2.3 (post-less and pre-less ordering relations)**.** Let $K : Z \rightrightarrows Z$ be a variable ordering structure imposed on a vector space Z. Then:

(i) The POST-LESS ORDERING RELATION $\preceq^{\text{post}}_{K[\cdot]}$ with respect to K is defined by

$$z \preceq_{K[u]} u \Longleftrightarrow z \in u - K[u]. \tag{2.6}$$

(ii) The PRE-LESS ORDERING RELATION $\preceq^{\text{pre}}_{K[\cdot]}$ is defined by

$$z \preceq_{K[z]} u \Longleftrightarrow u \in z + K[z]. \tag{2.7}$$

If K is a constant/nonvariable ordering structure $K[z] \equiv \Theta$ for some ordering set in Z containing the zero vector, there is no difference between the pre-less and post-less binary relations defined in (2.6) and (2.7); they both reduce to the Pareto ordering relation $\preceq_\Theta$ in (2.4). Note also that any given ordering relation $\preceq$ can be identified with the post-less binary relation $\preceq^{\text{post}}_{K[\cdot]}$ with respect to $K[z] := z - \mathcal{L}(z)$.

The next proposition provides sufficient conditions ensuring that both the binary relations (2.6) and (2.7) are preorders.

Proposition 2.4 (pre-less and post-less preorders)**.** *Let $K : Z \rightrightarrows Z$ be a variable ordering structure on a vector space Z, and let $0 \in K[z]$ for all $z \in Z$. The following hold:*

(i) *If K enjoys the* POST-LESS MONOTONICITY PROPERTY

$$\big[u \preceq_{K[z]} z\big] \Longrightarrow \big[K\,[u] + K\,[z] \subset K\,[z]\,\big],$$

then the post-less relation $\preceq^{\text{post}}_{K[\cdot]}$ is a preorder. Furthermore, the convexity and cone-valuedness of $K[z]$ ensure the inclusion $K\,[u] \subset K\,[z]$.

(ii) *If K enjoys the* PRE-LESS MONOTONICITY PROPERTY

$$\big[u \preceq_{K[u]} z\big] \Longrightarrow \big[K\,[z] + K\,[u] \subset K\,[u]\,\big],$$

then the pre-less relation $\preceq^{\text{pre}}_{K[\cdot]}$ is a preorder.

Proof. First we prove that $\preceq^{\text{post}}_{K[\cdot]}$ is a preorder. The reflexivity property follows from

$$0 \in K[z] \Longleftrightarrow z \in z - K[z] \Longleftrightarrow z \preceq_{K[z]} z$$

for any $z \in Z$. To check the transitivity property, take any vectors $z, z', z'' \in Z$ with $z \preceq_{K[z']} z' \Longleftrightarrow z \in z' - K[z']$ and $z' \preceq_{K[z'']} z'' \Longleftrightarrow z' \in z'' - K[z'']$. It yields

$$z \in z'' - K[z'] - K[z''], \quad \text{and so} \quad z \in z'' - K[z''] \tag{2.8}$$

by the post monotonicity condition for $z' \preceq_{K[z'']} z''$. This verifies the transitivity property and shows that $\preceq^{\text{post}}_{K[\cdot]}$ is a preorder. The proof of assertion (ii) is similar. $\square$

The question arises: is there any preordering structure with *nonconic* values satisfying the post-less monotonicity property? The following examples gives the affirmative answer.

Example 2.5 (nonconic preordering structure with the post-less monotonicity property)**.** Defined the variable ordering structure $K : I\!R^2 \rightrightarrows I\!R^2$ by

$$K[z] := \begin{cases} I\!R^2_+ & \text{if } z = 0, \\ \left\{(a,b) \in I\!R^2 \;\middle|\; a \geq 0,\ 0 \leq b \leq \left(e + \frac{1}{\|z\|}\right)^a - 1\right\} & \text{otherwise.} \end{cases}$$

It is obvious that $0 \in K[z]$ for all $z \in I\!R^2$ and that $u \in K[z]$ yields $\|u\| \geq \|z\|$. To verify the monotonicity property, take any pairs $(a,b) \in K[u]$ and $(c,d) \in K[z]$, which means that

$$a \geq 0,\ c \geq 0,\ b \leq \left(e + \frac{1}{\|z\|}\right)^a - 1, \text{ and } d \leq \left(e + \frac{1}{\|z\|}\right)^c - 1.$$

Since $c \geq 0$ and $\|u\| \geq \|z\|$, we have the relationships

$$\begin{aligned} & \frac{\left(e + \frac{1}{\|u\|}\right)^a - 1}{\left(e + \frac{1}{\|z\|}\right)^a - 1} \leq 1 \leq \left(e + \frac{1}{\|z\|}\right)^c \\ \Longleftrightarrow\quad & \left(e + \frac{1}{\|u\|}\right)^a - 1 \leq \left(e + \frac{1}{\|z\|}\right)^c \left(\left(e + \frac{1}{\|z\|}\right)^a - 1\right) \\ \Longleftrightarrow\quad & \left(e + \frac{1}{\|u\|}\right)^a - 1 \leq \left(e + \frac{1}{\|z\|}\right)^{a+c} - \left(e + \frac{1}{\|z\|}\right)^c \\ \Longleftrightarrow\quad & \left(e + \frac{1}{\|u\|}\right)^a - 1 + \left(e + \frac{1}{\|z\|}\right)^c - 1 \leq \left(e + \frac{1}{\|z\|}\right)^{a+c} - 1 \\ \Longrightarrow\quad & b + d \leq \left(e + \frac{1}{\|z\|}\right)^{(a+c)} - 1 \Longleftrightarrow (a,b) + (c,d) \in K[z]. \end{aligned}$$

Since (a,b) and (c,d) were chosen arbitrarily in $K[u]$ and $K[z]$, we get $K[u]+K[z] \subset K[z]$ verifying the monotonicity of K. It follows from Proposition 2.4 that $\preceq^{\text{post}}_{K[\cdot]}$ is a preorder.

Definition 2.6 (minimal points with respect to ordering structures)**.** Let $\Xi \neq \emptyset$ be a subset of a vector space Z equipped with a variable ordering structure $K : Z \rightrightarrows Z$. Then:

(i) $\bar{z} \in \Xi$ is a POST-LESS MINIMAL point of Ξ or a minimal point to Ξ with respect to the preorder $\preceq^{\text{post}}_{K[\cdot]}$ if there is no vector $z \in \Xi \setminus \{\bar{z}\}$ such that $z \preceq_{K[\bar{z}]} \bar{z}$, i.e.,

$$\Xi \cap \left(\bar{z} - K[\bar{z}]\right) = \{\bar{z}\}. \tag{2.9}$$

(ii) $\bar{z} \in \Xi$ is a PRE-LESS MINIMAL POINT to the set Ξ or a minimal point to Ξ with respect to the preorder $\preceq^{\text{pre}}_{K[\cdot]}$ if there is no vector $z \in \Xi \setminus \{\bar{z}\}$ such that $z \preceq_{K[z]} \bar{z}$, i.e.,

$$\forall\, z \in \Xi,\ z \not\preceq_{K[z]} \bar{z} \Longleftrightarrow \forall\, z \in \Xi,\ \bar{z} \notin z + K[z]. \tag{2.10}$$

It follows from (2.5) and (2.9) that $\operatorname{Min}(\Xi; \preceq^{\text{post}}_{K[\cdot]}) = \operatorname{Min}(\Xi; K[\bar{z}])$. According to [50, 51], a post-less minimal point is called an *extreme point* with respect to K, while a pre-less minimal point is called a *nondominated* point with respect to this structure.

3. Parametric minimal point theorem

This section is devoted to establishing a new minimal point theorem for preordered subsets of the Cartesian product of two spaces one of which (the decision space) is endowed with a quasimetric topological structure, while the other is an arbitrary set of parameters. We develop a constructive dynamical approach to prove the existence of minimal points for such sets employed to deriving set-valued variational principles in the subsequent sections.

Theorem 3.1 (parametric minimal point theorem in product spaces)**.** *Let (X,q) be a quasimetric space, let Z be a nonempty set of parameters, and let $\Xi\subset X\times Z$ be endowed with a preorder $\preceq$. Given $(x_0,z_0)\in\Xi$, define the (x_0,z_0)-level set of Ξ by*

$$\mathcal{L}:=\mathcal{L}(x_0,z_0)=\big\{(u,v)\in\Xi\big|\ (u,v)\preceq(x_0,z_0)\big\}$$

and assume that the following conditions hold:

(A1) Convergence monotonicity condition: *For any sequence $\{(x_k,z_k)\}\subset\mathcal{L}$ decreasing with respect to $\preceq$, we have that $q(x_k,x_{k+1})\to 0$ as $k\to\infty$.*

(A2) Limiting monotonicity condition: *for any sequence $\{(x_k,z_k)\}\subset\mathcal{L}$ decreasing with respect to $\preceq$, the Cauchy property of $\{x_k\}$ yields the existence of $(\bar{x},\bar{z})\in\mathcal{L}$ with*

$$(\bar{x},\bar{z})\preceq(x_k,z_k)\ \ \textit{for all}\ \ k\in I\!N. \tag{3.1}$$

(A3) Hausdorff monotonicity condition: *for any sequence $\{(x_k,z_k)\}\subset\mathcal{L}$ decreasing with respect to $\preceq$ and any $(\bar{u},\bar{v})\preceq(\bar{x},\bar{z})$, it follows from $q(x_k,\bar{x})\to 0$ and $q(x_k,\bar{u})\to 0$ as $k\to\infty$ that $\bar{x}=\bar{u}$.*

Then there is a decreasing sequence $\{(x_k,z_k)\}\subset\Xi$ starting at (x_0,z_0) and ending at a "partially" minimal point $(\bar{x},\bar{z})$ of Ξ with respect to $\preceq$ in the sense that if $(x,z)\in\Xi$ and $(x,z)\preceq(\bar{x},\bar{z})$, then $x=\bar{x}$. If furthermore $(\bar{x},\bar{z})$ satisfies the domination condition

$$\Big[(\bar{x},z)\preceq(\bar{x},\bar{z})\Longrightarrow z=\bar{z}\Big]\ \ \textit{for all}\ \ (\bar{x},z)\in\mathcal{L}, \tag{3.2}$$

then it can be chosen as a minimal point to the set Ξ with respect to $\preceq$.

Proof. Define the level-set mapping $\mathcal{L}:X\times Z\rightrightarrows X\times Z$ by

$$\mathcal{L}(x,z):=\big\{(u,v)\in\Xi\big|\ (u,v)\preceq(x,z)\big\}$$

and consider the projection of the set $\mathcal{L}(x,z)$ onto the space X given by

$$\mathcal{L}_X(x,z):=\mathrm{Proj}_X\big[\mathcal{L}(x,z)\big]=\big\{u\in X\big|\ (u,v)\in\mathcal{L}(x,z)\ \text{ for some }\ v\in Z\big\}.$$

It follows from the reflexivity and transitivity properties of $\preceq$ that

(a) $(x,z)\in\mathcal{L}(x,z)$ for all $(x,z)\in X\times Z$;
(b) if $(u,v)\preceq(x,z)$, then $\mathcal{L}(u,v)\subset\mathcal{L}(x,z)$ and thus $\mathcal{L}_X(u,v)\subset\mathcal{L}_X(x,z)$.

Denote $r(a;A) := \sup_{b\in A} q(a,b)$ and $\mathcal{L}_k := \mathcal{L}_X(x_k, z_k)$ for all $k \in \mathbb{N} \cup \{0\}$. Starting from $(x_0, z_0) \in \Xi$, we inductively construct a sequence $\{(x_k, z_k)\} \subset \Xi$ by

$$(3.3)\quad (x_k, z_k) \preceq (x_{k-1}, z_{k-1}) \ \text{ and } \ q(x_{k-1}, x_k) \geq r\big(x_{k-1}; \mathcal{L}_{k-1}\big) - 2^{-k}, \quad k \in \mathbb{N}.$$

It is clear from properties (a) and (b) of $\mathcal{L}$ that the iterative procedure (3.3) is well defined and that the generated sequence $\{(x_k, z_k)\}$ is decreasing with respect to $\preceq$.

It follows from (A1) that $q(x_k, x_{k+1}) \to 0$ as $k \to \infty$. Then the inequality in (3.3) implies that $r(x_k; \mathcal{L}_k) \to 0$. Using this and property (b) ensures that for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that for any $m \geq n \geq N_\varepsilon$ we have $x_m \in \mathcal{L}_m \subset \mathcal{L}_n$ due to (b) and

$$q(x_n, x_m) \leq r\big(x_n; \mathcal{L}_n\big) \leq \varepsilon \ \text{ whenever } \ m \geq n \geq N_\varepsilon,$$

which tells us that the sequence $\{x_k\} \subset X$ is Cauchy in (X, q). Condition (A2) yields the existence of $(\bar{x}, \bar{z}) \in \Xi$ satisfying (3.1), and thus we get that $q(x_k, \bar{x}) \to 0$ as $k \to \infty$.

To show now that $(\bar{x}, \bar{z})$ is a partial minimal point of Ξ with respect to $\preceq$, take an arbitrary pair $(u, v) \in \Xi \cap \mathcal{L}(\bar{x}, \bar{z})$ and deduce from (b) and (3.1) that

$$(u, v) \in \mathcal{L}(\bar{x}, \bar{z}) \subset \mathcal{L}(x_k, z_k) \ \text{ and thus } \ u \in \mathcal{L}_k, \quad k \in \mathbb{N}.$$

This justifies the validity of the inequality

$$q(x_k, u) \leq r\big(x_k; \mathcal{L}_k\big) \ \text{ for all } \ k \in \mathbb{N},$$

which implies in turn that $q(x_k, u) \to 0$ as $k \to \infty$, i.e., u is a limit of the sequence $\{x_k\}$. The Hausdorff monotonicity condition (A3) yields $u = \bar{x}$. Since (u, v) was chosen arbitrarily, this shows that $(\bar{x}, \bar{z})$ is a *partial* minimal point of Ξ with respect to $\preceq$.

Assume finally that condition (3.2) holds. This yields $v = \bar{z}$, and hence we arrive at

$$\Xi \cap \mathcal{L}(\bar{x}, \bar{z}) = \big\{(\bar{x}, \bar{z})\big\},$$

which means that $(\bar{x}, \bar{z})$ is a (fully) minimal point of Ξ with respect to $\preceq$. □

Remark 3.2 (on assumptions of Theorem 3.1)**.** Let us discuss the assumptions of this theorem that plays a crucial role in deriving variational principles in Sections 4 and 5.

(i) Condition (A1) holds automatically in all the results on variational principles established below. At the same time this condition is essential in general for the existence of minimal points in the case of Pareto preorders on $\mathbb{R}^2$; see Example 3.4.

(ii) In contrast to (A1), condition (A2) is very instrumental for the proofs of the set-valued extensions of the Ekeland variational principle developed in [1, 2].

(iii) Condition (A3) is automatic if (X, q) is a Hausdorff topological space.

(iv) Associating with Ξ the set-valued mapping $\Xi\colon X \rightrightarrows Z$ defined by $\Xi(x) := \{z \in Z |\ (x, z) \in \Xi\}$, the domination condition (3.2) says that $\bar{z} \in \operatorname{Min}(\Xi(\bar{x}); \preceq_{\bar{x}})$, i.e., $\bar{z}$ is a minimal point of $\Xi(\bar{x})$ with respect to the preorder $\preceq_{\bar{x}}$ defined by

$$u \preceq_{\bar{x}} z \text{ if and only if } \ (\bar{x}, u) \preceq (\bar{x}, z).$$

As a direct consequence of Theorem 3.1, we formulate now a nonparametric version of the minimal point theorem, which is an extension of the well-recognized result by Dancs, Hegedüs and Medvegyev [12, Theorem 3.2] to the case of quasi-metric spaces.

Corollary 3.3 (nonparametric quasimetric version of the minimal point theorem)**.** *Let (X, q) be a quasimetric space preordered by $\preceq$. Impose the following assumptions:*

(A1′) *For any sequence $\{x_k\} \subset X$ decreasing with respect to $\preceq$, we have $q(x_k, x_{k+1}) \to 0$.*

(A2′) *If the sequence $\{x_k\} \subset X$ is decreasing with respect to $\preceq$ and Cauchy, then there exists $\bar{x} \in X$ such that $\bar{x} \preceq x_k$ for all $k \in I\!N$.*

(A3′) *For any sequence $\{x_k\} \subset X$ decreasing with respect to $\preceq$, the conditions $q(x_k, \bar{x}) \to 0$, $q(x_k, \bar{u}) \to 0$, and $\bar{u} \preceq \bar{x}$ imply that $\bar{x} = \bar{u}$.*

Then X has a minimal point with respect to $\preceq$.

Proof. It is straightforward from Theorem 3.1 with $Z = \{\bar{z}\}$, $\Xi = X \times \{\bar{z}\}$, and the preorder $(x, \bar{z}) \preceq (u, \bar{z})$ on Ξ induced by $x \preceq u$. □

We are not familiar with any example in the literature showing that condition (A1′) is essential for the existence of the minimal point. Let us construct such an example.

Example 3.4 (convergence monotonicity is essential for the existence of minimal points)**.** Consider the square $\Xi := [-1, 1] \times [-1, 1] \subset I\!R^2$ in the standard Euclidean metric q and the Pareto preorder $\preceq_\Theta$ defined by (2.4) with $\Theta := I\!R_+ \times I\!R$. It is easy to check that for any point $(\bar{x}, \bar{z}) \in \Xi$ we have

$$\{\bar{x}\} \times [-1, 1] \subset \mathcal{L}(\bar{x}, \bar{z}) = \big\{(x, z) \in \Xi \big|\ (x, z) \preceq_\Theta (\bar{x}, \bar{z})\big\},$$

and so Ξ has no minimal point with respect to $\preceq_\Theta$. Let us show that condition (A1′) does not hold here. Indeed, we have $a := (-1, 1) \in \Xi$ and $b := (-1, -1) \in \Xi$ satisfying $a \preceq_\Theta b$ and $b \preceq_\Theta a$. Form the deceasing sequence $\{x_k\}$ by $x_{2k} := b$ and $x_{2k+1} := a$ and observe that $q(x_k, x_{k+1}) \equiv 2$ for all $k \in I\!N$.

Let us explore which kind of minimality can be obtained if condition (A1′) is not satisfied.

Proposition 3.5 (weak form of the minimal point theorem)**.** *Let (X, d) be a complete Hausdorff metric space preordered by $\preceq$. In addition to* (A2′)*, impose the boundedness condition: there is $x_0 \in X$ such that the level set $\mathcal{L}(x_0) := \{x \in X|\ x \preceq x_0\}$ is bounded, i.e., $d(x_0, x) \le \ell$ for all $x \in \mathcal{L}(x_0)$ with some $\ell > 0$. Then there exists $\bar{x} \in \mathcal{L}(x_0)$ satisfying*

$$d(x_0, \bar{x}) = r\big(x_0, \mathcal{L}(\bar{x})\big) := \sup\big\{d(x_0, x)\big|\ x \in \mathcal{L}(\bar{x})\big\}. \tag{3.4}$$

Proof. We proceed similarly to the proof of Theorem 3.1 while using a different iterative scheme. Observe first that the level-set mapping $\mathcal{L}(x) = \{u \in X|\ u \preceq x\}$ enjoys properties (a) and (b) listed in the previous theorem. Construct the sequence of iterations by

$$(x_k, z_k) \preceq (x_{k-1}, z_{k-1}) \ \text{ with } \ d(x_0, x_k) \ge r\big(x_0; \mathcal{L}(x_{k-1})\big) - 2^{-k}. \tag{3.5}$$

Since $\mathcal{L}(x_k) \subset \mathcal{L}(x_{k-1}) \subset \mathcal{L}(x_0)$ and $\mathcal{L}(x_0)$ is bounded by $\ell < \infty$, the numerical sequence $\{r_k\}$ defined by $r_k := r(x_0; \mathcal{L}(x_k))$, $k \in I\!N \cup \{0\}$, is decreasing and thus converges to some number $\bar r$. It follows from the inequality in (3.5) that $d(x_0, x_k) \to \bar r$ as $k \to \infty$. Furthermore, it is easy to check that $\{x_k\}$ is a Cauchy sequence in X, and so it converges to some $\bar x \in X$ with $d(x_0, \bar x) = \bar r$ due to the completeness of X. Employing (A2$'$) tells us that $\bar x \preceq x_k$ and therefore $\mathcal{L}(\bar x) \subset \mathcal{L}(x_k)$. It yields $r(x_0; \mathcal{L}(\bar x)) \le r_k$ for all $k \in I\!N$. Hence we have $\bar r \le r(x_0; \mathcal{L}(\bar x)) \le \bar r$, which verifies (3.4) and completes the proof of the theorem. $\square$

Let us illustrate Proposition 3.5 for the case of Example 3.4. If $x_0 = (a, b) \in \Xi = [-1, 1] \times [-1, 1]$ with $b \neq 0$, then $\bar x = \left(-1, \frac{-b}{\mathrm{sign}(b)}\right)$ satisfies (3.4) with $\mathcal{L}(\bar x) = \{-1\} \times [-1, 1]$. Otherwise, either $\bar x = (-1, 1)$ or $\bar x = (-1, -1)$ satisfies (3.4) with the same set $\mathcal{L}(\bar x)$.

The next result gives an extension from metric spaces to quasimetric spaces the minimal point theorem obtained by Tammer and Zălinescu [48, Theorem 1]. It is derived below as a consequence of the main Theorem 3.1.

Corollary 3.6 (minimal point theorem in product spaces with quasimetrics)**.** *Let (X, q) be a complete Hausdorff topological quasimetric space, let Z be a real topological vector space, and let $\Theta \subset Z$ be a proper convex cone. Consider a set-valued quasimetric $D : X \times X \rightrightarrows \Theta$ satisfying the following conditions:*

(D1) *$D(x_1, x_2) \subset \Theta$ for all $x_1, x_2 \in X$.*
(D2) *$0 \in D(x, x)$ for all $x \in X$.*
(D3) *$D(x_1, x_2) + D(x_2, x_3) \subset D(x_1, x_3) + \Theta$ for all $x_1, x_2, x_3 \in X$.*

Associate with this quasimetric a preorder $\preceq_D$ on $X \times Z$ defined by

$$(x_1, z_1) \preceq_D (x_2, z_2) \Longleftrightarrow z_2 \in z_1 + D(x_1, x_2) + \Theta.$$

Given $\emptyset \neq \Xi \subset X \times Z$, suppose that it satisfies the limiting monotonicity condition (A2) *with respect to $\preceq_D$ and that there is $z^* \in \Theta^+ := \{z^* \in Z^* |\ \langle z^*, z\rangle \ge 0 \ \text{ as } \ z \in \Theta\}$ such that*

$$\eta(\delta) := \inf\Big\{z^*(z)\Big|\ z \in \bigcup_{q(x,x') \ge \delta} D(x, x')\Big\} > 0 \tag{3.6}$$

for any $\delta > 0$ and that the boundedness from below condition holds:

$$\inf\big\{z^*(z)\big|\ \exists\, x \in X \ \text{ with } \ (x, z) \in \Xi\big\} > -\infty. \tag{3.7}$$

Then for any $(x_0, z_0) \in \Xi$ there exists $(\bar x, \bar z) \in \Xi$ for which $(\bar x, \bar z) \preceq_{D,z^*} (x_0, z_0)$ and $(\bar x, \bar z)$ is a minimal point of Ξ with respect to the partial order $\preceq_{D,z^*}$ defined by

$$(x_1, z_1) \preceq_{D,z^*} (x_2, z_2) \Longleftrightarrow \begin{cases} \text{either } (x_1, z_1) = (x_2, z_2), \quad \text{or} \\ (x_1, z_1) \preceq_D (x_2, z_2) \ \text{ and } \ z^*(z_1) < z^*(z_2). \end{cases}$$

Proof. It is easy to check that $\preceq_D$ is a preorder; see [48] for more details. Applying Theorem 3.1 in this setting requires checking first that assumption (A1) holds by taking into account that the other assumptions (A2) and (A3) are imposed in this corollary.

To check (A1) for the set Ξ with respect to the preorder $\preceq_D$, pick any sequence $\{(x_k, z_k)\}$ decreasing with respect to $\preceq_D$ and show that $q(x_k, x_{k+1}) \to 0$ as $k \to \infty$. Following [48] and arguing by contradiction, suppose that the sequence $\{q(x_k, x_{k+1})\}$ does not converge to zero and then find $\delta > 0$ and N_δ such that $q(x_k, x_{k+1}) > \varepsilon$ for $k \geq N_\delta$. Since $(x_{k+1}, z_{k+1}) \preceq_D (x_k, z_k)$ meaning that $z_{k+1} \in z_k + D(x_k, x_k + 1) + \Theta$, we get

$$z^*(z_k) - z^*(z_{k+1}) \geq \inf\big\{z^*(v)\big|\ v \in D(z_{k+1}, z_k)\big\} \geq \eta(\delta), \quad k \geq N_\delta.$$

It follows from $\eta(\delta) > 0$ by (3.6) that the sequence $z^*(z_k)$ tends to $-\infty$ as $k \to \infty$, which contradicts the boundedness condition (3.7) and thus verifies (A1).

Employing now Theorem 3.1 to the preordered set $(\Xi; \preceq_D)$ we find a "partial" minimal point $(\bar{x}, \bar{z})$ of Ξ with respect to $\preceq_D$. To complete the proof, it remains to show that $(\bar{x}, \bar{z})$ is a (full) minimal point of Ξ with respect to $\preceq_{D,z^*}$, which is in fact a partial order; see [48] for the verification of the reflexivity, transitivity, and antisymmetry properties of the latter ordering relation. Arguing by contradiction, suppose that $(\bar{x}, \bar{z})$ is not a minimal point of Ξ with respect to $\preceq_{D,z^*}$ and then find $(x, z) \in \Xi \setminus \{(\bar{x}, \bar{z})\}$ such that $(x, z) \preceq_{D,z^*} (\bar{x}, \bar{z})$, i.e.,

$$\begin{aligned}
& (x, z) \preceq_{D,z^*} (\bar{x}, \bar{z}) \ \text{ and } \ z^*(z) < z^*(\bar{z}) \\
\Longleftrightarrow\ & \bar{z} \in z + D(x, \bar{x}) + \Theta \ \text{ and } \ z^*(z) < z^*(\bar{z}) \\
\Longrightarrow\ & \bar{z} \in z + \Theta \ \text{ and } \ z^*(z) < z^*(\bar{z}) \\
\Longleftrightarrow\ & \exists\, \theta \in \Theta,\ \bar{z} \in z + \theta \ \text{ and } \ z^*(z) < z^*(\bar{z}) \\
\Longleftrightarrow\ & \exists\, \theta \in \Theta,\ z^*(\bar{z}) = z^*(z + \theta) = z^*(z) + z^*(\theta) \ \text{ and } \ z^*(z) < z^*(\bar{z}) \\
\Longrightarrow\ & z^*(\bar{z}) \preceq z^*(z) \ \text{ and } \ z^*(z) < z^*(\bar{z}),
\end{aligned}$$

where the first implication holds due to $D(x, \bar{x}) \subset \Theta$ and the convexity of the cone Θ, and the second one is satisfied by $z^* \in \Theta^+$. The obtained contradiction verifies the minimality of $(\bar{x}, \bar{z})$ for Ξ with respect to $\preceq_{D,z^*}$ and thus completes the proof. □

Another consequence of the obtained minimal point results (now of Corollary 3.3) is the following preorder principle recently established by Qui [39, Theorem 2.1].

Corollary 3.7 (preorder principle). *Let $(X, \preceq)$ be a preordered set, and let $\eta\colon X \to \overline{I\!R} := I\!R \cup \{\infty\}$ be an extended-real-valued function monotone with respect to $\preceq$, i.e.,*

$$u \preceq x \Longrightarrow \eta(u) \leq \eta(x) \ \textit{ for any } \ x, u \in X.$$

Suppose that the following conditions hold:

(Q1) $-\infty < \inf\big\{\eta(x)\big|\ x \in X\big\} < \infty$.

(Q2) *For any sequence $\{x_k\} \subset X$ decreasing with respect to $\preceq$, i.e., such that*

$$\eta(x_k) - \eta_k \to 0 \ \textit{ as } \ k \to \infty \ \textit{ with } \ \eta_k := \inf\big\{\eta(x)\big|\ x \preceq x_k\big\},$$

there is $\bar{x} \in X$ satisfying $\bar{x} \preceq x_k$ for all $k \in I\!N$.

Then there exists a point $\bar{x} \in X$ minimal for X with respect to the preorder $\preceq$.

Proof. Condition (Q1) tells us that $\operatorname{dom} \eta \neq \emptyset$, and we may assume without loss of generality that $\operatorname{dom} \eta = X$. Define the function $q : X \times X \to [0, \infty)$ by

$$q(x, u) := |\eta(x) - \eta(u)| \ \text{ for all } \ x, u \in X$$

and check easily that q is a quasimetric on X and that

$$u \preceq x \Longrightarrow \eta(u) \leq \eta(x) \Longrightarrow q(x, u) = \eta(x) - \eta(u). \tag{3.8}$$

Let us now verify that the imposed condition (Q1) and (Q2) ensure the fulfilment of all the assumptions (A1′), (A2′), and (A3′) in Corollary 3.3.

To proceed with (A1′), take a sequence $\{x_k\} \subset X$ decreasing with respect to $\preceq$. It follows from the relations in (3.8) that

$$q(x_k, x_{k+1}) = \eta(x_k) - \eta(x_{k+1}) \quad \text{for all} \quad k \in I\!N. \tag{3.9}$$

Summing up these equalities from $k = 1$ to m gives us

$$\sum_{k=1}^{m} q(x_k, x_{k+1}) = \eta(x_1) - \eta(x_{m+1}) \le \eta(x_1) - \ell \quad \text{with} \quad \ell := \inf_{x \in X} \eta(x).$$

We get further by passing to the limit as $m \to \infty$ that

$$\sum_{k=1}^{\infty} q(x_k, x_{k+1}) \le \eta(x_1) - \ell.$$

The boundedness from below of η in (Q1) ensures that this series is convergent, and thus $q(x_k, x_{k+1}) \to 0$ as $k \to \infty$, which justifies (A1′).

It follows directly from the construction in (3.9) that (Q2) implies the limiting monotonicity condition (A2′) in the quasimetric space (X, q) under consideration.

To verify (A3′), take a sequence $\{x_k\} \subset X$ decreasing with respect to $\preceq$ such that $q(x_k, \bar{x}) \to 0$, $q(x_k, \bar{u}) \to 0$, and $\bar{u} \preceq \bar{x}$. It follows from (3.8) that

$$q(x_k, \bar{x}) - q(x_k, \bar{u}) = \eta(x_k) - \eta(\bar{x}) - \big(\eta(x_k) - \eta(\bar{u})\big) = \eta(\bar{u}) - \eta(\bar{x}) \quad \text{for all} \quad k \in I\!N.$$

Since $q(x_k, \bar{x}) - q(x_k, \bar{u}) \to 0$ as $k \to \infty$, we clearly have $\bar{x} = \bar{u}$, which justifies (A3′).

To complete the proof, it remains to apply the result of Corollary 3.3 to the setting under consideration and arrive at the claimed conclusion for $(X, \preceq)$. □

4. Set-valued variational principles with variable ordering

The discovery of the new variational principle by Ivar Ekeland [15, 16] (called now the *Ekeland variational principle*, abbr. *EVP*) has been a major achievement in nonlinear analysis, especially in its variational aspects. Roughly speaking, it says that every lower semicontinuous function bounded from below on a complete metric space allows a slight perturbation of the distance type so that the perturbed function attains its strict global minimum at a point near the reference one. Since its appearance the EVP has founded numerous applications in analysis and other areas of mathematics. Over the last four decades many authors developed various extensions of this important principle and its equivalent formulations; e.g., Caristi's fixed point theorem, Takahashi's existence theorem, minimal point theorems, the petal and drop theorems, etc. We refer the reader to [8, 12, 17, 23, 19, 36, 40, 46, 47] and the bibliographies therein for more details, discussions, and applications. Among major extensions of the EVP to vector and set-valued mappings we mention the following.

- In [21, 22], Ha established new versions of the EVP for set-valued mappings that ensures the existence of a strict minimizer for a perturbed set-valued optimization problem, where the concept of optimality is understood in terms of Kuroiwa's set optimization criterion [33]. Further extensions of Ha's results has been recently obtained by Qiu [38, 39].

• Bednarczuk and Zagrodny [7] proved an extension of the EVP for vector mappings under a certain monotone semicontinuity assumption, which justifies the existence of the so-called H-near-to-minimal solution (in the sense of Németh [37]) of a perturbed mapping with the perturbation factor given by a convex subset H of the ordering cone multiplied by the distance function. This type of perturbation can also be founded in [20, 48].
• Gutiérrez, Jiménez and Novo [20] introduced a set-valued metric, which takes values in the family of all subsets of the ordering cone and satisfies the triangle inequality. By using it, they developed a new approach to extending the scalar EVP to vector mappings, where the perturbation contains a set-valued metric. Note that vector-valued metrics and the corresponding extensions of the EVP were earlier suggested by Khanh [30].
• Göpfer, Tammer and Zălinescu [18] and Tammer and Zălinescu [48] obtained minimal point theorems in product spaces with applications to vector and set-valued versions of the EVP, extending in this way the previous results by Isac and Tammer [27] and Ha [22].
• Bao and Mordukhovich [1, 2] derived enhanced extensions of the EVP to set-valued mappings under the limiting monotonicity condition with respect to a closed and convex ordering cone of the image space without any pointedness and solidness (nonempty interior) assumptions. In the subsequent development, Khanh and Quy [31] further extended these versions to the case of more general perturbations.
• Quite recently in [5, 6], Bao, Mordukhovich and Soubeyran developed variational principles of the Ekeland type for multiobjective problems with variable ordering structures, where each vector in the image space has its own ordering cone. Soleimani and Tammer [42] obtained, based on a scalarization technique, another vectorial version of the EVP for problems with solid variable ordering cones depending on points in the domain space.

In this section, we formulate and verify new (pre-less and post-less) versions of the Ekeland variational principle for set-valued mappings from quasimetric spaces to (real topological) vector spaces equipped with some variable ordering structure. Our proof involves the development of the dynamical approximation technique suggested in [1, 2] for problems with no variable structures and the application of the minimal point results of Section 3 whose derivation is also largely based on this technique. Note that this approach can be considered as a vector/set-valued counterpart (in the domain space) of the inductive procedure to prove the classical EVP suggested by Michael Crandall; see [17] and also [36, Theorem 2.26] with the commentaries therein. It is significantly different from the original transfinite induction arguments used by Ekeland [15, 16] and their vectorial modifications as well as from other techniques (e.g., scalarization, which requires nonempty interiority assumptions) developed by many authors to derive multiobjective versions of the EVP. We refer the reader to the survey in [20] and more recent publications [7, 31, 34, 38, 42, 48] with the bibliographies therein. Employing our technique, we are able to cover the vast majority of known results in this direction obtained by other methods.

In what follows we consider a general set-valued mapping $F\colon X \rightrightarrows Z$ from a quasimetric space X into a (real topological) vector space Z equipped with some variable ordering structure $K\colon Z \rightrightarrows Z$. As discussed in Section 2, there are two distinct preorders induced by K: the post-less and pre-less preorders with respect to K denoted by $\preceq^{\rm post}_{K[\cdot]}$ and $\preceq^{\rm pre}_{K[\cdot]}$, respectively. If no confusion arises, for simplicity we use the notation $\preceq_{K[\cdot]}$ for both preorders while emphasizing those results, which are specific for one or another structure.

Denote the *domain* and *graph* of F by, respectively,

$$\operatorname{dom} F := \big\{x \in X \big|\ F(x) \neq \emptyset\big\} \ \text{ and } \ \operatorname{gph} F := \big\{(x,z) \in X \times Z \big|\ z \in F(x)\big\}$$

and define some relevant notions regarding the variable ordering structure K, which are broadly used in set-valued analysis and vector optimization for problems with constant ordering structures; cf. [1, 2, 28, 35, 36]. We say that:

• F is (left-sequentially) *level-closed* if its z-level sets

$$\operatorname{Lev}(z;F) := \big\{x \in \operatorname{dom} F \big|\ \exists\, v \in F(x) \ \text{ with } \ v \preceq_{K[\cdot]} z\big\}$$

are (left-sequentially) closed in X for all $z \in Z$. Correspondingly, a set $\Xi \subset Z$ is level-closed if the function $I_\Xi(z) \equiv z$ on Ξ has this property.

• F is *level-decreasingly-closed* on $\operatorname{dom} F$ with respect to $\preceq_{K[\cdot]}$ if for any sequence $\{(x_k,z_k)\} \subset \operatorname{gph} F$ such that $x_k \to \bar x \in X$ as $k \to \infty$ and $\{z_k\}$ is a sequence decreasing with respect to $\preceq_{K[\cdot]}$, there is $\bar z \in \operatorname{Min}\big(F(\bar x);\preceq_{K[\cdot]}\big)$ satisfying $\bar z \preceq_{K[\cdot]} z_k$ for all $k \in I\!N$.

• F is *quasibounded from below* with respect to Θ if there is a bounded subset $M \subset Z$ such that $F(X) \subset M + \Theta$, where $F(X) := \cup_{x\in X} F(x)$. Correspondingly, a set $\Xi \subset Z$ is quasibounded from below if the constant mapping $F(x) \equiv \Xi$ has this property.

• F has the *domination property* at $\bar x \in \operatorname{dom} F$ if for every vector $z \in F(\bar x)$ there is $v \in \operatorname{Min}\big(F(\bar x);\preceq_{K[\cdot]}\big)$ such that $v \preceq_{K[\cdot]} z$. Correspondingly, a set $\Xi \subset Z$ has the domination property if it holds for constant mapping $F(x) \equiv \Xi$.

Remark 4.1 (on properties of sets and mappings)**.** Observe the following:

(i) When $F(x) \equiv \Xi$ for some $\Xi \subset Z$, the level-decreasing-closedness of F says that every decreasing sequence in Ξ has a lower bound in $\operatorname{Min}\big(\Xi;\preceq_{K[\cdot]}\big)$. This is more restrictive than the domination property of Ξ. Indeed, the union $\Xi := (AB) \cup (BC) \cup (CA)$ involving the three intervals without the ending points $A := (1,1)$, $B := (1,0)$, $C := (0,1)$ and ordered by the usual Pareto partial order $\preceq_{I\!R^2_+}$ has the domination property while not the decreasing lower-bound one since the decreasing sequence $\{(k^{-1},1)\} \subset \Xi$ converges to $(0,1) \notin \Xi$.

(ii) Let $F = f : X \to Z$ single-valued. It is clear that the level-closedness of f implies the level-deceasing-closedness. However, the converse implication does not hold in general. Indeed, define the vector function $f : I\!R \to I\!R^2$ by

$$f(x) := \begin{cases} (x,-x) & \text{if} \quad x < 0,\\ (x,-x+2) & \text{if} \quad x \geq 0, \end{cases}$$

and consider a constant Pareto ordering structure $K[z] \equiv I\!R^2_+$. It is easy to check that f is level-decreasingly-closed since for any deceasing sequence $\{f(z_k)\}$ with respect to $\preceq_{I\!R^2_+}$ we have $f(x_2) = f(x_3) = \ldots = f(x_{k-1}) = f(x_k) = \ldots$. Nevertheless, f is not level-closed since the $(1,1)$-level set of f is $[-1,0) \cup \{1\}$.

(iii) When $K[z] \equiv \Theta$ for some convex cone $\Theta \subset Z$, the level-decreasing-closedness of $F\colon X \rightrightarrows Z$ reduces to the limiting monotonicity property of F from [2] and the sequential submonotonicity property from [37] in the case of single-valued mappings.

Now we are ready to derive new versions of the EVP for problems with variable preference structures. Consider first the *post-less* setting.

Theorem 4.2 (set-valued Ekeland variational principle with variable post-less preorders)**.** *Let $F : X \rightrightarrows Z$ be a set-valued mapping between a complete Hausdorff quasimetric space (X, q) and a vector space Z, let $K : Z \rightrightarrows Z$ be a variable ordering structure on Z with the post-less preorder $\preceq_{K[\cdot]} = \preceq^{\text{post}}_{K[\cdot]}$ defined by* (2.6)*, let $\emptyset \neq \Theta \subset Z$ be a nontrivial* (*i.e., $\Theta \neq Z$*) *convex cone, and let $\xi \in Z$. Assume that:*

(H1) *For every $z \in Z$ we have that $0 \in K[z]$, the cone $K[z]$ is closed in Z, and*

$$K[z] + \operatorname{cone}(\xi) \subset K[z].$$

(H2) *The ordering structure K satisfies the postmonotonicity condition: for all $z, v \in Z$ we have the implication*

$$v \preceq_{K[z]} z \Longrightarrow K[v] + K[z] \subset K[z].$$

(H3) *F is quasibounded from below with respect to Θ.*

(H4) *F satisfies the level-deceasing-closedness condition on* $\operatorname{dom} F$ *with respect to the post-less preorder $\preceq_{K[\cdot]}$.*

(H5) $\xi \notin \operatorname{cl}\big(-\Theta - \operatorname{cone}(K[z_0])\big)$.

Then for every $\gamma > 0$ and every $(x_0, z_0) \in \operatorname{gph} F$ there is a pair $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ with $\bar{z} \in \operatorname{Min}\big(F(\bar{x}); K[\bar{z}]\big)$ satisfying the conditions

$$\bar{z} + \gamma q(x_0, \bar{x})\xi \preceq_{K[z_0]} z_0, \tag{4.1}$$

$$z + \gamma q(\bar{x}, x)\xi \not\preceq_{K[\bar{z}]} \bar{z} \;\text{ for all }\; (x, z) \in \operatorname{gph} F \setminus \{(\bar{x}, \bar{z})\}. \tag{4.2}$$

If furthermore, given arbitrary numbers $\varepsilon > 0$ and $\lambda > 0$, the starting point (x_0, z_0) is an $\varepsilon\xi$–approximate minimizer of F with respect to $K[z_0]$, i.e.,

$$\big(F(x) + \varepsilon\xi\big) \cap \big(z_0 - K[z_0]\big) = \emptyset \;\text{ for all }\; x \in \operatorname{dom} F, \tag{4.3}$$

then $\bar{x}$ can be chosen so that in addition to (4.1) *and* (4.2) *with $\gamma = \varepsilon/\lambda$ we have*

$$q(x_0, \bar{x}) \leq \lambda. \tag{4.4}$$

Proof. Without loss of generality, assume that $\gamma = 1$, observing that the general case can be easily reduced to the special one by applying the latter to the equivalent quasimetric $\widetilde{q}(x, x') := \gamma\, q(x, x')$. Define now a binary/ordering relation $\preceq$ on $\operatorname{gph} F \subset X \times Z$ by

$$(x', z') \preceq (x, z) \Longleftrightarrow z' + q(x, x')\xi \preceq_{K[z]} z.$$

Using this together with (H1) allows us to conclude that

$$z' + q(x, x')\xi \in z - K[z] \Longrightarrow z' \in z - K[z] \Longleftrightarrow z' \preceq_{K[z]} z.$$

Let us check that $\preceq$ is a preorder. Indeed, the reflexivity property follows from $z+q(x,x)\xi = z \preceq_{[z]} z$. To verify the transitivity property, pick any (x,z), (x',z'), and (x'',z'') in $\operatorname{gph} X\times Z$ so that $(x'',z'')\preceq(x',z')$ and $(x',z')\preceq(x,z)$, i.e.,

$$z''+q(x',x'')\xi \preceq_{K[z']} z' \;\text{ and }\; z'+q(x,x')\xi \preceq_{K[z]} z.$$

Then we get, by taking into account the triangle inequality for the quasimetric and using the condition $K[z]+K[z']+\operatorname{cone}(\xi)\subset K[z]$ by (H2), that

$$\begin{aligned} z''+q(x,x'')\xi =&\big(z'+q(x,x')\xi\big)+\big(z''+q(x',x'')\xi\big)\\ &+\big(q(x,x'')-q(x,x')-q(x',x'')\big)\xi - z'\\ \in&\big(z-K[z]\big)+\big(z'-K\left[z'\right]\big)-\operatorname{cone}(\xi)-z'\subset z\\ &-\big(K[z]+K\left[z'\right]+\operatorname{cone}(\xi)\big)\subset z-K[z], \end{aligned}$$

which implies that $z''+q(x,x'')\xi\leq_{K[z]} z$, i.e., $(x'',z'')\preceq(x,z)$ and thus $\preceq$ is a preorder.

Consider next the (x_0,z_0)-level set of $\operatorname{gph} F$ with respect to $\preceq$ given by

$$\Xi := \operatorname{Lev}\left((x_0,z_0);\preceq\right)=\big\{(x,z)\in\operatorname{gph} F\big|\;(x,z)\preceq(x_0,z_0)\big\}$$

and verify the validity for $(\Xi,\preceq)$ assumptions (A1) and (A2) of Theorem 3.1 in our setting, remembering that assumption (A3) holds automatically.

(A1) To justify the convergence monotonicity condition, take an arbitrary sequence $\{(x_k,z_k)\}$ in Ξ decreasing with respect to $\preceq$, i.e.,

$$(4.5)\quad (x_k,z_k)\preceq(x_{k-1},z_{k-1})\Longleftrightarrow z_k+q(x_{k-1},x_k)\xi\preceq_{K[z_{k-1}]} z_{k-1},\quad k\in\mathbb{N},$$

and show that $q(x_k,x_{k+1})\to 0$ as $k\to\infty$. By (H2) we get from $z_k\preceq_{K[z_{k-1}]} z_{k-1}$ that $K[z_k]+K[z_{k-1}]\subset K[z_{k-1}]$ for all $k\in\mathbb{N}$, and thus

$$(4.6)\qquad \sum_{k=0}^{m} K[z_k]\subset K[z_0]\quad\text{for all}\;\; m\in\mathbb{N}\cup\{0\}.$$

Summing up the inequality in (4.5) from $k=0$ to m gives us that

$$(4.7)\quad t_m\xi\in z_0-z_{m+1}-K[z_0]\subset z_0-M-\Theta-K[z_0],\quad m\in\mathbb{N}\cup\{0\},$$

where $t_m:=\sum_{k=0}^m q(x_k,x_{k+1})$, and where $M\subset Z$ is the bounded set in the definition of the quasiboundedness from below assumed in (H3).

Let us next prove by passing to the limit in (4.7) as $m\to\infty$ that

$$(4.8)\qquad \sum_{k=0}^{\infty} q(x_k,x_{k+1})<\infty.$$

Arguing by contradiction, suppose that (4.8) does not hold, i.e., $t_m\uparrow\infty$ as $m\to\infty$. By the inclusion in (4.7) and the boundedness of the set M therein, we find a bounded sequence $\{w_m\}\subset z_1-M$ satisfying the condition

$$(4.9)\quad t_m\xi-w_m\in-\Theta-K[z_1],\;\text{ i.e., }\;\xi-w_m/t_m\in-\Theta-K[z_1],\quad m\in\mathbb{N}\cup\{0\}.$$

Passing now to the limit as $m \to \infty$ and taking into account the boundedness of $\{w_m\}$ and that $t_m \uparrow \infty$, we arrive at $\xi \in \mathrm{cl}\big(-\Theta - K[z_0]\big)$, which contradicts the choice of ξ in (H5). Thus (4.8) holds, and we get $q(x_k, x_{k+1}) \to 0$ as $k \to \infty$.

(A2) To verify the limiting monotonicity condition on Ξ, take an arbitrary sequence $(x_k, z_k) \subset \Xi$ decreasing with respect to $\preceq$ so that $x_k \to \bar{x} \in \Xi$ and

$$(x_k, z_k) \preceq (x_{k-1}, z_{k-1}) \Longrightarrow z_k \leq_{K[z_{k-1}]} z_{k-1}, \quad k \in I\!N.$$

By the level-decreasing-closedness assumption (H4), there exists $\bar{z} \in \mathrm{Min}\big(F(\bar{x}); \preceq^{\mathrm{post}}_{K[\cdot]}\big) = \mathrm{Min}\big(F(\bar{x}); K[\bar{z}]\big)$ satisfying

$$\bar{z} \preceq_{K[z_k]} z_k \Longleftrightarrow \bar{z} \in z_k - K[z_k], \quad k \in I\!N.$$

Invoking now conditions (H1) and (H2) gives us $K[z_{k+n}] + K[z_k] + \mathrm{cone}\,(\xi) \subset K[z_k]$ for all $n,\ k \in I\!N$. It is easy to check that

$$\begin{array}{rcl} \bar{z} + q(x_k, \bar{x})\xi & \in & z_{k+n} - K[z_{k+n}] + q(x_k, \bar{x})\xi \\ & = & z_{k+n} + q(x_k, x_{k+n})\xi - K[z_{k+n}] + \big(q(x_k, \bar{x}) - q(x_k, x_{k+n})\big)\xi \\ & \subset & z_k - K[z_k] - K[z_{k+n}] + q(x_{k+n}, \bar{x})\xi - \mathrm{cone}\,(\xi) \\ & \subset & z_k + q(x_{k+n}, \bar{x})\xi - K[z_k]. \end{array}$$

Passing there to the limit as $n \to \infty$ with taking into account the closedness of $K[z_k]$ and the convergence $q(x_{k+n}, \bar{x}) \to 0$ as $n \to \infty$, we arrive at the equivalence

$$\bar{z} + q(x_k, \bar{x})\xi \in z_k - K[z_k] \Longleftrightarrow \bar{z} + q(x_k, \bar{x})\xi \leq_{K[z_k]} z_k \Longleftrightarrow (\bar{x}, \bar{z}) \preceq (x_k, z_k).$$

Since $k \in I\!N$ was chosen arbitrarily and since $\bar{z} \in \mathrm{Min}\big(F(\bar{x}); K[\bar{z}]\big)$, it follows that

$$(\bar{x}, z) \preceq (\bar{x}, \bar{z}) \Longleftrightarrow z = z + q(\bar{x}, \bar{x})\xi \preceq_{K[\bar{z}]} \bar{z} \Longleftrightarrow z = \bar{z},$$

which justifies the validity of assumption (A2) in Theorem 3.1.

After checking the assumptions, we can apply the first conclusion of Theorem 3.1 to the preordered set $(\Xi, \preceq)$. This gives us $(\bar{x}, \bar{z}) \in \mathrm{gph}\, F$ such that $(\bar{x}, \bar{z}) \preceq (x_0, z_0)$ and $(\bar{x}, \bar{z}) \in \mathrm{Min}\,(\Xi; \preceq)$, which clearly yields (4.1) and (4.2). It remains to estimate the quasidistance $q(x_0, \bar{x})$ by (4.4) when (x_0, z_0) is chosen as an approximate $\varepsilon\xi$-minimizer of F with $\gamma = (\varepsilon/\lambda)$. Arguing by contradiction, suppose that $q(x_0, \bar{x}) > \lambda$. Since $(\bar{x}, \bar{z})$ is "better" than (x_0, z_0) and since $K\,[z_0] + \mathrm{cone}\,(\xi) \subset K\,[z_0]$, we get

$$\bar{z} \in z_0 - \epsilon\xi - K\,[z_0] \subset z_0 - (\epsilon/\lambda)q(x_0, \bar{x})\xi - K\,[z_0]\,,$$

which contradicts the choice of (x_0, z_0) and completes the proof of the theorem. $\square$

When the variable ordering structure $K : Z \rightrightarrows Z$ is cone-valued, conditions (H1) and (H2) reduce, respectively, to:

(K1) $\xi \in \Theta_K := \cap_{z \in Z} K[z]$ and $K[z]$ is a proper, closed, and convex subcone of Z.

(K2) The ordering structure K enjoys the following *postmonotonicity condition*: if $v \preceq_{K[z]} z$, then $K[v] \subset F[z]$.

In this case the obtained Theorem 4.2 agrees with our previous result in [6, Theorem 3.1].

Next we deduce several corollaries of Theorem 4.2. The first one is [2, Theorem 3.4] when $K[x] \equiv \Theta$ for some closed convex cone Θ. Note that we do not require anymore that the spaces in question are Banach and the mapping F is level-closed with respect to $\preceq_\Theta$.

Corollary 4.3 (enhanced version of EVP for set-valued mappings with constant preferences)**.** *Let $F\colon (X,q) \rightrightarrows Z$ be a set-valued mapping from a complete and Hausdorff topological quasimetric space to a vector space equipped with a preorder $\preceq_\Theta$ induced by a proper, closed, and convex cone $\Theta \subset Z$. Assume that F is quasibounded from below with respect to Θ and level-decreasingly-closed on* $\operatorname{dom} F$ *with respect to $\preceq_\Theta$. Then for any $\gamma > 0$, $\xi \in \Theta \setminus (-\Theta)$, and $(x_0, z_0) \in \operatorname{gph} F$ there is $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ satisfying*

$$\bar{z} + \gamma q(x_0, \bar{x})\xi \preceq_\Theta z_0, \quad \bar{z} \in \operatorname{Min}\big(F(\bar{x}); \preceq_\Theta\big), \tag{4.10}$$

$$z + \gamma\, q(\bar{x}, x)\xi \not\preceq_\Theta \bar{z} \ \text{ for all } \ (x,z) \in \operatorname{gph} F \setminus \{(\bar{x}, \bar{z})\}. \tag{4.11}$$

If furthermore, given arbitrary numbers $\varepsilon > 0$ and $\lambda > 0$, the starting point (x_0, z_0) is an $\varepsilon\xi$-approximate minimizer of F with respect to Θ, i.e.,

$$\big(F(x) + \varepsilon\xi\big) \cap \big(z_0 - \Theta\big) = \emptyset \ \text{ for all } \ x \in \operatorname{dom} F,$$

then $\bar{x}$ can be chosen so that in addition to (4.10) *and* (4.11) *with $\gamma = \varepsilon/\lambda$ we have* (4.4)*.*

Proof. It is straightforward from Theorem 4.2. □

The next corollary is a specification of Theorem 4.2 for the case when the mapping F has a particular structure studied in [5, Theorem 3.4] with applications in behavioral sciences. Let (X, q) be a quasimetric, let $\overline{\Omega} \subset Y$ be a compact subset $\overline{\Omega} \subset Y$ of a Banach space, and let Z be a vector space equipped with a variable ordering structure $K : Z \rightrightarrows Z$ and the post-less preorder $\preceq^{\text{post}}_{K[\cdot]}$. Given a single-valued mapping $f : X \times Y \to Z$ and a set-valued mapping $\Omega : X \rightrightarrows \overline{\Omega}$, define $F : X \rightrightarrows Z$ by

$$F(x) := f\big(x, \Omega(x)\big) = \bigcup \Big\{ f(x,\omega) \in Z \,\Big|\, \omega \in \Omega(\omega) \Big\}. \tag{4.12}$$

Corollary 4.4 (specification of EVP for mappings of type (4.12)))**.** *In the setting described above, suppose the validity of conditions* $(K1)$ *and* $(K2)$ *with the common cone Θ_K defined therein. Impose in addition the following assumptions:*

- (B1) *f is quasibounded from below on* $\operatorname{gph} \Omega$ *with respect Θ.*
- (B2) *f is level-decreasingly-closed with respect to $\preceq_{K[\cdot]}$ on* $\operatorname{gph} \Omega$*; this condition is automatic provided that f is level-closed with respect to the same preorder.*
- (B3) *$f(x, \cdot)$ is continuous for each $x \in \operatorname{dom} \Omega$.*

Then for any $\gamma > 0$, $(x_0, \omega_0) \in \operatorname{gph}\Omega$, $\xi \in \Theta_K \setminus -\operatorname{cl}(\Theta + K[f_0])$, *and* $f_0 := f(x_0, \omega_0)$ *there is a pair* $(\bar{x}, \bar{\omega}) \in \operatorname{gph}\Omega$ *with* $\bar{f} := f(\bar{x}, \bar{\omega}) \in \operatorname{Min}\big(F(\bar{x}); K\,[\bar{f}]\,\big)$ *satisfying the relationships*

$$\bar{f} + \lambda\, q(x_0, \bar{x})\xi \ \leq_{K[f_0]}\ f_0, \text{ and} \tag{4.13}$$

$$f + \lambda\, q(\bar{x}, x)\xi \ \not\preceq_{K[\bar{f}]}\ \bar{f} \ \textit{ for all } \ (x, \omega) \in \operatorname{gph}\Omega \ \textit{ with } \ f := f(x, \omega) \neq \bar{f}. \tag{4.14}$$

If furthermore, given arbitrary numbers $\varepsilon > 0$ *and* $\lambda > 0$, *the starting point* (x_0, ω_0) *is an* $\varepsilon\xi$*-approximate minimizer of* f *over* $\operatorname{gph}\Omega$ *with respect to* $K\,[f_0]$, *then* $(\bar{x}, \bar{\omega})$ *can be chosen so that* (4.4) *holds together with* (4.13) *and* (4.14) *as* $\gamma = \varepsilon/\lambda$ *.*

Proof. It is easy to see that the conclusions of this corollary are induced by those in Theorem 4.2 for mappings F in form (4.12). Thus it remains to check that all the assumptions of Theorem 4.2 hold in the setting of the corollary. In fact, we only need to verify the validity of conditions (H3) and (H4) for F given in (4.12).

The quasiboundedness from below in (H3) for (4.12) obviously follows from (B1). To check the validity of the level-decreasing-closedness condition (H4), take an arbitrary sequence $\{(x_k, f_k)\} \subset \operatorname{gph} F$ with $f_k := f(x_k, \omega_k) \in F(x_k)$ for some $\omega_k \in \Omega(x_k)$ satisfying $x_k \to \bar{x}$ as $k \to \infty$ and $f_{k+1} \preceq_{K[f_k]} f_k$ for all $k \in I\!N$. We need to justify the existence of $\bar{\omega} \in \Omega(\bar{x})$ such that $\bar{f} = f(\bar{x}, \bar{\omega}) \in \operatorname{Min}\big(F(\bar{x}), K\,[\bar{f}]\,\big)$ and $\bar{f} \preceq_{K[f_k]} f_k$, $k \in I\!N$. Since the set $\overline{\Omega}$ is compact, the sequence $\{\omega_k\} \subset \overline{\Omega}$ has a subsequence converging to $\bar{\omega} \in \overline{\Omega}$. Then the level-decreasing-closedness of f over $\operatorname{gph}\Omega$ in (B2) ensures that $f(\bar{x}, \overline{\omega}) \preceq_{K[f_k]} f_k$.

Denoting $f_{\bar{\omega}} := f(\bar{x}, \bar{\omega})$ and forming the $f_{\bar{\omega}}$-level set of $f(\bar{x}, \cdot)$ over $\Omega(\bar{x})$ by

$$\Xi := \big\{\omega \in \Omega(\bar{x}) \big|\ f_\omega := f(\bar{x}, \omega) \preceq_{K[f_{\bar{\omega}}]} f(\bar{x}, \bar{\omega}) =: f_{\bar{\omega}}\big\}\,, \tag{4.15}$$

we get from the continuity of $f(\bar{x}, \cdot)$ that Ξ is compact with $\bar{\omega} \in \Xi$. Employ now the result by Luc [35, Corollary 3.8(c)], which ensures in our setting the existence of $\bar{\omega} \in \Xi$ such that

$$\bar{f} = f(\bar{x}, \bar{\omega}) \in \operatorname{Min}\big(f(\bar{x}, \Xi), K\,[f_{\bar{\omega}}]\big) \ \text{ with } \ f(\bar{x}, \Xi) := \bigcup_{\omega \in \Xi} \{f_\omega := f(\bar{x}, \omega) \in Z\}\,.$$

This means by the definition of minimality that

$$\big(\bar{f} - K\,[f_{\bar{\omega}}]\,\big) \cap f(\bar{x}, \Xi) = \big\{\bar{f}\big\}\,.$$

Since $\bar{f} \preceq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}}$, we have $K\,[\bar{f}] \subset K\,[f_{\bar{\omega}}]$ by condition (B2). Thus we get

$$(\bar{f} - K\,[\bar{f}]) \cap f(\bar{x}, \Xi) = \big\{\bar{f}\big\}\,, \ \text{ i.e., } \ \bar{f} \in \operatorname{Min}(f(x_*, \Xi), K\,[\bar{f}]).$$

Actually the following stronger conclusion holds:

$$\bar{f} \in \operatorname{Min}(F(\bar{x}), K\,[\bar{f}]) \ \text{ with } \ F(\bar{x}) = f\big(\bar{x}, \Omega(\bar{x})\big) \supset f(\bar{x}, \Xi). \tag{4.16}$$

To verify this, we argue by contradiction and suppose that (4.16) does not hold. It gives us $\omega \in \Omega(\bar{x}) \setminus \Xi$ such that $f_\omega \preceq_{K[\bar{f}]} \bar{f}$. Since $\bar{\omega} \in \Xi$, we get $\bar{f} \preceq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}}$. The transitivity property of the preorder $\preceq_{K[\cdot]}$ yields $f_\omega \preceq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}}$, and thus $\omega \in \Xi$ contradicting the choice of $\omega \in \Omega(\bar{x}) \setminus \Xi$ and hence justifying (4.16).

By $\bar{f} \preceq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}}$ and $f_{\bar{\omega}} \preceq_{K[f_k]} f_k$ we deduce from the preorder transitivity that $\bar{f} \preceq_{K[f_k]} f_k$ for all $k \in I\!N$. Thus the mapping F from (4.12) satisfies condition (H4) of Theorem 4.2, which completes the proof of the corollary. $\square$

The last result of this section presents a *pre-less* version of the EVP for problems with variable preferences, which is derived from Theorem 3.1 similarly to the post-less case above.

Theorem 4.5 (set-valued Ekeland variational principle with variable pre-less preorders)**.** *Consider the framework of Theorem* 4.2 *with replacing the post-less preorder by its pre-less counterpart* $\preceq_{K[\cdot]} = \preceq^{\rm pre}_{K[\cdot]}$ *defined in* (2.7) *and replacing assumptions* (H2), (H4), *and* (H5) *by, respectively, the following ones:*

(H2′) *The ordering structure* K *enjoys the premonotonicity condition: if* $v \preceq_{K[v]} z$, *then* $K[z] + K[v] \subset K[v]$.

(H4′) F *satisfies the level-decreasing-closedness condition on* $\operatorname{dom} F$ *with respect to the pre-less preorder* $\preceq_{K[\cdot]}$ *formulated as follows: for any decreasing sequence of pairs* $\{(x_k, z_k)\} \subset \operatorname{gph} F$ *with* $x_k \to \bar{x}$ *as* $k \to \infty$ *and* $z_{k+1} \preceq_{K[z_{k+1}]} z_k$ *for all* $k \in I\!N$ *there is* $\bar{z} \in \operatorname{Min}\big(F(\bar{x}); K[\bar{z}]\big)$ *such that* $\bar{z} \preceq_{K[\bar{z}]} z_k$, $k \in I\!N$.

(H5′) $\xi \notin \operatorname{cl}(-\Theta - \operatorname{cone}(\Theta^K))$ *with* $\Theta^K := \cup_{z \in Z} K[z]$.

Then for any number $\gamma > 0$ *and any starting point* $(x_0, z_0) \in \operatorname{gph} F$ *there is a pair* $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ *with* $\bar{z} \in \operatorname{Min}\big(F(\bar{x}); K[\bar{z}]\big)$ *satisfying the relationships*

$$\bar{z} \preceq_{K[\bar{z}]} z_0 - \gamma\, q(x_0, \bar{x})\xi \Longleftrightarrow z_0 - \gamma q(x_0, \bar{x})\xi \in \bar{z} + K[\bar{z}]. \tag{4.17}$$

$$z \not\preceq_{K[z]} \bar{z} - \gamma q(\bar{x}, x)\xi \quad \text{for all} \quad (x, z) \in \operatorname{gph} F \setminus \{(\bar{x}, \bar{z})\}. \tag{4.18}$$

If furthermore, given arbitrary numbers $\varepsilon, \lambda > 0$ *the starting pair* (x_0, z_0) *is an* $\varepsilon\xi$*–approximate minimizer of* F *with respect to* $K[\bar{z}]$ (*in particular, with respect to* Θ^K), *i.e.,*

$$\big(F(x) + \varepsilon\xi\big) \cap \big(z_0 - K[\bar{z}]\big) = \emptyset \quad \text{for all} \quad z \in \operatorname{dom} F,$$

then $\bar{x}$ *can be chosen so that in addition to* (4.17) *and* (4.18) *with* $\gamma = \varepsilon/\lambda$ *we have* (4.4).

Proof. We proceed in the proof lines of Theorem 4.2 while skipping some details. Let us highlight below the major differences.

Define the ordering relation $\preceq^{\rm pre}$ on $\operatorname{gph} F \subset X \times Z$ by

$$(u, v) \preceq^{\rm pre} (x, z) \Longleftrightarrow v \preceq_{K[v]} z - q(x, u)\xi \tag{4.19}$$

and denote $\Xi := \operatorname{Lev}((x_0, z_0); \preceq^{\rm pre}) = \{(x, z) \in \operatorname{gph} F |\ (x, z) \preceq^{\rm pre} (x_0, z_0)\}$. It is easy to check that $\preceq^{\rm pre}$ is a preorder on $\operatorname{gph} F$ under assumptions (H1) and (H2′). To justify the boundedness condition (A1) in Theorem 3.1, we replace (4.7) by

$$q(x_0, x_k)\xi \in z_0 - z_k - K[z_k] \subset z_0 - M - \Theta - \operatorname{cone}(\Theta^K),$$

which leads by the similar proof to a contradiction with (H5′). The fulfilment of condition (A2) in Theorem 3.1 is guaranteed by the level-decreasing-closedness condition (H4′). The rest of the proof is the same as given in Theorem 4.2. $\square$

Observe finally that Theorem 4.5 reduces to Corollary 4.3 (cf. also [2, Theorem 3.4]) provided that K is a constant ordering structure.

5. Variational principles with set-valued quasidistances

The main result of this section provides a new extension of the EVP concerning set-valued cost mappings with set-valued quasidistance perturbations. We derive this result by employing the minimal point theorem for preordered sets in product spaces established above in Corollary 3.3. Similarly to the previous variational principles given in this direction [20, 48, 38, 39], it is possible to deduce from the obtained result various new consequences in both cases of one-direction and multi-direction perturbations. However, we skip such consequences due to the size of the paper.

Recall first the definition of a set-valued Θ-quasimetric with respect to a convex subcone $\Theta \subset Z$ of a (real topological) vector space that was introduced in [20, Definition 3.1].

Definition 5.1 (set-valued Θ-quasimetrics)**.** Let X be a nonempty set, and let Θ be a convex subcone of a vector space Z. A mapping $D : X \times X \rightrightarrows \Theta$ is said to be a SET-VALUED Θ-QUASIMETRIC if it satisfies conditions (D1)–(D3) of Corollary 3.6.

If in addition to (D1)–(D3) the mapping D satisfies the *symmetry property*, i.e., $D(x_1, x_2) = D(x_2, x_1)$ for all $x_1, x_2 \in X$, then D is called a *set-valued* Θ*-metric*. Note also that in all the previous publications in this direction a stronger version of condition (D2) was imposed:

$$0 \in D(x_1, x_2) \Longrightarrow x_1 = x_2 \ \text{ for all } \ x_1, x_2 \in X.$$

It is important to emphasize that the usage set-valued quasidistances allows us to unify two kinds of perturbations known in various extensions of the EVP. Precisely, given a quasimetric space (X, q), one-directional and multi-directional perturbations are understood, respectively, in the following sense:

$$D_1(x_1, x_2) = q(x_1, x_2)\xi, \quad D_1(z_1, z_2) = q(x_1, x_2)H = \big\{q(x_1, x_2)h \big|\ h \in H\big\},$$

where $\xi \in \Theta$ is some positive direction in Z while $H \subset \Theta$ is a convex subset of Z with $0 \notin \operatorname{cl} H$. For brevity we skip further discussions on such perturbations referring the reader to the recent papers [38, 48]. Here is our main result involving set-valued perturbations.

Theorem 5.2 (variational principle with Θ-quasidistance perturbations)**.** *Let (X, q) be a complete Hausdorff topological quasimetric space, Z be a vector space, Θ be a closed and convex cone of Z, $D : X \times X \rightrightarrows \Theta$ be a Θ-quasimetric from Definition* 5.1*, and $F : X \times X \rightrightarrows Z$ be another set-valued mapping satisfying conditions* (D2) *and* (D3) *formulated above and the following assumptions:*

- (F1) BOUNDEDNESS CONDITION: *Given any vector $x \in X$, the mapping $F(x, \cdot)$ is quasibounded from below with respect to Θ.*
- (F2) LIMITING MONOTONICITY CONDITION: *For any decreasing sequence $\{x_k\} \subset X$, in the sense that $(F(x_k, x_{k+1}) + \gamma D(x_k, x_{k+1})) \cap (-\Theta) \neq \emptyset$ for all $k \in \mathbb{N}$, there is some $\bar{x} \in X$ such that $(F(x_k, \bar{x}) + \gamma D(x_k, \bar{x})) \cap (-\Theta) \neq \emptyset$.*

(F3) CONVERGENCE COMPARISON CONDITION: *Given any decreasing sequence* $\{x_k\} \subset X$ *in the sense of* (F2)*, the upper boundedness of the series*

$$\sum_{k=1}^{\infty} D(x_k, x_{k+1}) \subset M - \Theta \tag{5.1}$$

with some bounded set $M \subset Z$ *ensures that* $q(x_k, x_{k+1}) \to 0$ *as* $k \to \infty$.

Then for every $\gamma > 0$ *and every* $x_0 \in X$ *we find* $\bar{x} \in X$ *satisfying the relationships*

$$\big[F(x_0, \bar{x}) + \gamma D(x_0, \bar{x})\big] \cap (-\Theta) \neq \emptyset, \tag{5.2}$$

$$\big[\big(F(\bar{x}, x) + \gamma D(\bar{x}, x)\big) \cap (-\Theta) \neq \emptyset\big] \Longrightarrow \big[x = \bar{x} \ \textit{ for all } \ x \in X \setminus \{\bar{x}\}\big]. \tag{5.3}$$

Proof. Suppose without loss of generality that $\gamma = 1$ and define $\preceq_{F,D}$ by

$$x_1 \preceq_D x_2 \ \text{ if and only if } \ \big[F(x_1, x_2) + D(x_1, x_2)\big] \cap (-\Theta) \neq \emptyset. \tag{5.4}$$

Let us show that the ordering relation (5.4) is a preorder on X.

To verify the ordering *reflexivity* of $\preceq_{F,D}$, fix $x \in X$ and sum up the two imposed conditions $0 \in F(x, x)$ and $D(x, x) = \{0\}$. This gives us $0 \in F(x, x) + D(x, x)$ and so $0 \in [F(x, x) + D(x, x)] \cap (-\Theta)$, which justifies $x \preceq_{F,D} x$ and thus the reflexivity of $\preceq_{F,D}$.

To check the ordering *transitivity* of $\preceq_{F,D}$, pick any vectors $x_1, x_2, x_3 \in X$ satisfying $x_1 \preceq_{F,D} x_2$ and $x_2 \preceq_{F,D} x_3$, i.e., so that

$$\big[F(x_1, x_2) + \gamma D(x_1, x_2)\big] \cap (-\Theta) \neq \emptyset \ \text{ and } \ \big[F(x_2, x_3) + \gamma D(x_2, x_3)\big] \cap (-\Theta) \neq \emptyset.$$

This allows us to find directions $\theta_1, \theta_2 \in \Theta$ for which

$$-\theta_1 - \theta_2 \in F(x_1, x_2) + F(x_2, x_3) + \gamma\big(D(x_1, x_2) + D(x_2, x_3)\big).$$

Taking into account the triangle inclusion for the quasimetric D and the assumption (D3) on F, we get $\theta_3, \theta_4 \in \Theta$ such that

$$F(x_1, x_2) + F(x_2, x_3) - \theta_3 \subset F(x_1, x_3) \ \text{ and } \ D(x_1, x_2) + D(x_2, x_3) - \theta_4 \subset D(x_1, x_3).$$

Substituting these inclusions into the previous ones yields

$$-(\theta_1 + \theta_2 + \theta_3 + \theta_4) \in F(x_1, x_3) + \gamma D(x_1, x_3).$$

Since Θ is a convex cone, it follows that $(\theta_1 + \theta_2 + \theta_3 + \theta_4) \in \Theta$ and hence

$$-(\theta_1 + \theta_2 + \theta_3 + \theta_4) \in \big[F(x_1, x_3) + \gamma D(x_1, x_3)\big] \cap (-\Theta) \neq \emptyset,$$

which shows that $x_1 \preceq_{F,D}$ and thus justifies the transitivity of the relation $\preceq_{F,D}$.

Next we check the validity of all the assumption (A1′), (A2′), and (A3′) of the minimal point result of Corollary 3.3 for the x_0-level set $\Xi := \{x \in X |\ x \preceq_{F,D} x_0\}$ and the preorder $\preceq_D$. There is nothing to check for (A3′) by the Hausdorff property of X, while (A2′) is equivalent to (F2) due to the completeness of X. It remains to verify the limiting monotonicity condition (A1′). To proceed, pick an arbitrary decreasing sequence $\{x_k\}$ with respect to $\preceq_{F,D}$ and by (5.4) find $\{\theta_k\} \subset \Theta$ such that

$$-\theta_k \in F(x_k, x_{k+1}) + D(x_k, x_{k+1}) \ \text{ for all } \ k \in I\!\!N.$$

Summing up these inclusions from $k=1$ to m and using the transitive of F yield

$$(5.5)\qquad -\theta^m \in F(x_1,x_{m+1}) + \sum_{k=1}^{m} D(x_k,x_{k+1}) \ \text{ with } \ \theta^m := \sum_{k=1}^{m}\theta_k \in \Theta.$$

Since m was chosen arbitrarily, it follows from (F1) and the convexity of Θ that there is a bounded set M so that (5.1) holds. This tells us by (F3) that $\sum_{k=1}^{\infty} q(x_k,x_{k+1}) < \infty$ and hence $q(x_k,x_{k+1}) \to 0$ as $k \to \infty$, which justifies (A1′) for the preorder $(\Xi, \preceq_{FD})$.

Now we apply the assertions of Corollary 3.3 to $(X, \preceq_{F,D})$ and, given any point $x_0 \in X$, find $\bar{x} \in X$ so that $\bar{x} \preceq_{F,D} x_0$ and that $\bar{x}$ is a minimal point of Ξ with respect to $\preceq_{F,D}$. This is equivalent to (5.2) and (5.2), respectively, and completes the proof of the theorem. □

Note that if $F : X \times X \rightrightarrows Z$ is *separable*, i.e., $F(x_1,x_2) = G(x_2) - G(x_1)$ for some $G : X \rightrightarrows Z$, then F satisfies both conditions (D2) and (D3) imposed above.

The concluding result of this section provides effective sufficient conditions for the validity of the major convergence comparison condition (F3) of Theorem 5.2.

Theorem 5.3 (sufficient conditions for convergence comparison in the variational principle with quasimetric perturbations)**.** *Each of the following conditions ensures the validity of* (F3) *in the setting of Theorem* 5.2*:*

(a) *$D_\xi(x_1,x_2) = q(x_1,x_2)\xi$ for some direction $\xi \in \Theta \setminus (-\Theta)$.*
(b) *$D_H(x_1,x_2) = q(x_1,x_2)H$ for some convex set $H \subset \Theta$ with $0 \notin \operatorname{cl}(H+\Theta)$.*
(c) *There exist $z^* \in \Theta^+$ and $\eta\colon I\!\!R_+ \to I\!\!R_+$ such that*

$$(5.6)\qquad \inf\Big\{z^*(z)\Big|\ z \in \bigcup_{q(x_1,x_2)\ge\delta} D(x_1,x_2)\Big\} \ge \eta(\delta) > 0 \ \textit{ for all } \ \delta > 0.$$

(d) *Z is a normed space and the cone Θ satisfies the relationships*

$$(5.7)\qquad \|\theta_1+\theta_2\| \ge (1+\gamma)\max\big\{\|\theta_1\|, \|\theta_2\big\} \ \textit{ for some } \ \gamma > 0.$$

$$(5.8)\qquad \inf\big\{\|v\| \ \big|\ v \in D(x_1,x_2)\big\} \ge \lambda q(x_1,x_2) \ \textit{ for all } \ x_1,x_2 \in X.$$

Proof. We proceed case by case in verifying condition (F3).

(a) Arguing by contradiction, assume that (F3) does not hold and then find a sequence $\{x_k\} \subset X$ and a bounded set $M \subset Z$ such that

$$(5.9)\ \sum_{k=1}^{\infty} D_\xi(x_k,x_{k+1}) \subset M - \Theta \ \text{ and } \ q(x_k,x_{k+1}) \ge \delta > 0 \ \text{ for all } \ k \in I\!\!N.$$

By the definition of D_ξ we get the equality

$$\sum_{k=1}^{\infty} D_\xi(x_k,x_{k+1}) = \Big(\sum_{k=1}^{\infty} q(x_k,x_{k+1})\Big)\xi,$$

which ensures by (5.9) due to the boundedness of M and the conic structure of Θ that $\xi \in \mathrm{cl}\,(-\Theta) = -\Theta$. This contradicts the choice of ξ and thus verifies (F3) in this case.

(b) Arguing by contradiction as in case (a), suppose that (5.9) holds for D_H and then find, by the structure of D_H, a sequence $\{h_k\} \subset H$ and a bounded sequence $\{v_k\} \subset M$ such that

$$v_k \in \sum_{k=1}^{m} q(x_k, x_{k+1}) h_k + \Theta \quad \text{for all} \quad m \in I\!N.$$

Condition (5.9) allows us to form

$$\overline{h}_m := \sum_{k=1}^{m} \frac{q(x_k, x_{k+1})}{t_m} h_k \quad \text{with} \quad t_m := \sum_{k=1}^{m} q(x_k, x_{k+1}) > 0$$

and conclude by the convexity of H that

$$\frac{v_k}{t_m} \in \overline{h}_m + \Theta \subset H + \Theta \quad \text{for all} \quad m \in I\!N.$$

Passing to the limit as $m \to \infty$ while taking into account (5.9) and the boundedness of $\{v_m\}$, we deduce that $0 \in \mathrm{cl}\,(H + \Theta)$, which contradicts the choice of H and justifies (F3).

(c) Suppose by contradiction that there are $\{x_k\} \subset X$ and $M \subset Z$ such that (5.9) holds for the quasimetric D. Using (5.6) for any $d_k \in D(x_k, x_{k+1})$ and $k \in I\!N$ gives us

$$\begin{aligned} z^*(d_k) \quad &\geq \quad \inf\big\{z^*(v)\big|\; v \in F(x_k, x_{k+1})\big\} \\ &\geq \quad \inf\Big\{z^*(z)\Big|\; z \in \bigcup_{q(x_1,x_2)\geq\delta} D(x_1, x_2)\Big\} \\ &\geq \quad \eta(\delta) > 0. \end{aligned}$$

Fixing now a sequence $\{d_k\}$, we get $z^*(\sum_{k=1}^{\infty} d_k) = \infty$ and

$$\begin{aligned} z^*\Big(\sum_{k=1}^{\infty} d_k\Big) \quad &\leq \quad \sup\big\{z^*(v-\theta) = z^*(v) - z^*(\theta)\big|\; v \in M,\ \theta \in \Theta\big\} \\ &\leq \quad \sup\big\{z^*(v)\big|\; v \in M\big\} < \infty, \end{aligned}$$

where the first estimate holds due to $\sum_{k=1}^{\infty} d_k \subset M - \Theta$ in (5.9), the second one holds due to $z^* \in \Theta^+$, and the last is valid since M is bounded in Z. The obtained contradiction justifies that $q(x_k, x_{k+1}) \to 0$ as $k \to \infty$ in this case.

(d) As in case (c), suppose that there are $\{x_k\}$, $\{d_k\}$, M, and $\delta > 0$ satisfying (5.9). It follows from (5.8) that $\|d_k\| \geq \lambda q(x_k, x_{k+1}) \geq \lambda\delta$, $k \in I\!N$. We claim that the statement

$$\|s_n\| \geq \lambda\delta(1+\gamma)^n \quad \text{with} \quad s_n := \sum_{k=0}^{n} d_k \tag{5.10}$$

holds for all $n \in I\!N$ when (5.7) is fulfilled. Indeed, we have it for $n = 1$ since

$$\|s_1\| = \|d_0 + d_1\| \geq (1+\gamma)\max\big\{\|d_0\|,\ \|d_1\|\big\} \geq (1+\gamma)\max\big\{\lambda\delta,\ \lambda\delta\big\} \geq \lambda\delta(1+\gamma).$$

Assume now that (5.10) holds for $n = k$ and derive from it the relationships

$$\begin{aligned}\|s_{k+1}\| = \|s_k + d_{k+1}\| &\geq (1+\gamma)\max\left\{\|s_k\|,\ d_{k+1}\right\}\\ &\geq (1+\gamma)\max\left\{\delta\lambda(1+\gamma)^k,\ \delta\lambda\right\}\\ &= (1+\gamma)\delta\lambda(1+\gamma)^k = \delta\lambda(1+\gamma)^{k+1}.\end{aligned}$$

It shows that (5.10) is satisfied for $n = k+1$ and hence for any $n \in I\!N$. Passing there to the limit as $n \to \infty$ tells us that $\lim_{n\to\infty} \|s_n\| = \infty$, which contradicts (5.9) and thus justifies that $q(x_k, x_{k+1}) \to 0$ in this case as well. $\square$

Note that the sufficient condition (5.6) in case (c) is required in [20, 31, 38, 48] to establish a version of the Ekeland variational principle with perturbations containing multiple directions in the ordering cone Θ. The other conditions of Theorem 5.3 as well as the main one (F3) in Theorem 5.2 are new.

6. Concluding remarks

As mentioned in Section 1, our study of variational principles involving set-valued mappings on quasimetric spaces to topological spaces ordered by variable preference structures has been largely motivated by applications to adaptive dynamical models of behavioral sciences via the recent variational rationality approach of [43, 44]. Our previous results obtained in this direction [5, 6] were applied to goals systems in psychology and capability theory of human behavior. The new mathematical results established in this paper make a bridge to further applications; in particular, to building dynamical models in the reference dependent theory of psychological preferences developed earlier by Kahneman and Tversky [29, 49] and their numerous followers in static frameworks.

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Truong Q. Bao
Department of Mathematics & Computer Science, Northern Michigan University, Marquette, Michigan, USA
E-mail address: btruong@nmu.edu

Boris S. Mordukhovich
Department of Mathematics, Wayne State University, Detroit, Michigan, USA and King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia
E-mail address: boris@math.wayne.edu

Antone Soubeyran
Aix-Marseille School of Economics, Aix-Marseille University, CNRS & EHESS, Marseille, France
E-mail address: antoine.soubeyran@gmail.com