Journal of Nonlinear and Convex Analysis
Volume 16, Number 8, 2015, 1583–1591

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

DISCUSSION ON THE EQUIVALENCE OF W-DISTANCES WITH Ω-DISTANCES

ANTONIO-FRANCISCO ROLDÁN-LÓPEZ-DE-HIERRO AND ERDAL KARAPINAR

Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. In this manuscript, we study some relationships between w-distances on metric spaces and Ω-distances on G^*-metric spaces. Concretely we show that the class of all w-distances on metric spaces is a subclass of all Ω-distances on G^*-metric spaces. Then, researchers must be careful because some recent results about w-distances (for instances, in the topic of fixed point theory) can be seen as simple consequences of their corresponding results about Ω-distances. In this sense, we show how to translate some results between different metric models.

1. Introduction

The notion of metric plays a key role in nonlinear sciences. This concept has been generalized in various directions to get finer results in the related research areas. Some of them are the following ones: quasi-metrics, partial metrics, G-metrics, b-metrics, fuzzy metrics and probabilistic metrics. Among all, the notion of G-metric, introduced by Mustafa and Sims [12] have attracted attention of number of authors in the last decade. Recently, Samet *et al.* [15] and Jleli and Samet [7] proved that the topology of G-metric space and associated metric space coincides. Very recently, An *et al.* [1] re-proved the equivalence of G-metric topology with corresponding metric topology by repeating the same arguments as in [7, 15]. On the other hand, Kada *et al.* [8] introduced the concept of w-distance associated to a metric space and proved the existence and uniqueness of certain mappings in the setting of w-distance. Later, the notion of Ω-distance introduced by Saadati *et al.* [14]) as a natural generalization of w-distance in the context of G-metric space. In fact, this generalization is weak since the authors [14] use the rectangular inequality in the definition of Ω-distance although in corresponding part of the w-distance definition, the triangular inequality was not used. The literature on these topics has grown up very quickly (see e.g. [3, 4, 14, 16]).

After remarkable observations in [7, 15], it is quite natural to investigate the relation between w-distance and Ω-distance. Unexpectedly, we get there is a close relation.

In this paper, for our purposes, we will consider Ω-distances defined on a weak kind of spaces, that is, on G^*-metric spaces, firstly introduced by Roldán *et al* [13]. In this setting, we show that the class of all w-distances on metric spaces is a

---

2010 *Mathematics Subject Classification.* 54H25, 46T99, 47H10, 47H09.

Key words and phrases. w-distance, Ω-distance, relationships between different models, fixed point.

subclass of all Ω-distances on G^*-metric spaces. This paper can be considered as a continuation of [1,2,3,4].

2. Preliminaries

In this section, we recollect some fundamental definitions and basic results. Throughout the paper, for a non-empty set X, let X^2 be the product space $X \times X$ and $X^3 = X \times X \times X$.

Definition 2.1 (Mustafa and Sims [12])**.** A generalized metric (or a G-metric) on X is a mapping $G : X^3 \to [0, \infty)$ verifying, for all $x, y, z \in X$:

(G_1) $G(x, x, x) = 0$.
(G_2) $G(x, x, y) > 0$ if $x \neq y$.
(G_3) $G(x, x, y) \leq G(x, y, z)$ if $y \neq z$.
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables).
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (*rectangle inequality*).

Taking into account that the product space of G-metric spaces need not be a G-metric space, Roldán *et al.* introduced the following notion.

Definition 2.2 (Roldán and Karapınar [13])**.** A G^*-metric on X is a mapping $G : X^3 \to [0, \infty)$ verifying (G_1), (G_2), (G_4) and (G_5).

A G^*-metric on a set X lets us to consider a Hausdorff topology τ_G on X (see [13]).

Lemma 2.3. *If (X, d) is a metric space and we define, for all $x, y, z \in X$, $G_d(x, y, z) = \max(d(x, y), d(x, z), d(y, z))$, then G_d is a G-metric on X (and also a G^*-metric).*

Conversely, if (X, G) is a G^*-metric space and we define $d_G(x, y) = \max(G(x, y, y), G(y, x, x))$ for all $x, y \in X$, then d_G is a metric on X.

As we shall use henceforth, the following definition can also be considered in G^*-metric spaces.

Definition 2.4 (R. Saadati *et al.* [14])**.** Let (X, G) be a G-metric space. Then a function $\Omega : X^3 \to [0, \infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$;
(b) for any $x, y \in X$, $\Omega(x, y, \cdot), \Omega(x, \cdot, y) : X \to [0, \infty)$ are lower semi-continuous;
(c) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \varepsilon$.

The notion of w-distance was introduced by Kada *et al.* in [8].

Definition 2.5 (Kada *et al.* [8])**.** Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following properties:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(2) p is lower semi-continuous in its second variable, i.e., if $x \in X$ and $\{y_n\} \to y \in X$, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$;
(3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$, then $d(x, y) \leq \varepsilon$.

3. Relationships

A first approach to the problem of finding relationships between w-distances and Ω-distances is the following result.

Lemma 3.1. *Let Ω be a Ω-distance on a G^*-metric space (X, G) and define $q_\Omega : X^2 \to [0, \infty)$ by*

$$q_\Omega(x, y) = \Omega(x, y, y) + \Omega(y, x, x) \qquad \text{for all } x, y \in X.$$

Then q_Ω is a w-distance on the metric space (X, d_G^s), where

$$d_G^s(x, y) = G(x, x, y) + G(y, x, x) \qquad \text{for all } x, y \in X.$$

Proof. We prove three properties.

(a) For all $x, y \in X$ we have that

$$\begin{aligned} q_\Omega(x, z) &= \Omega(x, z, z) + \Omega(z, x, x) \\ &\le \Omega(x, y, y) + \Omega(y, z, z) + \Omega(z, y, y) + \Omega(y, x, x) \\ &= [\Omega(x, y, y) + \Omega(y, x, x)] + [\Omega(y, z, z) + \Omega(z, y, y)] \\ &= q_\Omega(x, y) + q_\Omega(y, z). \end{aligned}$$

(b) The result follows Definition 2.4 (b).

(c) Let $\varepsilon > 0$ and let $\delta > 0$ such that

$$\left.\begin{array}{l} \Omega(x, a, a) \le \delta \\ \Omega(a, y, z) \le \delta \end{array}\right\} \Rightarrow G(x, y, z) \le \frac{\varepsilon}{2}.$$

Now, let $x, y, z \in X$ be such that $q_\Omega(z, x) \le \delta$ and $q_\Omega(z, y) \le \delta$. Then

$$\left.\begin{array}{l} \Omega(x, z, z) \le \Omega(z, x, x) + \Omega(x, z, z) = q_\Omega(z, x) \le \delta \\ \Omega(z, y, y) \le \Omega(z, y, y) + \Omega(y, z, z) = q_\Omega(z, y) \le \delta \end{array}\right\} \Rightarrow G(x, y, y) \le \frac{\varepsilon}{2},$$

$$\left.\begin{array}{l} \Omega(y, z, z) \le \Omega(z, y, y) + \Omega(y, z, z) = q_\Omega(z, y) \le \delta \\ \Omega(z, x, x) \le \Omega(z, x, x) + \Omega(x, z, z) = q_\Omega(z, x) \le \delta \end{array}\right\} \Rightarrow G(y, x, x) \le \frac{\varepsilon}{2}.$$

Therefore $d_G^s(x, y) = G(x, x, y) + G(y, x, x) \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

The last result has two drawbacks. Firstly, it is only possible to generate symmetric w-distances. Furthermore, it is impossible to get the original Ω-distance using q_Ω since different Ω-distances induce the same w-distance. For instance, if $X = [0, \infty)$ provided with the G-metric

$$G(x, y, z) = \max(|x - y|, |x - z|, |y - z|) \quad \text{for all } x, y, z \in X,$$

then the mappings $\Omega_1(x, y, z) = x+2y+3z$ and $\Omega_2(x, y, z) = 2x+2y+2z$, defined for all $x, y, z \in X$, are Ω-distances on (X, G). However, $q_{\Omega_1}(x, y) = q_{\Omega_2}(x, y) = 6x + 6y$ for all $x, y \in X$.

To overcome these drawbacks, we present the following results.

Theorem 3.2. *Let p be a w-distance on a metric space (X, d) and define $\Omega_p : X^3 \to [0, \infty)$ by*

$$\Omega_p(x, y, z) = p(x, x) + p(x, y) + p(x, z) \qquad \text{for all } x, y, z \in X.$$

Then Ω_p is a Ω-distance on (X, G_d). Moreover, for all $x, y, z, a \in X$, the following properties hold.

(P_1) $\dfrac{2}{3}\,\Omega_p(x,x,x) \leq \Omega_p(x,x,y)$.

(P_2) $\Omega_p(x,y,z) \leq \Omega_p(x,a,a) + \Omega_p(a,y,z) - \dfrac{1}{3}\,\Omega_p(a,a,a)$.

(P_3) $\Omega_p(x,x,y) - \dfrac{2}{3}\,\Omega_p(x,x,x) = \dfrac{1}{2}\left[\,\Omega_p(x,y,y) - \dfrac{1}{3}\,\Omega_p(x,x,x)\,\right]$.

(P_4) *For all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left.\begin{array}{l} \Omega_p(z,z,x) - \dfrac{2}{3}\,\Omega_p(z,z,z) \leq \delta \\ \Omega_p(z,z,y) - \dfrac{2}{3}\,\Omega_p(z,z,z) \leq \delta \end{array}\right\} \Rightarrow\ G_d(x,x,y) \leq \varepsilon.$$

(P_5) $\Omega_p(x,y,z) = \Omega_p(x,x,y) + \Omega_p(x,x,z) - \Omega_p(x,x,x)$.

(P_6) $\Omega_p(x,y,z) = \Omega_p(x,z,y)$.

(P_7) $\Omega_p(x,y,y) + \Omega_p(x,z,z) = 2\Omega_p(x,y,z)$.

In particular, for all $x, y, z \in X$,

$$p(x,y) = \Omega_p(x,x,y) - \frac{2}{3}\,\Omega_p(x,x,x) = \frac{1}{2}\left[\,\Omega_p(x,y,y) - \frac{1}{3}\,\Omega_p(x,x,x)\,\right] \tag{3.1}$$

Proof. Clearly, $\Omega_p(x,y,z) \geq 0$ for all $x, y, z \in X$. First, we prove the three properties that define a Ω-distance.

(a) Concretely, we prove (P_2). Applying (1), we have that

$$\begin{aligned} \Omega_p(x,y,z) &= p(x,x) + p(x,y) + p(x,z) \\ &\leq p(x,x) + p(x,a) + p(a,y) + p(x,a) + p(a,z) \\ &\leq [p(x,x) + p(x,a) + p(x,a)] + [p(a,a) + p(a,y) + p(a,z)] - p(a,a) \\ &= \Omega_p(x,a,a) + \Omega_p(a,y,z) - \frac{1}{3}\,\Omega_p(a,a,a). \end{aligned}$$

In particular, $\Omega_p(x,y,z) \leq \Omega_p(x,a,a) + \Omega_p(a,y,z)$.

(b) Given $x, y, z \in X$, the mappings $\Omega_p(x,\cdot,z) = p(x,x) + p(x,\cdot) + p(x,z)$ and $\Omega_p(x,y,\cdot) = p(x,x) + p(x,y) + p(x,\cdot)$ are lower semi-continuous since p is lower semi-continuous in its second variable.

(c) Fix $\varepsilon > 0$ arbitrary. Applying (3) to $\varepsilon/2$, there exists $\delta > 0$ such that

$$\left.\begin{array}{l} p(z,x) \leq \delta \\ p(z,y) \leq \delta \end{array}\right\} \Rightarrow\ d(x,y) \leq \frac{\varepsilon}{2}.$$

Let $x, y, z, a \in X$ verifying $\Omega_p(x,a,a) \leq \delta$ and $\Omega_p(a,y,z) \leq \delta$. Then

$$\left.\begin{array}{l} p(x,x) \leq p(x,x) + p(x,a) + p(x,a) = \Omega_p(x,a,a) \leq \delta \\ p(x,a) \leq p(x,x) + p(x,a) + p(x,a) = \Omega_p(x,a,a) \leq \delta \end{array}\right\} \Rightarrow\ d(x,a) \leq \frac{\varepsilon}{2};$$

$$\left.\begin{array}{l} p(a,a) \leq p(a,a) + p(a,y) + p(a,z) = \Omega_p(a,y,z) \leq \delta \\ p(a,y) \leq p(a,a) + p(a,y) + p(a,z) = \Omega_p(a,y,z) \leq \delta \end{array}\right\} \Rightarrow\ d(a,y) \leq \frac{\varepsilon}{2};$$

$$\left.\begin{array}{l} p(a,a) \leq p(a,a)+p(a,y)+p(a,z)=\Omega_p(a,y,z) \leq \delta \\ p(a,z) \leq p(a,a)+p(a,y)+p(a,z)=\Omega_p(a,y,z) \leq \delta \end{array}\right\} \Rightarrow\ d(a,z) \leq \frac{\varepsilon}{2};$$

$$\left.\begin{array}{l} p(a,y) \leq p(a,a)+p(a,y)+p(a,z)=\Omega_p(a,y,z) \leq \delta \\ p(a,z) \leq p(a,a)+p(a,y)+p(a,z)=\Omega_p(a,y,z) \leq \delta \end{array}\right\} \Rightarrow\ d(y,z) \leq \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} d(x,y) &\leq d(x,a)+d(a,y) \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad \text{and} \\ d(x,z) &\leq d(x,a)+d(a,z) \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

Hence

$$G_d(x,y,z)=\max(d(x,y),d(x,z),d(y,z)) \leq \max(\varepsilon,\varepsilon,\varepsilon/2)=\varepsilon.$$

We conclude that Ω_p is a Ω-distance on (X,G_d). Next, we prove all announced properties.

- (P_1) It is clear that $\Omega_p(x,x,y)-\frac{2}{3}\,\Omega_p(x,x,x)=2p(x,x)+p(x,y)-\frac{2}{3}\,3p(x,x)=p(x,y)\geq 0$.
- (P_2) Already proved.
- (P_3) On the one hand

$$\frac{1}{2}\,\Omega_p(x,y,y)-\frac{1}{6}\,\Omega_p(x,x,x)=\frac{1}{2}\,\left[p(x,x)+2p(x,y)\right]-\frac{1}{6}\,3p(x,x)=p(x,y),$$

and on the other hand

$$\Omega_p(x,x,y)-\frac{2}{3}\,\Omega_p(x,x,x)=\left[2p(x,x)+p(x,y)\right]-\frac{2}{3}\,3p(x,x)=p(x,y).$$

This also proved (3.1).

- (P_4) Notice that

$$\begin{aligned} &\Omega_p(z,z,x)-\frac{2}{3}\,\Omega_p(z,z,z)=2p(z,z)+p(z,x)-\frac{2}{3}\,3p(z,z)=p(z,x), \\ &\Omega_p(z,z,y)-\frac{2}{3}\,\Omega_p(z,z,z)=2p(z,z)+p(z,y)-\frac{2}{3}\,3p(z,z)=p(z,y), \\ &G_d(x,x,y)=\max\left(d(x,x),d(x,y)\right)=d(x,y). \end{aligned}$$

Therefore, property (P_4) is equivalent to axiom (3) of a w-distance.

- (P_5) It follows from

$$\begin{aligned} &\Omega_p(x,x,y)+\Omega_p(x,x,z)-\Omega_p(x,x,x) \\ &\quad=\left[2p(x,x)+p(x,y)\right]+\left[2p(x,x)+p(x,z)\right]-3p(x,x) \\ &\quad=p(x,x)+p(x,y)+p(x,z)=\Omega_p(x,y,z). \end{aligned}$$

- (P_6) It is obvious.
- (P_7) It is clear that

$$\begin{aligned} \Omega_p(x,y,y)+\Omega_p(x,z,z)&=\left[p(x,x)+2p(x,y)\right]+\left[p(x,x)+2p(x,z)\right] \\ &=2\left[p(x,x)+p(x,y)+p(x,z)\right]=2\Omega_p(x,y,z). \end{aligned}$$

□

Next, we study the converse of the previous result.

Theorem 3.3. *Let Ω be a Ω-distance on a G^*-metric space (X,G) verifying properties (P_1)-(P_4) of Theorem 3.2, and define $p_\Omega : X^2 \to [0,\infty)$ by*

$$p_\Omega(x,y) = \Omega(x,x,y) - \frac{2}{3}\,\Omega(x,x,x) \qquad \text{for all } x,y \in X.$$

Then p_Ω is a w-distance on the metric space (X,d_G). Furthermore, if Ω also verifies (P_5), then

$$\Omega_{p_\Omega} = \Omega.$$

Proof. By property (P_1), $p_\Omega(x,y) \geq 0$ for all $x,y \in X$. We prove three properties.

(1) (P_2) and (P_3) yield to

$$\begin{aligned} p_\Omega(x,z) &= \Omega(x,x,z) - \frac{2}{3}\,\Omega(x,x,x) = \frac{1}{2}\,\left[\,\Omega(x,z,z) - \frac{1}{3}\,\Omega(x,x,x)\,\right] \\ &\leq \frac{1}{2}\,\left[\,\left(\Omega(x,y,y) + \Omega(y,z,z) - \frac{1}{3}\,\Omega(y,y,y)\right) - \frac{1}{3}\,\Omega(x,x,x)\,\right] \\ &= \frac{1}{2}\,\left[\,\left(\Omega(x,y,y) - \frac{1}{3}\,\Omega(x,x,x)\right) + \left(\Omega(y,z,z) - \frac{1}{3}\,\Omega(y,y,y)\right)\,\right] \\ &= p_\Omega(x,y) + p_\Omega(y,z). \end{aligned}$$

(2) Clearly, $p_\Omega(x,\cdot) = \Omega(x,x,\cdot) - \frac{2}{3}\,\Omega(x,x,x)$ is lower semi-continuous in its second variable.

(3) Let $\varepsilon > 0$. By property (P_4), there is $\delta > 0$ such that

$$\left.\begin{aligned} p_\Omega(z,x) &= \Omega(z,z,x) - \frac{2}{3}\,\Omega(z,z,z) \leq \delta \\ p_\Omega(z,y) &= \Omega(z,z,y) - \frac{2}{3}\,\Omega(z,z,z) \leq \delta \end{aligned}\right\} \;\Rightarrow\; G(x,x,y) \leq \varepsilon.$$

As the conditions are symmetric on x and y,

$$\left.\begin{aligned} p_\Omega(z,x) &\leq \delta \\ p_\Omega(z,y) &\leq \delta \end{aligned}\right\} \;\Rightarrow\; d_G(x,y) = \max(G(x,x,y),G(y,y,x)) \leq \varepsilon.$$

Now suppose that (P_5) holds. Then, for all $x,y,z \in X$, we have that

$$\begin{aligned} \Omega_{p_\Omega}(x,y,z) &= p_\Omega(x,x) + p_\Omega(x,y) + p_\Omega(x,z) = \left(\Omega(x,x,x) - \frac{2}{3}\,\Omega(x,x,x)\right) \\ &\quad + \left(\Omega(x,x,y) - \frac{2}{3}\,\Omega(x,x,x)\right) + \left(\Omega(x,x,z) - \frac{2}{3}\,\Omega(x,x,x)\right) \\ &= \Omega(x,x,y) + \Omega(x,x,z) - \Omega(x,x,x) = \Omega(x,y,z). \end{aligned}$$

□

Corollary 3.4. *If p is a w-distance on a metric space (X,d), then $p_{\Omega_p} = p$.*

Proof. For all $x,y \in X$,

$$p_{\Omega_p}(x,y) = \Omega_p(x,x,y) - \frac{2}{3}\,\Omega_p(x,x,x) =$$

$$= 2p(x, x,) + p(x, y) - \frac{2}{3}\, 3p(x, x) = p(x, y).$$

□

Corollary 3.5. *The notion of w-distance on a metric space is a particularization of the notion of Ω-distance on a G^*-metric space.*

However, the class of Ω-distances are bigger that the class of w-distances since the following Ω-distance do not generate a w-distance.

Example 3.6. Let $X = [0, \infty)$ provided with the Euclidean distance $d_0(x, y) = |x - y|$ for all $x, y \in X$, and the G^*-metric associated to d_0, that is, $G_{d_0}(x, y, z) = \max(|x - y|, |x - z|, |y - z|)$ for all $x, y, z \in X$. Define $\Omega(x, y, z) = x + 2y + 3z$ for all $x, y, z \in X$. Then Ω is a Ω-distance on (X, G_{d_0})[1]. However, it does not come from a w-distance because it is not symmetric in its two last variables.

4. Translations between fixed point theorems using w-distances and Ω-distances

As application of the previous results, we are going to show how we can translate some results involving Ω-distances to statements using w-distances, and viceversa (in some cases). In particular, we apply the introduced relationships to the field of fixed point theory, but our results can also be applied to other areas: Topology, equations theory, etc.

Firstly, we show how we can translate every fixed point result in the setting of Ω-distances to the framework of w-distances. This procedure allows us to present a new class of contractivity conditions, as in the following illustrative example.

In [14], the authors proved the following result (necessary preliminaries can be found therein).

Theorem 4.1 (Saadati *et al.*, 2010, Theorem 2.2)**.** *Let $(X, \preccurlyeq)$ be a partially ordered set. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Let X be Ω-bounded. Suppose that there exists $k \in [0, 1)$ such that*

$$\Omega(Tx, T^2x, Tw) \leq k\ \Omega(x, Tx, w) \quad \text{for all } x \preccurlyeq Tx \text{ and } w \in X.$$

Also for every $x \in X$

$$\inf(\ \Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, T^2x, y) : x \preccurlyeq Tx\ ) > 0$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point. Moreover, if $u = Tv$, then $\Omega(u, v, v) = 0$.

Taking into account that if a subset is p-bounded, then it is also Ω_p-bounded, then we can deduce the following consequence.

Corollary 4.2. *Let p be a w-distance in a complete metric space (X, d) and let $\preccurlyeq$ be a partial order on X. Assume that X is p-bounded. Let $T : X \to X$ be a $\preccurlyeq$-non-decreasing mapping from X into itself and suppose that there exists $k \in [0, 1)$ such that, for all $x \preccurlyeq Tx$ and $w \in X$,*

$$p(Tx, Tx) + p(Tx, T^2x) + p(Tx, Tw) \leq k\,(p(x, x) + p(x, Tx) + p(x, w))\,.$$

[1]A proof can be found on page ??.

Also for every $x \in X$

$$\inf\left(\ 4p(x,x)+3p(x,y)+p(x,Tx)+p(x,T^2x) : x \preccurlyeq Tx\ \right) > 0$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point. Moreover, if $u = Tv$, then $p(u,u) = p(u,v) = 0$.

Proof. It is only necessary to apply Theorem 4.1 to the Ω-distance Ω_p defined in Theorem 3.2, taking into account that (X, G_d) is a complete G-metric space. □

Using the introduced relationships, the converse procedure is only possible when the Ω-distance verify some properties, as we show using the following result given in [6].

Theorem 4.3 (Ilić and Rakočević, 2008, Theorem 3.1)**.** *Let X be a complete metric space with metric d and let p be a w-distance on X. Let $f, g : X \to X$ commutes, satisfy $g(X) \subset f(X)$ and suppose that there exists a constant $k \in (0,1)$ such that, for every $x, y \in X$,*

$$p(gx, gy) \leq \lambda\ M_p(x,y) \qquad \text{where}$$

$$M_p(x,y) = \max\left(p(fx,fy), p(fx,gx), p(fy,gy), p(fx,gy), p(fy,gx)\right).$$

Also assume that for every $y \in X$ with $f(y) \neq g(y)$, we have that

$$\inf\left(p(fx,y) + p(fx,gx) : x \in X\right) > 0.$$

Then f and g have a common unique fixed point u in X (that is, a point $u \in X$ such that $fu = gu = u$) and $p(u,u) = 0$.

Taking into account that if (X, G) is a complete G-metric space, then (X, d_G) is a complete metric space, the previous result can be enunciated in the following way.

Corollary 4.4. *Let Ω be a Ω-distance verifying properties (P_1)-(P_5) of Theorem 3.2 on a G^*-metric space (X, G). Let $f, g : X \to X$ be two commuting mappings such that $g(X) \subset f(X)$ and suppose that there exists a constant $k \in (0,1)$ verifying that, for every $x, y \in X$,*

$$\Omega(gx,gx,gy) - \frac{2}{3}\ \Omega(gx,gx,gx) \leq \lambda\ M_\Omega^{f,g}(x,y) \qquad \text{where}$$

$$\begin{aligned} M_\Omega^{f,g}(x,y) = \max\Bigg(&\Omega(fx,fx,fy) - \frac{2}{3}\ \Omega(fx,fx,fx), \\ &\Omega(fx,fx,gx) - \frac{2}{3}\ \Omega(fx,fx,fx),\ \Omega(fy,fy,gy) - \frac{2}{3}\ \Omega(fy,fy,fy), \\ &\Omega(fx,fx,gy) - \frac{2}{3}\ \Omega(fx,fx,fx),\ \Omega(fy,fy,gx) - \frac{2}{3}\ \Omega(fy,fy,fy)\Bigg). \end{aligned}$$

Also assume that for every $y \in X$ with $f(y) \neq g(y)$, we have that

$$\inf\left(\Omega(fx,fx,y) + \Omega(fx,fx,gx) - \frac{4}{3}\ \Omega(fx,fx,fx) : x \in X\right) > 0.$$

Then f and g have a common unique fixed point u in X and $\Omega(u,u,u) = 0$.

Proof. It is only necessary to apply Theorem 4.3 to the w-distance p_Ω (that exists by Theorem 3.3). □

References

[1] T. V. An, N. V. Dung and V. T. L. Hang, *A new approach to fixed point theorems on G-metric spaces*, Topology and its Appl. **160** (2013), 1486–1493.
[2] J. García-Falset, L. Guran and E. Llorens-Fuster, *Fixed points for multivalued contractions with respect to a w -distance*, Sci. Math. Jpn. **71** (2010), 83–91.
[3] L. Gholizadeh, *A fixed point theorem in generalized ordered metric spaces with application*, J. Nonlinear Sci. Appl. **6** (2013), 244–251.
[4] L. Gholizadeh, R. Saadati, W. Shatanawi and S. M. Vaezpour, *Contractive mapping in generalized ordered metric spaces with application in integral equations*, Math. Probl. Eng. **2011**, Article ID 380784, 14 pages.
[5] R. H. Haghi, Sh. Rezapour and N. Shahzad, *Be careful on partial metric fixed point results*, Topology and its Appl. **160** (2013), 450–454.
[6] D. Ilić and V. Rakočević, *Common fixed points for maps on metric space with w-distance*, Applied Mathematics and Computation **199** (2008), 599–610.
[7] M. Jleli and B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appl. 2012, 2012:210 (22 November 2012)
[8] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete matrix spaces*, Math. Japonica **44** (1996), 381–391.
[9] H. Lakzian, H. Aydi and B.E. Rhoades, *Fixed points for* (φ, ψ, p)*-weakly contractive mappings in metric spaces with w-distance*, Applied Mathematics and Computation **219** (2013), 6777–6782.
[10] L.-J. Lin and C.-S. Chuang, *Some new fixed point theorems of generalized nonlinear contractive multivalued maps in complete metric spaces*, Computers and Mathematics with Applications **62** (2011), 3555–3566.
[11] J. Marín, S. Romaguera and P. Tirado, *Weakly contractive multivalued maps and w-distances on complete quasi-metric spaces*, Fixed Point Theory Appl. 2011, 2011:2.
[12] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), 289–297.
[13] A. Roldán and E. Karapınar, *Some multidimensional fixed point theorems on partially preordered* G^**-metric spaces under* (ψ, φ)*-contractivity conditions*, Fixed Point Theory Appl. 2013 2013:158.
[14] R. Saadati, S. M. Vaezpour, P. Vetro and B .E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Model. **52** (2010), 797–801.
[15] B. Samet, C. Vetro and F. Vetro, *Remarks on -Metric Spaces*, Int. J. Anal. **2013** (2013), Article ID 917158, 6 pages
[16] W. Shatanawi and A. Pitea, *Ω-Distance and coupled fixed point in G-metric spaces*, Fixed Point Theory Appl., doi: 10.1186/1687-1812-2013-208.

Manuscript received August 30, 2014
revised November 30, 2014

A.-F. Roldán-López-de-Hierro
Department of Mathematics, University of Jaén, Campus las Lagunillas s/n, 23071, Jaén, Spain
E-mail address: `afroldan@ujaen.es, aroldan@ugr.es`

E. Karapinar
Department of Mathematics, Atilim University 06836, Incek, Ankara, Turkey and; Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: `ekarapinar@atilim.edu.tr, erdalkarapinar@yahoo.com`