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RANGES OF PERTURBED QUASIBOUNDED MAXIMAL MONOTONE OPERATORS

IN-SOOK KIM

Dedicated to Professor Wataru Takahashi on his 70th birthday

Abstract. Let X be a real reflexive separable Banach space with dual space X^* and let L be a dense subspace of X. We study the solvability of the inclusion of the type $p^* \in Tx + Sx + Cx$, where $T : D(T) \subset X \to 2^{X^*}$ is a strongly quasibounded maximal monotone operator, $S : D(S) = L \subset X \to X^*$ is a strongly quasibounded maximal monotone operator, and $C : D(C) \subset X \to X^*$ satisfies condition $(S_+)_L$. The method is to use a topological degree theory for the sum $T + S + C$, based on the Kartsatos-Skrypnik degree for densely defined operators. We show that a given pathwise connected set is included in the range of the operator $T + S + C$. This implies an invariance of domain result from which the above inclusion can be solved for every $p^* \in X^*$ under a coercivity condition. Moreover, the existence of zeros for the inclusion is also discussed, with a regularization method.

1. Introduction

As one of the main objects in nonlinear analysis, the theory of monotone operators has been studied by many researchers in various ways, with applications to evolution equations and elliptic equations; see *e.g.*, [5, 17, 18]. It was mainly based on topological degree theories for suitable classes of nonlinear operators; see [6, 7, 11, 12, 16, 17].

Let X be a real reflexive separable Banach space with dual space X^* and let L be a dense subspace of X. We consider the inclusion of the type

$$p^* \in Tx + Sx + Cx,$$

where $T : D(T) \subset X \to 2^{X^*}$ is a strongly quasibounded maximal monotone operator, $S : D(S) = L \subset X \to X^*$ is a maximal monotone operator, and $C : D(C) \subset X \to X^*$ satisfies condition $(S_+)_L$. Kartsatos and Skrypnik [11] defined a topological degree for densely defined operators satisfying condition $(S_+)_{0,L}$, with application to nonlinear Dirichlet elliptic problems. Moreover, the structure of the class $(S_+)_L$ and a new construction of the degree for this class were considered in [3]. The class $(S_+)_L$ can be regarded as a natural extension of the classical definition of class (S_+); see Lemmas 3.2 and 5.4 below.

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Kartsatos and Quarcoo [10] introduced a degree theory for the sum $T + C$. Adhikari and Kartsatos [2] defined a topological degree for the sum $T+S+C$ provided that C is strongly quasibounded with respect to S and satisfies a generalized (S_+)-condition. Thus, a certain quasiboundedness condition on the perturbation C is usually required; see [11, 12]. Instead, we are now interested in solving the above inclusion in the case where S is strongly quasibounded.

The purpose of this paper is to study the solvability of the inclusion about $(S_+)_L$-perturbations of the sum of strongly quasibounded maximal monotone operators, which can be of wide application. To this end, we first introduce a topological degree theory for the sum $T + S + C$ following the basic line of [10]. Namely, the degree is defined as the limit of the Kartsatos-Skrypnik degree of the operators $T_t + S + C$ developed in [11], where $T_t : X \to X^*$ is the approximant introduced by Brézis, Crandall, and Pazy [4]. Next, applying the degree theory, we show that a given pathwise connected set is included in the range of the perturbed strongly quasibounded maximal monotone operator. This implies an invariance of domain result for $(S_+)_L$-perturbations of the sum of strongly quasibounded maximal monotone operators. As a consequence of this result, we see that under a weak coercivity condition the above inclusion can be solved for every $p^* \in X^*$. Moreover, in a more concrete situation, we establish the existence of zeros for the inclusion with a regularization method. From this, a surjectivity result is deduced. For earlier results on ranges of perturbed maximal monotone operators, we refer to [2, 9, 12, 13].

This paper is organized as follows: In Section 2, we list some definitions and notations and give useful properties concerning maximal monotone operators which will be needed. In Section 3, we introduce a topological degree theory for $(S_+)_L$-perturbations of the sum of strongly quasibounded maximal monotone operators based on the Kartsatos-Skrypnik degree, as a key tool of our results. Section 4 is devoted to ranges of perturbed strongly quasibounded maximal monotone operators including openness and surjectivity. We illustrate our results by a simple example. In Section 5, we deal with the existence of zeros for the above inclusion and surjectivity on the sum is also discussed.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and $(X^*, \|\cdot\|)$ be its dual space with dual pairing $\langle\cdot,\cdot\rangle$. Given a nonempty subset Ω of X, let $\overline{\Omega}$, $\operatorname{int}\Omega$, and $\partial\Omega$ denote the closure, the interior, and the boundary of Ω in X, respectively. Let $B_r(x)$ denote the open ball in X or X^* of radius $r > 0$ centered at x. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Let $T : D(T) \subset X \to 2^{X^*}$ be a multi-valued operator, where $D(T) := \{x \in X : Tx \neq \emptyset\}$ is called the *effective domain* of T. Then the operator T is said to be

(a) *monotone* if $\langle u^* - v^*, x - y\rangle \geq 0$ for every $x, y \in D(T)$ and every $u^* \in Tx, v^* \in Ty$.

(b) *strictly monotone* if $\langle u^* - v^*, x - y\rangle > 0$ for every $x, y \in D(T)$ with $x \neq y$ and every $u^* \in Tx, v^* \in Ty$.

(c) *maximal monotone* if it is monotone and it follows from $(x, u^*) \in X \times X^*$ and

$$\langle u^* - v^*, x - y \rangle \geq 0 \quad \text{for every } y \in D(T) \text{ and every } v^* \in Ty$$

that $x \in D(T)$ and $u^* \in Tx$.

(d) *strongly quasibounded* if for every $\ell > 0$ there exists a constant $K(\ell) > 0$ such that for all $x \in D(T)$ with $\|x\| \leq \ell$ and $\langle u^*, x \rangle \leq \ell$ where $u^* \in Tx$, we have $\|u^*\| \leq K(\ell)$.

We say that a single-valued operator $C : D(C) \subset X \to X^*$ satisfies condition (S_+) on a set $\Omega \subset D(C)$ if for every sequence $\{x_n\}$ in Ω with $x_n \rightharpoonup x_0$ and $\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \leq 0$, we have $x_n \to x_0$.

An operator $C : D(C) \subset X \to X^*$ is said to be *bounded* if C maps bounded subsets of $D(C)$ into bounded subsets of X^*. The operator C is said to be *demicontinuous* if for every $x_0 \in D(C)$ and for every sequence $\{x_n\}$ in $D(C)$ with $x_n \to x_0$, we have $Cx_n \rightharpoonup Cx_0$. The operator C is said to be *completely continuous* if for every $x_0 \in D(C)$ and for every sequence $\{x_n\}$ in $D(C)$ with $x_n \rightharpoonup x_0$, we have $Cx_n \to Cx_0$.

It is known in [8, Proposition 14] that a monotone operator $T : D(T) \subset X \to 2^{X^*}$ on the reflexive Banach space X is strongly quasibounded provided that the origin 0 is an interior point of its effective domain $D(T)$.

A typical example of strongly quasibounded maximal monotone operators is the subdifferential as follows: Let K be a closed convex subset of a reflexive Banach space X with $0 \in \operatorname{int} K$. If $\chi : X \to [0, +\infty]$ is defined to be 0 for $x \in K$ and $+\infty$ for $x \in X \setminus K$, then the subdifferential $\partial\chi : D(\partial\chi) \subset X \to 2^{X^*}$ is maximal monotone, $D(\partial\chi) = K$, $0 \in \partial\chi(0)$ and hence it is obviously strongly quasibounded; see [14, 18]. Here, $u^* \in \partial\chi(x)$ if and only if $\chi(x) \neq +\infty$ and

$$\chi(y) \geq \chi(x) + \langle u^*, y - x \rangle \qquad \text{for all } y \in X.$$

The following result on the sum is taken from Browder and Hess [8, Theorem 9].

Lemma 2.1. *Let $T : D(T) \subset X \to 2^{X^*}$ and $S : D(S) \subset X \to 2^{X^*}$ be two maximal monotone operators on the reflexive Banach space X with $0 \in D(T) \cap D(S)$. If T is (strongly) quasibounded, then the sum $T + S$ is maximal monotone.*

Throughout this paper, X will always be an infinite-dimensional real reflexive separable Banach space which has been renormed so that X and X^* are locally uniformly convex.
It is known that in this case the normalized duality operator $J : X \to X^*$ is bounded, continuous, surjective, strictly monotone, maximal monotone and satisfies condition (S_+), and such that $\langle Jx, x \rangle = \|x\|^2$ and $\|Jx\| = \|x\|$ for $x \in X$; see *e.g.*, [18].

For our aim, we need the following important properties concerning the class of maximal monotone operators.

Lemma 2.2. *Let $T : D(T) \subset X \to 2^{X^*}$ be a maximal monotone operator. Then the following statements hold:*

(a) *For each* $t \in (0,\infty)$, *the operator* $T_t \equiv (T^{-1}+tJ^{-1})^{-1} : X \to X^*$ *is bounded, demicontinuous, and maximal monotone. Thus,* $T_t x \in T(x - tJ^{-1}T_t x)$ *for* $x \in X$.

(b) *If, in addition,* $0 \in D(T)$ *and* $0 \in T(0)$, *then the operator* $(0,\infty) \times X \to X^*, (t,x) \mapsto T_t x$ *is continuous on* $(0,\infty) \times X$.

Proof. Statement (a) is due to Brézis, Crandall, and Pazy [4, Lemma 1.3]. For statement (b), we refer to [12, Lemma 3.1]. □

Lemma 2.3. *Suppose that* $T : D(T) \subset X \to 2^{X^*}$ *is a strongly quasibounded maximal monotone operator with* $0 \in D(T)$ *and* $0 \in T(0)$. *Then the following statements hold:*

(a) *If* $\{t_n\}$ *is a sequence in* $(0,\infty)$ *and* $\{x_n\}$ *is a sequence in* X *such that*

$$\|x_n\| \leq K \qquad \text{and} \qquad \langle T_{t_n} x_n, x_n\rangle \leq K_1,$$

where K, K_1 *are positive constants, then the sequence* $\{T_{t_n} x_n\}$ *is bounded in* X^*.

(b) *If, in addition,* $S : D(S) \subset X \to X^*$ *is maximal monotone and strongly quasibounded with* $0 \in D(S)$ *and* $S(0) = 0$ *and if* $\{t_n\}$ *is a sequence in* $(0,\infty)$ *and* $\{x_n\}$ *is a sequence in* $D(S)$ *such that*

$$\|x_n\| \leq K \qquad \text{and} \qquad \langle T_{t_n} x_n + Sx_n, x_n\rangle \leq K_1,$$

where K, K_1 *are positive constants, then the sequences* $\{T_{t_n} x_n\}$ *and* $\{Sx_n\}$ *are bounded in* X^*.

Proof. For statement (a), we refer to [10, Lemma D].
(b) Since the operators T_{t_n} and S are monotone and $T_{t_n}(0) = 0 = S(0)$, we have by the hypothesis

$$\langle T_{t_n} x_n, x_n\rangle \leq K_1 \qquad \text{and} \qquad \langle Sx_n, x_n\rangle \leq K_1.$$

By (a) and the strong quasiboundedness of the operator S, it is obvious that the sequences $\{T_{t_n} x_n\}$ and $\{Sx_n\}$ are bounded in X^*. □

Lemma 2.4. *Let* $T : D(T) \subset X \to 2^{X^*}$ *and* $S : D(S) \subset X \to 2^{X^*}$ *be two maximal monotone operators with* $0 \in D(T) \cap D(S)$ *and* $0 \in T(0) \cap S(0)$ *such that* T *is strongly quasibounded. Suppose that* $\{t_n\}$ *is a sequence in* $(0,\infty)$ *with* $t_n \downarrow 0$ *and* $\{x_n\}$ *is a sequence in* $D(S)$ *such that* $x_n \rightharpoonup x_0 \in X$ *and* $T_{t_n} x_n + w_n^* \rightharpoonup z_0^* \in X^*$, *where* $w_n^* \in Sx_n$. *Then the following statements hold:*

(a) *The inequality* $\liminf_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0\rangle \geq 0$ *is true.*

(b) *If* $\lim_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0\rangle = 0$, *then* $x_0 \in D(T+S)$ *and* $z_0^* \in (T+S)x_0$.

(c) *In particular, if* $x_n \to x_0$ *and* $w_n^* \rightharpoonup w_0^*$, *then* $x_0 \in D(S)$ *and* $w_0^* \in Sx_0$.

Here $D(T+S)$ *denotes the intersection of two effective domains* $D(T)$ *and* $D(S)$.

Proof. For the sake of convenience, we give the proof in a precise manner which is of course substantially analogous to that of Lemma 1 in [1].
(a) Assume on the contrary that there is a subsequence of $\{n\}$, again denoted by $\{n\}$, such that

$$\lim_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0\rangle < 0. \tag{2.1}$$

Set $v_n^* := T_{t_n} x_n$. By (2.1), we have

$$\limsup_{n\to\infty} \langle v_n^* + w_n^*, x_n\rangle = \limsup_{n\to\infty} [\langle v_n^* + w_n^*, x_n - x_0\rangle + \langle v_n^* + w_n^*, x_0\rangle] < \langle z_0^*, x_0\rangle, \tag{2.2}$$

which implies along with $\langle w_n^*, x_n\rangle \geq 0$

$$\limsup_{n\to\infty} \langle v_n^*, x_n\rangle \;\leq\; \limsup_{n\to\infty} \langle v_n^* + w_n^*, x_n\rangle \;<\; \langle z_0^*, x_0\rangle.$$

Since $\{x_n\}$ is bounded in X, it follows from Lemma 2.3(a) that the sequence $\{v_n^*\}$ is bounded in X^* and hence $\lim_{n\to\infty} t_n\|v_n^*\| = 0$.

For every $x \in D(T+S)$ and every $u^* = u_1^* + u_2^* \in Tx + Sx$, we have by the monotonicity of the operator T with $v_n^* \in T(x_n - t_n J^{-1} v_n^*)$

$$\langle v_n^* - u_1^*, x_n - t_n J^{-1} v_n^* - x\rangle \geq 0$$

and so

$$\langle v_n^* - u_1^*, x_n - x\rangle \geq t_n \langle v_n^*, J^{-1} v_n^*\rangle - \langle u_1^*, t_n J^{-1} v_n^*\rangle \geq -t_n \|v_n^*\|\,\|u_1^*\|.$$

Hence it follows from $\langle w_n^* - u_2^*, x_n - x\rangle \geq 0$ that

$$\langle v_n^* + w_n^* - u^*, x_n - x\rangle = \langle v_n^* - u_1^*, x_n - x\rangle + \langle w_n^* - u_2^*, x_n - x\rangle \geq -t_n\|v_n^*\|\,\|u_1^*\|,$$

which implies

(2.3)

$$\begin{aligned} \liminf_{n\to\infty} \langle v_n^* + w_n^*, x_n\rangle \;&\geq\; \liminf_{n\to\infty} [\langle v_n^* + w_n^*, x\rangle - \langle u^*, x\rangle + \langle u^*, x_n\rangle - t_n\|v_n^*\|\,\|u_1^*\|] \\ &\geq\; \langle z_0^*, x\rangle - \langle u^*, x\rangle + \langle u^*, x_0\rangle. \end{aligned}$$

Combining (2.2) with (2.3), we get

$$\langle z_0^* - u^*, x_0 - x\rangle > 0 \qquad \text{for every } x \in D(T+S) \text{ and every } u^* \in (T+S)x. \tag{2.4}$$

Since the sum $T + S$ is maximal monotone by Lemma 2.1, we obtain

$$x_0 \in D(T+S) \qquad \text{and} \qquad z_0^* \in (T+S)x_0.$$

Letting $x = x_0 \in D(T+S)$ in (2.4), we arrive at a contradiction. Therefore, we have shown that

$$\liminf_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0\rangle \geq 0.$$

(b) Suppose that

$$\lim_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0\rangle = 0. \tag{2.5}$$

Let $x \in D(T+S)$ and $u^* = u_1^* + u_2^* \in Tx + Sx$. By the monotonicity of the operator S and (2.5), we have

$$\limsup_{n\to\infty} \langle T_{t_n} x_n, x_n\rangle \leq \limsup_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n\rangle \leq \langle z_0^*, x_0\rangle. \tag{2.6}$$

In view of Lemma 2.3(a), the sequence $\{T_{t_n} x_n\}$ is bounded in X^* and so $\lim_{n\to\infty}$ $t_n\|T_{t_n} x_n\| = 0$. This implies as above that

$$\liminf_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n\rangle \;\geq\; \langle z_0^*, x\rangle - \langle u^*, x\rangle + \langle u^*, x_0\rangle.$$

Combining this with (2.6), we obtain

$$\langle z_0^* - u^*, x_0 - x\rangle \geq 0 \qquad \text{for every } x \in D(T+S) \text{ and every } u^* \in (T+S)x.$$

By the definition of maximal monotonicity of the operator $T+S$, we have

$$x_0 \in D(T+S) \qquad \text{and} \qquad z_0^* \in (T+S)x_0.$$

Statement (c) is a special case of statement (b) with $T \equiv 0$. This completes the proof. ☐

3. Degree theory

In this section, we introduce a topological degree theory for $(S_+)_L$-perturbations of the sum of strongly quasibounded maximal monotone operators, based on the degree theory of Kartsatos and Skrypnik [11]. It was motivated by the works of Kartsatos and Quarcoo [10] and Adhikari and Kartsatos [2].

Let L be a dense subspace of X and let $\mathcal{F}(L)$ denote the class of all finite-dimensional subspaces of L. Let $\{F_n\}$ be a sequence in the class $\mathcal{F}(L)$ such that for each $n \in \mathbb{N}$ we have

$$F_n \subset F_{n+1}, \quad \dim F_n = n, \quad \text{and} \quad \overline{\bigcup_{n\in\mathbb{N}} F_n} = X. \tag{3.1}$$

Set $L\{F_n\} := \bigcup_{n\in\mathbb{N}} F_n$.

Definition 3.1. Let $C : D(C) \subset X \to X^*$ be a single-valued operator with $L \subset D(C)$. We say that the operator C satisfies condition $(S_+)_{0,L}$ if for every sequence $\{F_n\}$ in $\mathcal{F}(L)$ satisfying (3.1) and for every sequence $\{x_n\}$ in L with

$$x_n \rightharpoonup x_0, \quad \limsup_{n\to\infty} \langle Cx_n, x_n\rangle \le 0, \quad \text{and} \quad \lim_{n\to\infty} \langle Cx_n, y\rangle = 0 \quad \text{for all } y \in L\{F_n\},$$

we have $x_n \to x_0$, $x_0 \in D(C)$, and $Cx_0 = 0$. We say that the operator C satisfies condition $(S_+)_L$ if the operator $C_h : D(C) \to X^*$, defined by $C_h x := Cx - h$, satisfies condition $(S_+)_{0,L}$ for every $h \in X^*$.

The condition $(S_+)_{0,L}$ was first introduced by Kartsatos and Skrypnik [11] and the structure of the class $(S_+)_L$ was studied by Berkovits [3]. The following result shows that the class $(S_+)_L$ is a natural extension of the classical definition of class (S_+); see also Lemma 5.4 below.

Lemma 3.2. *Let L be a dense subspace of X. Then the following relations hold:*

(a) *If $C : X \to X^*$ is a strongly quasibounded demicontinuous operator that satisfies condition (S_+) on L, then the operator C satisfies condition $(S_+)_L$.*

(b) *If $C : D(C) \subset X \to X^*$ is bounded with $L \subset D(C)$ and satisfies condition $(S_+)_L$, then $D(C) = X$ and $C : X \to X^*$ is demicontinuous and satisfies condition (S_+) on L.*

Proof. (a) Let $h \in X^*$ be given. Suppose that $\{x_n\}$ is any sequence in L such that

$$x_n \rightharpoonup x_0, \quad \limsup_{n\to\infty} \langle Cx_n - h, x_n\rangle \le 0, \quad \text{and} \quad \lim_{n\to\infty} \langle Cx_n - h, y\rangle = 0 \quad \text{for all } y \in L\{F_n\}. \tag{3.2}$$

Then $\{\langle Cx_n, x_n\rangle\}$ is bounded from above. Otherwise, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \langle Cx_{n_k}, x_{n_k}\rangle = +\infty,$$

which implies with the first and second of (3.2)

$$+\infty = \limsup_{k\to\infty} \langle Cx_{n_k}, x_{n_k}\rangle \;=\; \limsup_{k\to\infty} [\langle Cx_{n_k} - h, x_{n_k}\rangle + \langle h, x_{n_k}\rangle] \;\le\; \langle h, x_0\rangle.$$

This is a contradiction. By the strong quasiboundedness of the operator C, the sequence $\{Cx_n\}$ is bounded in X^*. Since $L\{F_n\}$ is dense in the reflexive Banach space X, it follows from the third of (3.2) that $Cx_n \rightharpoonup h$. Hence we obtain from the first and second of (3.2) that

$$\begin{aligned}\limsup_{n\to\infty} &\langle Cx_n, x_n - x_0\rangle \\ &\le\; \limsup_{n\to\infty} \langle Cx_n - h, x_n\rangle - \lim_{n\to\infty} \langle Cx_n - h, x_0\rangle + \lim_{n\to\infty} \langle h, x_n - x_0\rangle \\ &\le\; 0.\end{aligned}$$

Since C satisfies condition (S_+) on L and is demicontinuous, we have

$$x_n \to x_0, \qquad x_0 \in X = D(C), \qquad \text{and} \qquad Cx_0 - h = 0.$$

We conclude that the operator C satisfies condition $(S_+)_L$.
For statement (b), we refer to [3, Theorem 3.3]. This completes the proof. □

Assume that $T : D(T) \subset X \to 2^{X^*}$ is a multi-valued operator, $S : D(S) = L \subset X \to X^*$ is a single-valued operator, and $C : D(C) \subset X \to X^*$ is a single-valued operator with $L \subset D(C)$ such that

- (t1) T is maximal monotone and strongly quasibounded with $0 \in D(T)$ and $0 \in T(0)$.
- (s1) S is maximal monotone and strongly quasibounded with $S(0) = 0$.
- (s2) For every $F \in \mathcal{F}(L)$ and $v \in L$, the function $s(F, v) : F \to \mathbb{R}$, defined by $s(F, v)(x) := \langle Sx, v\rangle$, is continuous on F.
- (c1) C satisfies condition $(S_+)_L$.
- (c2) For every $F \in \mathcal{F}(L)$ and $v \in L$, the function $c(F, v) : F \to \mathbb{R}$, defined by $c(F, v)(x) := \langle Cx, v\rangle$, is continuous on F.
- (c3) There exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\langle Cx, x\rangle \ge -\psi(\|x\|) \qquad \text{for each } x \in D(C).$$

To define a topological degree for $(S_+)_L$-perturbations of the sum of maximal monotone operators, we need some auxiliary results.

Proposition 3.3. *Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : D(S) = L \subset X \to X^*$ satisfies (s1), and $C : D(C) \subset X \to X^*$ satisfies (c1) and (c3) with $L \subset D(C)$. Then the operator $T_t + S + C$ satisfies condition $(S_+)_{0,L}$ for each $t \in (0,\infty)$.*

Proof. Let $t \in (0,\infty)$ be given. Suppose that $\{F_n\}$ is any sequence in $\mathcal{F}(L)$ satisfying (3.1) and $\{x_n\}$ is any sequence in L such that $x_n \rightharpoonup x_0$,

$$(3.3)\quad \limsup_{n\to\infty} \langle T_t x_n + Sx_n + Cx_n, x_n\rangle \le 0, \quad \text{and} \quad \lim_{n\to\infty} \langle T_t x_n + Sx_n + Cx_n, y\rangle = 0$$

for all $y \in L\{F_n\}$. From the second of (3.3), we know that the sequence $\{\langle T_t x_n + Sx_n + Cx_n, x_n\rangle\}$ is bounded from above by a positive constant K_1. If K denotes a positive upper bound for the bounded sequence $\{\|x_n\|\}$, then we have by (c3)

$$\langle T_t x_n + Sx_n, x_n\rangle \le -\langle Cx_n, x_n\rangle + K_1 \le \psi(\|x_n\|) + K_1 \le \psi(K) + K_1.$$

According to Lemma 2.3(b), the sequences $\{T_t x_n\}$ and $\{Sx_n\}$ are bounded in X^*. Notice that each bounded sequence in a reflexive Banach space has a weakly convergent subsequence. Passing to a subsequence, if necessary, we may suppose that $T_t x_n \rightharpoonup v^*$ and $Sx_n \rightharpoonup w^*$ for some $v^*, w^* \in X^*$. Then, from the last equality of (3.3), along with

$$\langle Cx_n, y\rangle = \langle T_t x_n + Sx_n + Cx_n, y\rangle - \langle T_{t_n} x_n + Sx_n, y\rangle,$$

we obtain

$$(3.4)\qquad \lim_{n\to\infty} \langle Cx_n + v^* + w^*, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\}.$$

In view of (3.3) with the equality

$$\begin{aligned}\langle Cx_n + v^* + w^*, x_n\rangle &= \langle Cx_n + T_t x_n + Sx_n, x_n\rangle - \langle T_t x_n + Sx_n, x_n - x_0\rangle \\ &\quad - \langle T_t x_n + Sx_n, x_0\rangle + \langle v^* + w^*, x_n\rangle,\end{aligned}$$

we have by Lemma 2.4(a)

$$(3.5)\qquad \limsup_{n\to\infty} \langle Cx_n + v^* + w^*, x_n\rangle \ \le\ -\liminf_{n\to\infty} \langle T_t x_n + Sx_n, x_n - x_0\rangle \ \le\ 0.$$

Since the operator C satisfies condition $(S_+)_L$, it follows from (3.4) and (3.5) that

$$x_n \to x_0, \quad x_0 \in D(C), \quad \text{and} \quad Cx_0 + v^* + w^* = 0.$$

Since T_t is demicontinuous and S is maximal monotone, we obtain from Lemma 2.4(c) that $v^* = T_t x_0$, $x_0 \in D(S)$, and $w^* = Sx_0$ and therefore

$$x_0 \in D(T_t + S + C) \quad \text{and} \quad T_t x_0 + Sx_0 + Cx_0 = 0.$$

We conclude that the operator T_t+S+C satisfies condition $(S_+)_{0,L}$. This completes the proof. □

Proposition 3.4. *Let T, S, and C be the operators as in Proposition 3.3. Let Ω be a bounded open subset of X. Suppose that*

$$0 \notin Tx + Sx + Cx \qquad \text{for all } x \in D(T) \cap L \cap \partial\Omega.$$

Then there exists a positive number t_0 such that

$$T_t x + Sx + Cx \neq 0 \qquad \text{for all } (t,x) \in (0, t_0] \times (L \cap \partial\Omega).$$

Proof. Assume that the assertion is not true. Then there are sequences $\{t_n\}$ in $(0,\infty)$ with $t_n \downarrow 0$ and $\{x_n\}$ in $L \cap \partial\Omega$ such that

$$T_{t_n}x_n + Sx_n + Cx_n = 0. \tag{3.6}$$

If K is a positive upper bound for the sequence $\{\|x_n\|\}$, then (3.6) and (c3) imply that

$$\langle T_{t_n}x_n + Sx_n, x_n\rangle = -\langle Cx_n, x_n\rangle \le \psi(\|x_n\|) \le \psi(K) =: K_1.$$

In view of Lemma 2.3(b), we may suppose that

$$x_n \rightharpoonup x_0 \in X \qquad \text{and} \qquad T_{t_n}x_n + Sx_n \rightharpoonup z_0^* \in X^*.$$

By (3.6), we have $Cx_n \rightharpoonup -z_0^*$ and hence

$$\lim_{n\to\infty} \langle Cx_n + z_0^*, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\}. \tag{3.7}$$

Observing the equality

$$\begin{aligned}\langle Cx_n + z_0^*, x_n\rangle &= \langle Cx_n + T_{t_n}x_n + Sx_n, x_n\rangle - \langle T_{t_n}x_n + Sx_n, x_n - x_0\rangle \\ &\quad - \langle T_{t_n}x_n + Sx_n, x_0\rangle + \langle z_0^*, x_n\rangle,\end{aligned}$$

it follows from (3.6) and Lemma 2.4(a) that

$$\limsup_{n\to\infty} \langle Cx_n + z_0^*, x_n\rangle \ \le\ -\liminf_{n\to\infty} \langle T_{t_n}x_n + Sx_n, x_n - x_0\rangle \ \le\ 0. \tag{3.8}$$

Since the operator C satisfies condition $(S_+)_L$, it follows from (3.7) and (3.8) that

$$x_n \to x_0 \in D(C) \qquad \text{and} \qquad Cx_0 + z_0^* = 0.$$

Moreover, $T_{t_n}x_n + Sx_n \rightharpoonup z_0^*$ implies that

$$\lim_{n\to\infty} \langle T_{t_n}x_n + Sx_n, x_n - x_0\rangle = 0.$$

Hence we obtain from Lemma 2.4(b) that $x_0 \in D(T+S)$ and $z_0^* \in (T+S)x_0$ and therefore

$$x_0 \in D(T) \cap L \cap \partial\Omega \qquad \text{and} \qquad 0 \in Tx_0 + Sx_0 + Cx_0,$$

which contradicts the hypothesis on the boundary. Consequently, the assertion is true. This completes the proof. □

Proposition 3.5. *Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : D(S) = L \to X^*$ satisfies (s1) and (s2), and $C : D(C) \subset X \to X^*$ satisfies (c1), (c2), and (c3) with $L \subset D(C)$. Let Ω be a bounded open subset of X. If*

$$0 \notin Tx + Sx + Cx \qquad \textit{for all } x \in D(T) \cap L \cap \partial\Omega,$$

then the degree $d(T_t + S + C, \Omega, 0)$ is constant for all $t \in (0, t_0]$, where t_0 is a fixed positive number determined by Proposition 3.4. Here, the symbol d denotes the Kartsatos-Skrypnik degree introduced in [11].

Proof. In view of Propositions 3.3 and 3.4, the degree $\mathrm{d}(T_t + S + C, \Omega, 0)$ is well defined for every $t \in (0, t_0]$. It suffices to prove that for any two numbers $t_1, t_2 \in (0, t_0]$ we have

$$\mathrm{d}(T_{t_1} + S + C, \Omega, 0) = \mathrm{d}(T_{t_2} + S + C, \Omega, 0). \tag{3.9}$$

Let t_1, t_2 be any two points in the interval $(0, t_0]$ with $t_1 < t_2$. Consider a continuous function $\alpha : [0,1] \to \mathbb{R}$ given by

$$\alpha(t) := (1-t)t_1 + tt_2.$$

For $t \in [0,1]$, let $A_t : D(A_t) \subset X \to X^*$ be an operator defined by

$$A_t x := T_{\alpha(t)}x + Sx + Cx,$$

where $D(A_t) = L$. According to Proposition 3.4, $A_t x \neq 0$ for all $(t,x) \in [0,1] \times (D(A_t) \cap \partial\Omega)$. For every finite-dimensional space $F \subset L\{F_n\}$ and every $v \in L\{F_n\}$, the function $\tilde{a}(F,v) : [0,1] \times F \to \mathbb{R}$, defined by $\tilde{a}(F,v)(t,x) := \langle A_t x, v\rangle$, is continuous on $[0,1] \times F$. This follows from Lemma 2.2(b), (s2), and (c2). To show that A_0 and A_1 are homotopic with respect to Ω in the sense of Definition 4.2 of [11], it remains to verify that the family $\{A_t\}$ satisfies condition $(S_+)^{(t)}_{0,L}$.
Suppose that $\{t_n\}$ is any sequence in $[0,1]$ and $\{x_n\}$ is any sequence in $L\{F_n\}$ such that $t_n \to \tilde{t}$, $x_n \rightharpoonup x_0$, and

$$\limsup_{n\to\infty} \langle A_{t_n}x_n, x_n\rangle \le 0, \quad \text{and} \quad \lim_{n\to\infty} \langle A_{t_n}x_n, y\rangle = 0 \quad \text{for all } y \in L\{F_n\}. \tag{3.10}$$

From the first of (3.10), we obtain that the sequence $\{\langle A_{t_n}x_n, x_n\rangle\}$ is bounded from above by a positive constant K_1. Then we have

$$\langle T_{\alpha(t_n)}x_n + Sx_n, x_n\rangle \le -\langle Cx_n, x_n\rangle + K_1 \le \psi(K) + K_1,$$

where K is a positive upper bound for the sequence $\{\|x_n\|\}$. By Lemma 2.3(b), the sequences $\{T_{\alpha(t_n)}x_n\}$ and $\{Sx_n\}$ are bounded in X^*. Without loss of generality, we may suppose that $T_{\alpha(t_n)}x_n \rightharpoonup v^*$ and $Sx_n \rightharpoonup w^*$ for some $v^*, w^* \in X^*$. Then the second of (3.10) implies that

$$\lim_{n\to\infty} \langle Cx_n + v^* + w^*, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\}. \tag{3.11}$$

Since the operator $T_{\alpha(t_n)}$ is monotone and $T_{\alpha(t_n)}x_0 \to T_{\alpha(\tilde{t})}x_0$ by Lemma 2.2(b), we have

$$\liminf_{n\to\infty} \langle T_{\alpha(t_n)}x_n, x_n - x_0\rangle \ \ge\ \liminf_{n\to\infty} \langle T_{\alpha(t_n)}x_0, x_n - x_0\rangle \ =\ 0,$$

which implies together with $\liminf_{n\to\infty} \langle Sx_n, x_n - x_0\rangle \ge 0$

$$\liminf_{n\to\infty} \langle T_{\alpha(t_n)}x_n + Sx_n, x_n - x_0\rangle \ge 0. \tag{3.12}$$

From (3.10), (3.12), and the equality

$$\begin{aligned}\langle Cx_n + v^* + w^*, x_n\rangle \ &= \langle Cx_n + T_{\alpha(t_n)}x_n + Sx_n, x_n\rangle - \langle T_{\alpha(t_n)}x_n + Sx_n, x_n - x_0\rangle \\ &\quad - \langle T_{\alpha(t_n)}x_n + Sx_n, x_0\rangle + \langle v^* + w^*, x_n\rangle,\end{aligned}$$

it follows that

$$\limsup_{n\to\infty} \langle Cx_n + v^* + w^*, x_n\rangle \ \le\ -\liminf_{n\to\infty} \langle T_{\alpha(t_n)}x_n + Sx_n, x_n - x_0\rangle \ \le\ 0. \tag{3.13}$$

Since the operator C satisfies condition $(S_+)_L$, it follows from (3.11) and (3.13) that

$$x_n \to x_0 \in D(C) \qquad \text{and} \qquad Cx_0 + v^* + w^* = 0.$$

By Lemma 2.2(b), we have $T_{\alpha(t_n)}x_n \to T_{\alpha(\tilde{t})}x_0$ and so $v^* = T_{\alpha(\tilde{t})}x_0$. Since S is maximal monotone, we have $x_0 \in D(S)$ and $w^* = Sx_0$ and hence

$$x_0 \in D(A_{\tilde{t}}) \qquad \text{and} \qquad A_{\tilde{t}}x_0 = T_{\alpha(\tilde{t})}x_0 + Sx_0 + Cx_0 = 0.$$

We conclude that the family $\{A_t\}$ satisfies condition $(S_+)^{(t)}_{0,L}$.
Since A_0 and A_1 are thus homotopic with respect to Ω, Theorem 4.1 of [11] states that

$$\mathrm{d}\,(A_0, \Omega, 0) = \mathrm{d}\,(A_1, \Omega, 0),$$

that is, (3.9) holds, what we wanted to prove. This completes the proof. $\square$

We are now ready to define a topological degree for $(S_+)_L$-perturbations of the sum of strongly quasibounded maximal monotone operators, based on the Kartsatos-Skrypnik degree theory in [11].

Definition 3.6. Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : D(S) = L \subset X \to X^*$ satisfies (s1) and (s2), and $C : D(C) \subset X \to X^*$ satisfies (c1), (c2), and (c3) with $L \subset D(C)$. Let Ω be a bounded open set in X. If $0 \notin (T + S + C)(D(T) \cap L \cap \partial\Omega)$, then we can define a degree function by

$$\deg\,(T + S + C, \Omega, 0) := \lim_{t \downarrow 0} \mathrm{d}\,(T_t + S + C, \Omega, 0).$$

If $p^* \in X^*$ is such that $p^* \notin (T + S + C)(D(T) \cap L \cap \partial\Omega)$, then we define

$$\deg\,(T + S + C, \Omega, p^*) := \deg\,(T + S + C - p^*, \Omega, 0).$$

According to Proposition 3.5, the degree function is well defined and this definition seems to be almost the same as that in [10], except that the operator $T + C$ there is replaced by $T+S+C$. The attempt was inspired by Adhikari and Kartsatos [2]. The advantage is that our definition would be more applicable.

We give some of the fundamental properties of the above degree.

Theorem 3.7. *Let L be a dense subspace of X. Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : L \to X^*$ satisfies (s1) and (s2), and $C : D(C) \subset X \to X^*$ satisfies (c1), (c2), and (c3) with $L \subset D(C)$. Let Ω be a bounded open set in X. Then we have the following properties:*

(a) *If $0 \in \Omega$, then deg $(\varepsilon J, \Omega, 0) = 1$ for each $\varepsilon > 0$.*
(b) *If $p^* \notin (T+S+C)(D(T) \cap L \cap \partial\Omega)$ and deg $(T+S+C, \Omega, p^*) \neq 0$, then the inclusion $p^* \in (T + S + C)x$ has at least one solution in $D(T) \cap L \cap \Omega$.*
(c) *If $0 \in \Omega$ and $0 \notin H(t, \cdot)(D(T) \cap L \cap \partial\Omega)$ for all $t \in [0, 1]$, where*

$$H(t, x) := t(T + S + C)x + (1 - t)\tilde{C}x,$$

then deg $(T + S + C, \Omega, 0) =$ deg $(\tilde{C}, \Omega, 0)$. Here, $\tilde{C} : X \to X^*$ is bounded, demicontinuous, strictly monotone and satisfies condition (S_+) on L, $\tilde{C}(0) = 0$, and $\langle \tilde{C}x, x\rangle \geq \phi(\|x\|)$ for all $x \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing, continuous, and $\phi(0) = 0$.
(d) *If $\alpha : [0, 1] \to X^*$ is a continuous curve and $0 \notin H(t, \cdot)(D(T) \cap L \cap \partial\Omega)$ for all $t \in [0, 1]$, where*

$$H(t, x) := (T + S + C)x - \alpha(t),$$

then the degree deg$(H(t,\cdot),\Omega,0)$ *is constant.*

Proof. Noticing by Lemma 2.1 that the sum $T+S$ is maximal monotone and strongly quasibounded, the proof might be essentially the same as that in [10, Theorem 3]. To make clear the difference, we perform the proof of property (c) in detail.
(c) Consider a map H_1 given by

$$H_1(t,s,x) := s(T_t + S + C)x + (1-s)\tilde{C}x \qquad \text{for } t \in (0,\infty) \text{ and } s \in [0,1].$$

First, we will prove that there exists a positive number t_0 such that the equation $H_1(t,s,x) = 0$ has no solution in $L \cap \partial\Omega$ for all $t \in (0, t_0]$ and all $s \in [0,1]$. Assume the contrary. Then there are sequences $\{t_n\}$ in $(0,\infty)$, $\{s_n\}$ in $[0,1]$, and $\{x_n\}$ in $L \cap \partial\Omega$ such that $t_n \downarrow 0$, $s_n \to s_0$, and

$$s_n(T_{t_n} + S + C)x_n + (1-s_n)\tilde{C}x_n = 0. \tag{3.14}$$

Then $s_n \in (0,1]$ for all $n \in \mathbb{N}$, by the injectivity of $\tilde{C}$ with $\tilde{C}(0) = 0$. Moreover, we have $s_0 \in (0,1]$. Indeed, if $s_0 = 0$, then (3.14), (c3), and the monotonicity of the operator $T_{t_n} + S$ with $(T_{t_n} + S)(0) = 0$ imply that

$$\begin{aligned}(1-s_n)\phi(\|x_n\|) &\le s_n\langle T_{t_n}x_n + Sx_n, x_n\rangle + (1-s_n)\langle \tilde{C}x_n, x_n\rangle \\ &= -s_n\langle Cx_n, x_n\rangle \le s_n(\psi(\|x_n\|))\end{aligned}$$

and hence $\phi(\|x_n\|) \to 0$ and therefore $x_n \to 0 \in \Omega$, which is a contradiction.
It follows from (3.14) and the monotonicity of $\tilde{C}$ with $\tilde{C}(0) = 0$ that

$$\langle T_{t_n}x_n + Sx_n, x_n\rangle = -\langle Cx_n, x_n\rangle - \Big(\frac{1-s_n}{s_n}\Big)\langle \tilde{C}x_n, x_n\rangle \le \psi(\|x_n\|).$$

In view of Lemma 2.3(b) and the boundedness of $\tilde{C}$, we may suppose that

$$x_n \rightharpoonup x_0, \quad T_{t_n}x_n + Sx_n \rightharpoonup z_0^*, \quad \text{and} \quad \tilde{C}x_n \rightharpoonup c^*$$

for some $x_0 \in X$ and some $z_0^*, c^* \in X^*$. Set $\tilde{s}_n := (1-s_n)/s_n$ and $\tilde{s}_0 := (1-s_0)/s_0$. Since we have by (3.14) $Cx_n \rightharpoonup -z_0^* - \tilde{s}_0 c^*$, it is obvious that

$$\lim_{n\to\infty}\langle Cx_n + z_0^* + \tilde{s}_0c^*, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\}. \tag{3.15}$$

It is known in [18, Proposition 32.7] that each demicontinuous monotone operator $A : X \to X^*$ on the real reflexive Banach space X is maximal monotone. Since the operator $\tilde{C}$ is maximal monotone, Lemma 2.4(a) implies that

$$\liminf_{n\to\infty}\langle T_{t_n}x_n + Sx_n + \tilde{s}_n\tilde{C}x_n, x_n - x_0\rangle \ge 0. \tag{3.16}$$

From (3.14), (3.16), and the equality

$$\begin{aligned}\langle Cx_n + z_0^* + \tilde{s}_0c^*, x_n\rangle = &\ \langle Cx_n + T_{t_n}x_n + Sx_n + \tilde{s}_n\tilde{C}x_n, x_n\rangle \\ &- \langle T_{t_n}x_n + Sx_n + \tilde{s}_n\tilde{C}x_n, x_n - x_0\rangle \\ &- \langle T_{t_n}x_n + Sx_n + \tilde{s}_n\tilde{C}x_n, x_0\rangle + \langle z_0^* + \tilde{s}_0c^*, x_n\rangle,\end{aligned}$$

it follows that

$$\limsup_{n\to\infty}\langle Cx_n + z_0^* + \tilde{s}_0c^*, x_n\rangle \le -\liminf_{n\to\infty}\langle T_{t_n}x_n + Sx_n + \tilde{s}_n\tilde{C}x_n, x_n - x_0\rangle \le 0. \tag{3.17}$$

Since the operator C satisfies condition $(S_+)_L$, we obtain from (3.15) and (3.17) that

$$x_n \to x_0, \quad x_0 \in D(C), \quad \text{and} \quad Cx_0 + z_0^* + \tilde{s}_0 c^* = 0.$$

As $\lim_{n\to\infty}\langle T_{t_n}x_n + Sx_n, x_n - x_0\rangle = 0$, Lemma 2.4(b) states that $x_0 \in D(T+S)$ and $z_0^* \in (T+S)x_0$. By the demicontinuity of $\tilde{C}$, we have $c^* = \tilde{C}x_0$ and hence

$$x_0 \in D(T)\cap L\cap\partial\Omega \quad \text{and} \quad 0 \in s_0(Tx_0 + Sx_0 + Cx_0) + (1-s_0)\tilde{C}x_0 = H(s_0, x_0),$$

which contradicts the hypothesis that $0 \notin H(s,\cdot)(D(T)\cap L\cap\partial\Omega)$ for all $s \in [0,1]$. Therefore, there exists $t_0 > 0$ such that the equation $H_1(t,s,x) = 0$ has no solution in $L\cap\partial\Omega$ for any $(t,s) \in (0,t_0]\times[0,1]$.

Next, we want to show that for each fixed $t \in (0,t_0]$, we have

$$\deg(H_1(t,1,\cdot),\Omega,0) = \deg(H_1(t,0,\cdot),\Omega,0). \tag{3.18}$$

Fix $t \in (0,t_0]$ and consider a family of operators $A_s : D(A_s) \subset X \to X^*$ given by

$$A_s x := H_1(t,s,x),$$

where $D(A_s) = X$ for $s = 0$ and $D(A_s) = L$ for $s \in (0,1]$. By Lemma 3.2 and Proposition 3.3, the operators $A_0 = \tilde{C}$ and $A_1 = T_t + S + C$ satisfy condition $(S_+)_{0,L}$. To show that A_0 and A_1 are homotopic with respect to Ω, we only have to check that the family $\{A_s\}$ satisfies condition $(S_+)^{(s)}_{0,L}$. For this, suppose that $\{s_n\}$ is any sequence in $[0,1]$ and $\{x_n\}$ is any sequence in $L\{F_n\}$ such that $s_n \to s_0$, $x_n \rightharpoonup x_0$, and

$$\limsup_{n\to\infty}\langle A_{s_n}x_n, x_n\rangle \le 0, \quad \text{and} \quad \lim_{n\to\infty}\langle A_{s_n}x_n, y\rangle = 0 \quad \text{for all } y \in L\{F_n\}. \tag{3.19}$$

There are two cases to consider. Let $s_0 = 0$. Obviously, we have

$$\begin{aligned} s_n\langle Cx_n, x_n\rangle &= \langle A_{s_n}x_n, x_n\rangle - s_n\langle T_t x_n + Sx_n, x_n\rangle - (1-s_n)\langle \tilde{C}x_n, x_n\rangle \\ &\le \langle A_{s_n}x_n, x_n\rangle - (1-s_n)\phi(\|x_n\|). \end{aligned} \tag{3.20}$$

Since the sequence $\{\|x_n\|\}$ is bounded, it has a convergent subsequence, denoted again by $\{\|x_n\|\}$. Noting that $\{\psi(\|x_n\|)\}$ is also bounded, it follows from the first of (3.19) and (3.20) that

$$0 = -\lim_{n\to\infty} s_n\psi(\|x_n\|) \le \limsup_{n\to\infty} s_n\langle Cx_n, x_n\rangle \le -\lim_{n\to\infty}\phi(\|x_n\|) \le 0$$

and so $\lim_{n\to\infty} x_n = 0$. Hence $x_0 = 0 \in D(A_{s_0})$ and $A_{s_0}x_0 = 0$.
Now let $s_0 \in (0,1]$. We may suppose that $s_n > 0$ for all $n \in \mathbb{N}$. Set $\tilde{s}_n := (1-s_n)/s_n$ and $\tilde{s}_0 := (1-s_0)/s_0$. We may rewrite (3.19) in the form:

$$\limsup_{n\to\infty}\langle T_t x_n + Sx_n + Cx_n + \tilde{s}_n\tilde{C}x_n, x_n\rangle \le 0 \tag{3.21}$$

and

$$\lim_{n\to\infty}\langle T_t x_n + Sx_n + Cx_n + \tilde{s}_n\tilde{C}x_n, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\}. \tag{3.22}$$

From (3.21), we know that the sequence $\{\langle T_{t_n}x_n + Sx_n + Cx_n + \tilde{s}_n\tilde{C}x_n, x_n\rangle\}$ is bounded from above by a positive upper bound K_1. This and the monotonicity of $\tilde{C}$ imply that

$$\langle T_t x_n + Sx_n, x_n\rangle \le -\langle Cx_n, x_n\rangle - \tilde{s}_n\langle\tilde{C}x_n, x_n\rangle + K_1 \le \psi(\|x_n\|) + K_1.$$

By Lemma 2.3(b), the sequences $\{T_t x_n\}$ and $\{Sx_n\}$ are bounded in X^*. We may suppose that $T_t x_n \rightharpoonup v^*$, $Sx_n \rightharpoonup w^*$, and $\tilde{C}x_n \rightharpoonup h^*$ for some $v^*, w^*, h^* \in X^*$. Then (3.22) implies that

$$\lim_{n\to\infty} \langle Cx_n + v^* + w^* + \tilde{s}_0 h^*, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\}. \tag{3.23}$$

As above, it follows from (3.21) and Lemma 2.4(a) that

$$\begin{aligned}\limsup_{n\to\infty} \langle Cx_n + v^* + w^* + \tilde{s}_0 h^*, x_n\rangle &\le -\liminf_{n\to\infty} \langle T_t x_n + Sx_n + \tilde{s}_n \tilde{C}x_n, x_n - x_0\rangle \\ &\le 0.\end{aligned} \tag{3.24}$$

Since the operator C satisfies condition $(S_+)_L$, it follows from (3.23) and (3.24) that

$$x_n \to x_0 \in D(C) \qquad \text{and} \qquad Cx_0 + v^* + w^* + \tilde{s}_0 h^* = 0.$$

Since the operator S is maximal monotone and the operators T_t, $\tilde{C}$ are demicontinuous, it is clear that

$$x_0 \in D(S), \qquad w^* = Sx_0, \qquad v^* = T_t x_0, \qquad \text{and} \qquad h^* = \tilde{C}x_0$$

and therefore

$$x_0 \in D(A_{s_0}) \qquad \text{and} \qquad A_{s_0}x_0 \; = \; s_0(T_t x_0 + Sx_0 + Cx_0) + (1-s_0)\tilde{C}x_0 = 0.$$

We have shown that the family $\{A_s\}$ satisfies condition $(S_+)^{(s)}_{0,L}$.
Since A_0 and A_1 are thus homotopic with respect to Ω, Theorem 4.1 of [11] implies that $\mathrm{d}\,(A_1, \Omega, 0) = \mathrm{d}\,(A_0, \Omega, 0)$, that is, (3.18) holds. Therefore, for every $t \in (0, t_0]$, we have

$$\mathrm{d}\,(T_t + S + C, \Omega, 0) = \mathrm{d}\,(\tilde{C}, \Omega, 0).$$

By Definition 3.6, we conclude that

$$\deg\,(T + S + C, \Omega, 0) = \mathrm{d}\,(\tilde{C}, \Omega, 0).$$

This completes the proof of (c). □

Remark 3.8. The difference between our definition and Definition 5 of [2] is that the assumption of strong quasiboundedness on the operator C in [2] was moved to that of the operator S. Moreover, a generalized (S_+)-condition in [2] was replaced by condition $(S_+)_L$.

We present an example of operators satisfying condition $(S_+)_{0,L}$ which is a particular form of Theorem 5.1 in [11].

Example 3.9. Let G is a bounded open set in $\mathbb{R}^N$ and let $2 \le p < \infty$ and set $p' := p/(p-1)$. Suppose that $\rho : \mathbb{R} \to \mathbb{R}$ is a continuous function on $\mathbb{R}$ and there is a positive constant c such that

$$0 \le \rho(t) \le c\left[\,\Big|\int_0^t \rho(s)ds\Big| + 1\right]^r \qquad \text{for all } t \in \mathbb{R},$$

where r is an exponent with $0 \le r < N/(N-2)$. Let $A : D(A) \subset W_0^{1,p}(G) \to [W_0^{1,p}(G)]^*$ be an operator setting by

$$\langle Au, \varphi \rangle = \sum_{i=1}^{N} \int_G \Big[\rho^2(u) \frac{\partial u}{\partial x_i} + \Big| \frac{\partial u}{\partial x_i} \Big|^{p-2} \frac{\partial u}{\partial x_i} \Big] \frac{\partial \varphi(x)}{\partial x_i}\, dx,$$

where

$$D(A) = \{u \in W_0^{1,p}(G) : \rho^2(u) \frac{\partial u}{\partial x_i} \in L^{p'}(G)\}.$$

Taking into account the p-Laplace operator Δ_p defined by

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \Big(\Big| \frac{\partial u}{\partial x_i} \Big|^{p-2} \frac{\partial u}{\partial x_i} \Big),$$

Theorem 5.1 of [11] implies that the operator A satisfies condition $(S_+)_{0,L}$ with respect to the space $L = C_0^\infty(G)$.

According to [11], the degree theory for densely defined operators satisfying condition $(S_+)_{0,L}$ is used to solve the Dirichlet boundary value problem for the elliptic equation

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \Big[\rho^2(u) \frac{\partial u}{\partial x_i} + \Big| \frac{\partial u}{\partial x_i} \Big|^{p-2} \frac{\partial u}{\partial x_i} \Big] = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} f_i(x),$$

where $f_i \in L^{p'}(G)$ for $i = 1, \cdots, N$.

4. Ranges

This section is devoted to ranges of perturbed strongly quasibounded maximal monotone operators including openness and surjectivity, by using the degree theory in Section 3.

The following result says that a given pathwise connected set is included in the range of the sum of strongly quasibounded maximal monotone operators by a $(S_+)_L$-perturbation. The case when the perturbation C is strongly quasibounded with respect to S and satisfies a generalized (S_+)-condition can be found in [2, Theorem 8].

Theorem 4.1. *Let L be a dense subspace of X. Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : L \to X^*$ satisfies (s1) and (s2), and $C : D(C) \subset X \to X^*$ satisfies (c1), (c2), and (c3) with $L \subset D(C)$. Let Ω be a bounded open subset of X with $0 \in \Omega$. Suppose that M is a pathwise connected set in X^* such that*

(4.1)
$$[(T+S+C)(D(T)\cap L\cap\Omega)]\cap M \neq \emptyset \quad \text{and} \quad [(T+S+C+\varepsilon J)(D(T)\cap L\cap\partial\Omega)]\cap M = \emptyset$$

for every $\varepsilon \ge 0$. Then we have

$$M \subset (T+S+C)(D(T) \cap L \cap \Omega).$$

Proof. Take an element q^* from the set $[(T+S+C)(D(T)\cap L\cap\Omega)]\cap M$. Let p^* be an arbitrary element of M with $p^*\neq q^*$. Let $\alpha:[0,1]\to M$ be a continuous path in X^* such that $\alpha(0)=q^*$ and $\alpha(1)=p^*$. First, we consider a homotopy $H_{1,n}:[0,1]\times(D(T)\cap L)\to 2^{X^*}$ defined by

$$H_{1,n}(t,x):=Tx+Sx+Cx+\frac{1}{n}Jx-\alpha(t).$$

We will show that for all large n, the inclusion

$$0\in H_{1,n}(t,x) \tag{4.2}$$

has no solution in $[0,1]\times(D(T)\cap L\cap\partial\Omega)$. For this, assume the contrary. Then there is a subsequence $\{n_k\}$ of $\{n\}$ such that the inclusion $0\in H_{1,n_k}(t,x)$ has a solution in $[0,1]\times(D(T)\cap L\cap\partial\Omega)$ for each $k\in\mathbb{N}$. For brevity, we may suppose that for each $n\in\mathbb{N}$, there exist $t_n\in[0,1], x_n\in D(T)\cap L\cap\partial\Omega$, and $v_n^*\in Tx_n$ such that

$$v_n^*+Sx_n+Cx_n+\frac{1}{n}Jx_n=\alpha(t_n). \tag{4.3}$$

Passing to a subsequence, if necessary, we may suppose that $t_n\to t_0$ and $x_n\rightharpoonup x_0$ for some $t_0\in[0,1]$ and some $x_0\in X$. Since the sequences $\{x_n\},\{\psi(\|x_n\|)\}$, and $\{\alpha(t_n)\}$ are bounded and

$$\begin{aligned}\langle v_n^*+Sx_n,x_n\rangle &= -\langle Cx_n,x_n\rangle-\frac{1}{n}\langle Jx_n,x_n\rangle+\langle\alpha(t_n),x_n\rangle\\ &\le \psi(\|x_n\|)+\|\alpha(t_n)\|\,\|x_n\|,\end{aligned}$$

we can choose a positive constant ℓ such that

$$\|x_n\|\le\ell,\quad \langle v_n^*,x_n\rangle\le\ell,\quad\text{and}\quad\langle Sx_n,x_n\rangle\le\ell\qquad\text{for all } n\in\mathbb{N}.$$

By the strong quasiboundedness of the operators T and S, the sequences $\{v_n^*\}$ and $\{Sx_n\}$ are bounded in X^*. We may suppose that $v_n^*\rightharpoonup v^*$ and $Sx_n\rightharpoonup s^*$ for some $v^*,s^*\in X^*$. By (4.3), we have $Cx_n\rightharpoonup -v^*-s^*+\alpha(t_0)$ and hence

$$\lim_{n\to\infty}\langle Cx_n+v^*+s^*-\alpha(t_0),y\rangle=0\qquad\text{for all } y\in L\{F_n\}. \tag{4.4}$$

Since the operator $T+S$ is maximal monotone, Lemma 2.4(a) implies that

$$\liminf_{n\to\infty}\langle v_n^*+Sx_n,x_n-x_0\rangle\ge 0$$

and therefore

$$\liminf_{n\to\infty}\langle v_n^*+Sx_n+\frac{1}{n}Jx_n-\alpha(t_n),x_n-x_0\rangle\ge 0. \tag{4.5}$$

From (4.3), (4.5), and the equality

$$\begin{aligned}\langle Cx_n+v^*+s^*-\alpha(t_0),x_n\rangle = &\ \Big\langle Cx_n+v_n^*+Sx_n+\frac{1}{n}Jx_n-\alpha(t_n),x_n\Big\rangle\\ &-\Big\langle v_n^*+Sx_n+\frac{1}{n}Jx_n-\alpha(t_n),x_n-x_0\Big\rangle\\ &-\Big\langle v_n^*+Sx_n+\frac{1}{n}Jx_n-\alpha(t_n),x_0\Big\rangle\\ &+\langle v^*+s^*-\alpha(t_0),x_n\rangle\end{aligned}$$

it follows that

$$\begin{aligned}(4.6)\qquad &\limsup_{n\to\infty}\langle Cx_n + v^* + s^* - \alpha(t_0), x_n\rangle \\ &\quad \le -\liminf_{n\to\infty}\left\langle v_n^* + Sx_n + \frac{1}{n}Jx_n - \alpha(t_n), x_n - x_0\right\rangle \\ &\quad \le 0.\end{aligned}$$

Since the operator C satisfies condition $(S_+)_L$, we obtain from (4.4) and (4.6) that

$$x_n \to x_0, \quad x_0 \in D(C), \quad \text{and} \quad Cx_0 + v^* + s^* - \alpha(t_0) = 0.$$

By the maximal monotonicity of the operator $T + S$, Lemma 2.4(c) states that

$$x_0 \in D(T+S) \qquad \text{and} \qquad v^* + s^* \in Tx_0 + Sx_0.$$

Therefore, we have

$$\alpha(t_0) \in Tx_0 + Sx_0 + Cx_0 \qquad \text{and} \qquad x_0 \in D(T) \cap L \cap \partial\Omega,$$

which contradicts the hypothesis (4.1) with $\varepsilon = 0$. Until now, we have shown that assertion (4.2) holds for all large n, that is, $0 \notin H_{1,n}(t,\cdot)(D(T) \cap L \cap \partial\Omega)$ for all $t \in [0,1]$.

Note that the operator $\hat{T}_n := T + (1/n)J$ is maximal monotone by Lemma 2.1, strongly quasibounded, and $0 \in \hat{T}_n(0)$. In view of (4.2), we obtain from Theorem 3.7(d) that

$$(4.7)\qquad \deg(H_{1,n}(1,\cdot), \Omega, 0) = \deg(H_{1,n}(0,\cdot), \Omega, 0) \qquad \text{for all large } n.$$

Next, we consider another homotopy $H_{2,n}$ given by

$$H_{2,n}(t,x) := t(Tx + Sx + Cx + \frac{1}{n}Jx - q^*) + (1-t)Jx.$$

We now prove that for all large n, the inclusion

$$(4.8)\qquad 0 \in H_{2,n}(t,x)$$

has no solution in $[0,1] \times (D(T) \cap L \cap \partial\Omega)$. If this is not true, then we may suppose that there are sequences $\{t_n\}$ in $[0,1]$, $\{x_n\}$ in $D(T) \cap L \cap \partial\Omega$, and $\{v_n^*\}$ in X^* with $v_n^* \in Tx_n$ such that

$$(4.9)\qquad t_n(v_n^* + Sx_n + Cx_n + \frac{1}{n}Jx_n - q^*) + (1-t_n)Jx_n = 0.$$

We may suppose that $t_n \to t_0$ and $x_n \rightharpoonup x_0$, where $t_0 \in [0,1]$ and $x_0 \in X$. By the injectivity of the duality operator J, we have $t_n \in (0,1]$. Moreover, the limit t_0 belongs to $(0,1]$. In fact, if $t_0 = 0$, then (4.9), (c3), and the monotonicity of the operator $T + S + (1/n)J$ imply that

$$\begin{aligned}(1-t_n)\|x_n\|^2 &\le t_n\langle v_n^* + Sx_n + \frac{1}{n}Jx_n, x_n\rangle + (1-t_n)\langle Jx_n, x_n\rangle \\ &= -t_n\langle Cx_n - q^*, x_n\rangle \le t_n[\psi(\|x_n\|) + \|q^*\|\,\|x_n\|]\end{aligned}$$

and so $\lim_{n\to\infty} x_n = 0$, which contradicts the fact that 0 is not contained in the closed set $\partial\Omega$.
It follows from (t1), (s1), and

$$\begin{aligned}\langle v_n^* + Sx_n, x_n\rangle &= -\langle Cx_n, x_n\rangle + \langle q^*, x_n\rangle - \Big(\frac{1}{n} + \frac{1-t_n}{t_n}\Big)\langle Jx_n, x_n\rangle \\ &\le \psi(\|x_n\|) + \|q^*\|\,\|x_n\|\end{aligned}$$

that the sequence $\{v_n^* + Sx_n\}$ is bounded in X^*. Together with the boundedness of the duality operator J, we may suppose without loss of generality that $v_n^* + Sx_n \rightharpoonup w^*$ and $Jx_n \rightharpoonup j^*$ for some $w^*, j^* \in X^*$. Set $\tilde{t}_0 := (1-t_0)/t_0$. Since we have by (4.9) $Cx_n \rightharpoonup -w^* + q^* - \tilde{t}_0 j^*$, we can show as above that

$$\lim_{n\to\infty} \langle Cx_n + w^* - q^* + \tilde{t}_0 j^*, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\} \tag{4.10}$$

and

$$\limsup_{n\to\infty} \langle Cx_n + w^* - q^* + \tilde{t}_0 j^*, x_n\rangle \le 0. \tag{4.11}$$

The relation (4.11) follows, in view of Lemma 2.4(a), from

$$\liminf_{n\to\infty} \langle v_n^* + Sx_n - q^* + \frac{1}{n} Jx_n + \Big(\frac{1-t_n}{t_n}\Big) Jx_n, x_n - x_0\rangle \ge 0.$$

Since the operator C satisfies condition $(S_+)_L$, we obtain from (4.10) and (4.11) that

$$x_n \to x_0, \quad x_0 \in D(C), \quad \text{and} \quad Cx_0 + w^* - q^* + \tilde{t}_0 j^* = 0.$$

Since the sum $T+S$ is maximal monotone and J is continuous, we have

$$x_0 \in D(T+S), \qquad w^* \in (T+S)x_0, \qquad \text{and} \qquad j^* = Jx_0.$$

Consequently, we get

$$q^* \in Tx_0 + Sx_0 + Cx_0 + \tilde{t}_0 Jx_0 \quad \text{and} \quad x_0 \in D(T) \cap L \cap \partial\Omega,$$

which contradicts the hypothesis (4.1) with $\varepsilon = \tilde{t}_0$. Therefore, assertion (4.8) is true for all large n, that is, $0 \notin H_{2,n}(t,\cdot)(D(T)\cap L\cap\partial\Omega)$ for all $t \in [0,1]$.
Recall that the duality operator $J : X \to X^*$ is bounded, continuous, strictly monotone and satisfies condition (S_+), and $\langle Jx, x\rangle = \phi(\|x\|)$ for all $x \in X$, where $\phi(t) := t^2$. Moreover, the operator $\hat{C} : D(\hat{C}) \subset X \to X^*$, defined by $\hat{C}x := Cx - q^*$, satisfies condition $(S_+)_L$ and other conditions with $\hat{c}(F,v)(x) := \langle \hat{C}x, v\rangle$ for $x \in F$ and $\langle \hat{C}x, x\rangle \ge -\hat{\psi}(\|x\|)$ for $x \in D(\hat{C})$, where $\hat{\psi}(t) := (1+\|q^*\|)\max\{\psi(t), t\}$. Taking (4.8) into account, Theorem 3.7(c) implies that

$$\deg(H_{2,n}(1,\cdot), \Omega, 0) \;=\; \deg(H_{2,n}(0,\cdot), \Omega, 0) \qquad \text{for all large } n. \tag{4.12}$$

Combining (4.7) with (4.12), we get by Theorem 3.7(a),(b)

$$\begin{aligned}\deg(T+S+C+\frac{1}{n}J - p^*, \Omega, 0) &= \deg(T+S+C+\frac{1}{n}J - q^*, \Omega, 0) \\ &= \deg(J, \Omega, 0) \;=\; 1,\end{aligned}$$

which implies that the inclusion $p^* \in Tx + Sx + Cx + (1/n)Jx$ has a solution x_n in $D(T) \cap L \cap \Omega$. Thus, for all large n, we have

$$v_n^* + Sx_n + Cx_n + \frac{1}{n}Jx_n = p^*, \tag{4.13}$$

where $v_n^* \in Tx_n$. From $\langle v_n^* + Sx_n, x_n\rangle \le \psi(\|x_n\|) + \|p^*\|\,\|x_n\|$, we obtain that the sequence $\{v_n^* + Sx_n\}$ is bounded in X^*. Without loss of generality, we may suppose that $x_n \rightharpoonup x_0$ and $v_n^* + Sx_n \rightharpoonup w_0^*$ for some $x_0 \in X$ and some $w_0^* \in X^*$. Then we have by (4.13) $Cx_n \rightharpoonup -w_0^* + p^*$ and so $\lim_{n\to\infty}\langle Cx_n + w_0^* - p^*, y\rangle = 0$ for all $y \in L\{F_n\}$. As before, we get

$$\limsup_{n\to\infty} \langle Cx_n + w_0^* - p^*, x_n\rangle \le 0.$$

A similar argument establishes that $x_n \to x_0 \in D(T) \cap L \cap \overline{\Omega}$, $w_0^* \in Tx_0 + Sx_0$, and $Cx_0 + w_0^* - p^* = 0$ and hence

$$p^* \in Tx_0 + Sx_0 + Cx_0.$$

From the second of (4.1) with $\varepsilon = 0$, it is clear that $x_0 \in D(T) \cap L \cap \Omega$. Since p^* was arbitrary in M, we conclude that

$$M \subset (T + S + C)(D(T) \cap L \cap \Omega).$$

This completes the proof. □

Recall that an operator $A : D(A) \subset X \to 2^{X^*}$ is said to be *locally injective* on a set $\Omega \subset X$ if for every $x \in D(A) \cap \Omega$, there exists an open ball $B_r(x)$ in X such that A is injective on $D(A) \cap \overline{B_r(x)}$. If $\Omega = X$, we simply say that A is locally injective. As a consequence of Theorem 4.1, we obtain an invariance of domain result for $(S_+)_L$-perturbations of strongly quasibounded maximal monotone operators.

Corollary 4.2. *Let L be a dense subspace of X. Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : L \to X^*$ satisfies (s1) and (s2), and $C : D(C) \subset X \to X^*$ satisfies (c1), (c2), and (c3) with $L \subset D(C)$ and $C(0) = 0$. Let Ω be a bounded open subset of X. If $T + S + C + \varepsilon J$ is injective on $D(T) \cap L \cap \overline{\Omega}$ for every $\varepsilon > 0$ and $T + S + C$ is locally injective on Ω, then the set $(T + S + C)(D(T) \cap L \cap \Omega)$ is open in X^*.*

Proof. Let $p^* \in (T + S + C)(D(T) \cap L \cap \Omega)$ be arbitrary. Then we have

$$p^* = v^* + Sz + Cz \qquad \text{for some } z \in D(T) \cap L \cap \Omega \text{ and some } v^* \in Tz.$$

For simplicity, we may suppose that $z = 0$, $v^* = 0$, and $p^* = 0$. It is possible if we consider the sets $D(\widetilde{T}) \equiv D(T) - z$, $D(\widetilde{C}) \equiv D(C) - z$, and $\widetilde{\Omega} \equiv \Omega - z$, and the operators $\widetilde{T}x \equiv T(x + z) - Tz$, $\widetilde{S}x \equiv S(x + z) - Sz$, and $\widetilde{C}x \equiv C(x + z) - Cz$. Since $T + S + C$ is locally injective on the open set Ω, there exists an open ball $B_s(0)$ in X such that $\overline{B_s(0)} \subset \Omega$ and $T + S + C$ is injective on $D(T) \cap L \cap \overline{B_s(0)}$. To apply Theorem 4.1, we have to show that there is a positive number r such that

$$[(T + S + C + \varepsilon J)(D(T) \cap L \cap \partial B_s(0))] \cap B_r(0) = \emptyset \qquad \text{for every } \varepsilon \ge 0, \tag{4.14}$$

where $B_r(0) \subset X^*$. Assume on the contrary that for a sequence $\{r_n\}$ in $(0, \infty)$ with $r_n \to 0$, there exist the corresponding sequences $\{\varepsilon_n\}$ in $[0, \infty)$, $\{p_n^*\}$ in X^* with $p_n^* \in B_{r_n}(0)$, $\{x_n\}$ in $D(T) \cap L \cap \partial B_s(0)$, and $\{v_n^*\}$ in X^* with $v_n^* \in Tx_n$ such that

$$v_n^* + Sx_n + Cx_n + \varepsilon_n Jx_n = p_n^* \qquad \text{for each } n \in \mathbb{N}. \tag{4.15}$$

Then $\{x_n\}$ is bounded, $p_n^* \to 0$, and $\langle v_n^* + Sx_n, x_n\rangle \le \psi(s) + s\|p_n^*\|$. By the strong quasiboundedness of the operator $T+S$, the sequence $\{v_n^* + Sx_n\}$ is bounded in X^*. We may suppose without loss of generality that $\varepsilon_n \to \varepsilon_0$, $x_n \rightharpoonup x_0 \in X$, $v_n^* + Sx_n \rightharpoonup w^* \in X^*$, and $Jx_n \rightharpoonup j^* \in X^*$. Note that $\varepsilon_0 \in [0,\infty)$. Indeed, if $\varepsilon_0 = \infty$, then the monotonicity of $T+S$ implies that

$$\varepsilon_n s^2 = \varepsilon_n \|x_n\|^2 \le \langle v_n^* + Sx_n, x_n\rangle + \varepsilon_n\langle Jx_n, x_n\rangle \le \psi(s) + s\|p_n^*\| < K$$

for some positive constant K, which yields a contradiction.
Since we have by (4.15) $Cx_n \rightharpoonup -w^* - \varepsilon_0 j^*$ and the operator C satisfies condition $(S_+)_L$, a standard argument proves that $x_n \to x_0 \in D(C)$ and $Cx_0 + w^* + \varepsilon_0 j^* = 0$. Since the operator $T+S$ is maximal monotone and J is continuous, we obtain that $x_0 \in D(T+S)$, $w^* \in (T+S)x_0$, and $j^* = Jx_0$ and hence $x_0 \in D(T) \cap L \cap \partial B_s(0)$ and $0 \in Tx_0 + Sx_0 + Cx_0 + \varepsilon_0 Jx_0$. On the other hand, we have $0 \in (T+S+C+\varepsilon_0 J)(D(T) \cap L \cap B_s(0))$. This contradicts the injectivity of the operator $T+S+C+\varepsilon_0 J$ on the set $D(T)\cap L \cap \overline{B_s(0)}$. Thus, assertion (4.14) holds. Applying Theorem 4.1 with $\Omega = B_s(0)$ and $M = B_r(0)$, we have

$$B_r(0) \subset (T+S+C)(D(T) \cap L \cap B_s(0)).$$

Hence it follows from $B_s(0) \subset \Omega$ that the set $(T+S+C)(D(T)\cap L\cap\Omega)$ is open in X^*. This completes the proof. □

Remark 4.3. When $T \equiv 0$ and C is strongly quasibounded with respect to S and satisfies a generalized (S_+)-condition, Theorem 4.1 and Corollary 4.2 were studied in [13, Theorem 7], with the aid of the degree theory for densely defined maps involving operators of type (S_+) in [11]. For the case where $S \equiv 0$ and C is strongly quasibounded and generalized pseudomonotone, we refer to Theorem 6.3 of [12].

From Corollary 4.2, we get a surjectivity result on the sum $T+S+C$ under a weak coercivity condition.

Corollary 4.4. *Let L, T, S, and C be as in Corollary 4.2. Suppose that $T+S+C+\varepsilon J$ is injective on $D(T)\cap L$ for every $\varepsilon > 0$ and $T+S+C$ is locally injective. If $T+S+C$ is weakly coercive, that is,*

$$\inf\{\|v^* + Sx + Cx\| : v^* \in Tx\} \to \infty \quad \text{as } \|x\| \to \infty,\ x \in D(T)\cap L,$$

then the operator $T+S+C$ is surjective.

Proof. Applying Corollary 4.2 with $\Omega = X$, the set $(T+S+C)(D(T)\cap L)$ is nonempty and open in X^*. To prove the surjectivity of the operator $T+S+C$, it is sufficient to verify that this set is closed in the connected space X^*. Let $\{x_n\}$ be any sequence in $D(T)\cap L$ such that

$$\lim_{n\to\infty} v_n^* + Sx_n + Cx_n = y^* \qquad \text{for some } y^* \in X^*, \tag{4.16}$$

where $v_n^* \in Tx_n$. Then the sequence $\{x_n\}$ is bounded in X. In fact, assume on the contrary that there is a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that $\lim_{n\to\infty}\|x_n\| = \infty$. By the weak coercivity of the operator $T+S+C$, we get

$$\lim_{n\to\infty} \|v_n^* + Sx_n + Cx_n\| = \infty,$$

which contradicts (4.16). Set $y_n^* := v_n^* + Sx_n + Cx_n$. Since $\{x_n\}$ is thus bounded in X and

$$\langle v_n^* + Sx_n, x_n\rangle = -\langle Cx_n, x_n\rangle + \langle y_n^*, x_n\rangle \leq \psi(\|x_n\|) + \|y_n^*\| \, \|x_n\|,$$

the strong quasiboundedness of $T + S$ implies that the sequence $\{v_n^* + Sx_n\}$ is bounded in X^*. So we may suppose that $x_n \rightharpoonup x_0$ and $v_n^* + Sx_n \rightharpoonup w^*$ for some $x_0 \in X$ and some $w^* \in X^*$. Since $Cx_n \rightharpoonup y^* - w^*$, the operator C satisfies condition $(S_+)_L$, and the operator $T + S$ is maximal monotone, we can show that

$$x_n \to x_0, \quad x_0 \in D(C) \cap D(T + S), \quad w^* \in (T + S)x_0, \quad \text{and} \quad Cx_0 - y^* + w^* = 0.$$

From $y^* \in Tx_0 + Sx_0 + Cx_0 \subset (T + S + C)(D(T) \cap L)$, it is clear that the set $(T + S + C)(D(T) \cap L)$ is closed in X^*. We conclude that the operator $T + S + C$ is surjective. This completes the proof. □

We close this section by illustrating the above results by the following simple example.

Example 4.5. Let G be a bounded open set in $\mathbb{R}^N$ and let $2 \leq p < \infty$. Set $X = W_0^{1,p}(G)$. Let $S, C : X \to X^*$ be two operators setting by

$$\langle Su, \varphi\rangle = \int_G |u|^{p-2} u\varphi \, dx,$$

$$\langle Cu, \varphi\rangle = \sum_{i=1}^{N} \int_G \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx.$$

Then it is obvious that the operator S is completely continuous, monotone, $S(0) = 0$ and the operator C is bounded, continuous, uniformly monotone, $C(0) = 0$, and satisfies condition (S_+) on X; see [15, Theorem 2.2] and [18, Proposition 26.10]. In particular, the operator S is maximal monotone, bounded and the operator C is injective and coercive. If we take $T = \partial\chi$ in Section 2, then Corollaries 4.2 and 4.4 apply, on observing that the operator $T + S + C + \varepsilon J$ is injective on $D(T)$ for every $\varepsilon \geq 0$ and the operator $T + S + C$ is (weakly) coercive.

Remark 4.6. In [11], we see that condition $(S_+)_{0,L}$ on the whole abstract operator must be verified to solve a given differential equation; see Example 3.9. In our approach, each term of the possible whole operator is at first characterized. In many cases, it is more convenient when applying.

5. Zeros

In this section, we deal with the solvability for a nonlinear inclusion of the form $T + S + C$, based on the degree theory stated in Section 3.

We now establish the existence of zeros for the inclusion with a regularization method which is a generalization of Theorem 4 in [10].

Theorem 5.1. *Let L be a dense subspace of X. Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : L \to X^*$ satisfies (s1) and (s2), and $C : D(C) \subset X \to X^*$ satisfies (c1), (c2), and (c3) with $L \subset D(C)$. Let Ω be a bounded open subset of X with $0 \in \Omega$. Let Λ be a positive number with $\Lambda > Q$, where $Q_1 := \inf\{\|x\| : x \in \partial\Omega\}$, $Q_2 := \psi(\sup\{\|x\| : x \in \partial\Omega\})$ and $Q := Q_2/Q_1^2$. If*

$$0 \notin Tx + Sx + Cx + \lambda Jx \qquad \text{for all } (\lambda, x) \in (0, \Lambda) \times (D(T) \cap L \cap \partial\Omega), \tag{5.1}$$

then the inclusion

$$0 \in Tx + Sx + Cx$$

has a solution x in $D(T) \cap L \cap \overline{\Omega}$. If, in addition, (5.1) holds for $\lambda = 0$, then the solution x belongs to the set $D(T) \cap L \cap \Omega$.

Proof. Let ε_0 be a positive number such that $Q + \varepsilon_0 < \Lambda$. Fix $\varepsilon \in (0, \varepsilon_0]$ and consider a homotopy given by

$$H(t, x) := t(Tx + Sx + Cx + \varepsilon Jx) + (1 - t)Jx.$$

First, we will prove that the inclusion $0 \in H(t, x)$ has no solution in $D(T) \cap L \cap \partial\Omega$ for every $t \in [0, 1]$. Assume that the assertion is false. Then there exist sequences $\{t_n\}$ in $[0, 1]$, $\{x_n\}$ in $D(T) \cap L \cap \partial\Omega$, and $\{v_n^*\}$ in X^* with $v_n^* \in Tx_n$ such that

$$t_n(v_n^* + Sx_n + Cx_n + \varepsilon Jx_n) + (1 - t_n)Jx_n = 0. \tag{5.2}$$

We may suppose that $t_n \to t_0$ and $x_n \rightharpoonup x_0$ for some $t_0 \in [0, 1]$ and some $x_0 \in X$. Then we have $t_n \in (0, 1]$ for all $n \in \mathbb{N}$ and the limit t_0 belongs to $(0, 1]$. Indeed, if $t_0 = 0$, then (5.2), (c3), and the monotonicity of the operator $T + S + \varepsilon J$ imply that

$$\begin{aligned}(1 - t_n)Q_1^2 &\le (1 - t_n)\|x_n\|^2 \le t_n\langle v_n^* + Sx_n + \varepsilon Jx_n, x_n\rangle + (1 - t_n)\langle Jx_n, x_n\rangle \\ &= -t_n\langle Cx_n, x_n\rangle \le t_n Q_2\end{aligned} \tag{5.3}$$

and so $Q_1 = 0$, which is a contradiction. Since the sum $T + S$ is strongly quasi-bounded and we have

$$\langle v_n^* + Sx_n, x_n\rangle = -\langle Cx_n, x_n\rangle - \Big(\varepsilon + \frac{1 - t_n}{t_n}\Big)\langle Jx_n, x_n\rangle \le Q_2,$$

the sequence $\{v_n^* + Sx_n\}$ is clearly bounded in X^*. We may suppose without loss of generality that $v_n^* + Sx_n \rightharpoonup w^*$ and $Jx_n \rightharpoonup j^*$ for some $w^*, j^* \in X^*$. Set $\tilde{t}_0 := (1 - t_0)/t_0$. From $Cx_n \rightharpoonup -w^* - (\varepsilon + \tilde{t}_0)j^*$, it is obvious that

$$\lim_{n\to\infty}\langle Cx_n + w^* + (\varepsilon + \tilde{t}_0)j^*, y\rangle = 0 \qquad \text{for all } y \in L\{F_n\}. \tag{5.4}$$

In view of Lemma 2.4(a), we get

$$\liminf_{n\to\infty}\langle v_n^* + Sx_n + \Big(\varepsilon + \frac{1 - t_n}{t_n}\Big)Jx_n, x_n - x_0\rangle \ge 0,$$

which implies along with (5.2)

$$\limsup_{n\to\infty}\langle Cx_n + w^* + (\varepsilon + \tilde{t}_0)j^*, x_n\rangle \le 0. \tag{5.5}$$

Since the operator C satisfies condition $(S_+)_L$, it follows from (5.4) and (5.5) that

$$x_n \to x_0, \quad x_0 \in D(C), \quad \text{and} \quad Cx_0 + w^* + (\varepsilon + \tilde{t}_0)j^* = 0.$$

Since the operator $T+S$ is maximal monotone and J is continuous, we have

$$x_0 \in D(T+S), \quad w^* \in (T+S)x_0, \quad \text{and} \quad j^* = Jx_0.$$

Therefore, we obtain

$$x_0 \in D(T) \cap L \cap \partial\Omega \quad \text{and} \quad 0 \in Tx_0 + Sx_0 + Cx_0 + (\varepsilon + \tilde{t}_0)Jx_0,$$

which contradicts the hypothesis (5.1) with $\varepsilon + \tilde{t}_0 \in (0, \Lambda)$, on observing from (5.3) that $\tilde{t}_0 = \lim_{n\to\infty}(1-t_n)/t_n \le Q$. Thus, we have shown that $0 \notin H(t,\cdot)(D(T) \cap L \cap \partial\Omega)$ for all $t \in [0,1]$.
Note that the operator $\hat{T}_\varepsilon := T + \varepsilon J$ is maximal monotone by Lemma 2.1, strongly quasibounded, and $0 \in \hat{T}_\varepsilon(0)$. Using some properties of the degree stated in Theorem 3.7, we obtain that

$$\deg(T+S+C+\varepsilon J, \Omega, 0) = \deg(J, \Omega, 0) = 1$$

and hence the inclusion

$$0 \in Tx + Sx + Cx + \varepsilon Jx$$

has a solution x_ε in $D(T) \cap L \cap \Omega$.
For a sequence $\{\varepsilon_n\}$ in $(0, \varepsilon_0]$ with $\varepsilon_n \to 0$, let $\{x_{\varepsilon_n}\}$ be the corresponding sequence in $D(T) \cap L \cap \Omega$ such that $0 \in Tx_{\varepsilon_n} + Sx_{\varepsilon_n} + Cx_{\varepsilon_n} + \varepsilon_n Jx_{\varepsilon_n}$. Setting $x_n := x_{\varepsilon_n}$, it can be rewritten as

$$v_n^* + Sx_n + Cx_n + \varepsilon_n Jx_n = 0,$$

where $v_n^* \in Tx_n$. Since $T+S$ is strongly quasibounded, it follows from $\langle v_n^* + Sx_n, x_n\rangle \le \psi(\|x_n\|)$ that the sequence $\{v_n^* + Sx_n\}$ is bounded in X^*. We may suppose that $x_n \rightharpoonup x_0$ and $v_n^* + Sx_n \rightharpoonup w^*$ for some $x_0 \in X$ and some $w^* \in X^*$. As $Cx_n \rightharpoonup -w^*$, we can show as above that

$$\limsup_{n\to\infty} \langle Cx_n + w^*, x_n\rangle \le 0 \quad \text{and} \quad \lim_{n\to\infty} \langle Cx_n + w^*, y\rangle = 0 \quad \text{for all } y \in L\{F_n\}.$$

Since C satisfies condition $(S_+)_L$ and $T+S$ is maximal monotone, we obtain that

$$x_n \to x_0, \quad x_0 \in D(C) \cap D(T+S), \quad Cx_0 + w^* = 0, \quad \text{and} \quad w^* \in (T+S)x_0.$$

We conclude that $x_0 \in D(T) \cap L \cap \overline{\Omega}$ and $0 \in Tx_0 + Sx_0 + Cx_0$. If, in addition, hypothesis (5.1) holds for $\lambda = 0$, then it is trivial that the limit x_0 belongs to the open set Ω. This completes the proof. □

From Theorem 5.1, we deduce a surjectivity result under a coercivity condition.

Corollary 5.2. *Let L be a dense subspace of X. Suppose that $T : D(T) \subset X \to 2^{X^*}$ satisfies (t1), $S : L \to X^*$ satisfies (s1) and (s2), and $C : D(C) \subset X \to X^*$ satisfies (c1), (c2), and (c3) with $L \subset D(C)$. If the following coercivity condition*

$$\lim_{\substack{\|x\|\to\infty \\ x\in D(T)\cap L,\, v^*\in Tx}} \frac{\langle v^* + Sx + Cx, x\rangle}{\|x\|} = +\infty$$

holds, then the operator $T+S+C$ is surjective.

Proof. Let $h^* \in X^*$ be arbitrary but fixed. By the coercivity condition, there exists an open ball $B_s(0)$ in X such that

$$\frac{\langle v^* + Sx + Cx, x\rangle}{\|x\|} > \|h^*\|$$

for all $x \in D(T) \cap L \cap \partial B_s(0)$ and all $v^* \in Tx$, which implies

$$\langle v^* + Sx + Cx + \lambda Jx - h^*, x\rangle \;\geq\; \langle v^* + Sx + Cx, x\rangle - \|h^*\|\,\|x\| \;>\; 0$$

for all $\lambda \in [0, \infty)$. Note that

$$0 \notin Tx + Sx + Cx + \lambda Jx - h^* \qquad \text{for all } (\lambda, x) \in [0, \infty) \times (D(T) \cap L \cap \partial B_s(0))$$

and the operator $\hat{C} : D(\hat{C}) \subset X \to X^*$, defined by $\hat{C}x := Cx - h^*$, satisfies condition $(S_+)_L$ and other conditions, as we observed in the proof of Theorem 4.1. Applying Theorem 5.1 with $C = \hat{C}$ and $\Omega = B_s(0)$, the inclusion $h^* \in Tx + Sx + Cx$ has a solution in $D(T) \cap L \cap B_s(0)$. Therefore, the operator $T + S + C$ is surjective. This completes the proof. □

Definition 5.3. Let $C : D(C) \subset X \to X^*$ be a single-valued operator with $L \subset D(C)$. We say that the operator C satisfies condition $(S_+)_{0,D(C)}$ if for every sequence $\{F_n\}$ in $\mathcal{F}(L)$ satisfying (3.1) and for every sequence $\{x_n\}$ in $D(C)$ with

$$x_n \rightharpoonup x_0, \quad \limsup_{n\to\infty} \langle Cx_n, x_n\rangle \leq 0, \quad \text{and} \quad \lim_{n\to\infty} \langle Cx_n, y\rangle = 0 \quad \text{for all } y \in L\{F_n\},$$

we have $x_n \to x_0$, $x_0 \in D(C)$, and $Cx_0 = 0$. We say that the operator C satisfies condition $(S_+)_{D(C)}$ if the operator $C_h : D(C) \to X^*$, defined by $C_h x := Cx - h$, satisfies condition $(S_+)_{0,D(C)}$ for every $h \in X^*$.

From Definitions 3.1 and 5.3 and Example 3.2 of [3], it is obvious that $(S_+)_{D(C)}$ is a proper subclass of $(S_+)_L$. We give an analogue of Lemma 3.2 about $(S_+)_{D(C)}$, as a slight modification of Theorem 3.3 in [3].

Lemma 5.4. *Let L be a dense subspace of X. Then the following relations hold:*

(a) *If $C : D(C) = X \to X^*$ is a strongly quasibounded demicontinuous operator that satisfies condition (S_+) on $D(C)$, then the operator C satisfies condition $(S_+)_{D(C)}$.*

(b) *If $C : D(C) \subset X \to X^*$ is bounded with $L \subset D(C)$ and satisfies condition $(S_+)_{D(C)}$, then $D(C) = X$ and $C : X \to X^*$ is demicontinuous and satisfies condition (S_+) on X.*

Remark 5.5. When $S \equiv 0$, Theorem 5.1 and Corollary 5.2 reduce to Theorem 4 and Corollary 1 of [10], respectively. However, stronger condition $(S_+)_{D(C)}$ on the operator C was required in [10].

Remark 5.6. If the operators T, S, C are the same as in Example 4.5, then Theorem 5.1 and Corollary 5.2 also apply, on observing from the strict monotonicity of the operator C that $0 \notin Tx + Sx + Cx + \lambda Jx$ for all $(\lambda, x) \in [0, \infty) \times (D(T) \cap L \cap \partial\Omega)$.

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In-Sook Kim
Department of Mathematics, Sungkyunkwan University, Suwon 440–746, Korea
E-mail address: iskim@skku.edu