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WEAK *P*-PROPERTY ON PRODUCT SPACES AND RELATED MULTIDIMENSIONAL BEST PROXIMITY POINT THEOREMS

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Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. In recent times, many authors proved several coupled, tripled, quadrupled and, in general, multidimensional fixed point theorems. In many cases, these results become to be simple consequences of their corresponding unidimensional theorems. In this paper, we show how the weak P-property can be induced in product spaces and how to use it to enunciate some kind of multidimensional (including coupled and tripled) best proximity (and fixed) point theorems.

1. Introduction

In recent times, many authors proved several coupled, tripled, quadrupled and, in general, multidimensional fixed point theorems (see, for instance, [1, 3, 6, 7, 9–12, 14, 15, 18–24]). In many cases, these results become to be simple consequences of their corresponding unidimensional theorems (see, e.g., [2, 5, 13, 16]). In this paper, we show how the weak P-property can be induced in product spaces and how to use it to enunciate some kind of multidimensional (including coupled and tripled) best proximity (and fixed) point results.

2. Preliminaries

Let n be a positive integer. Given a non-empty set X, let denote by X^n the product space $X \times X \times \overset{(n)}{\cdots} \times X$ of n identical copies of X. If A is a non-empty subset of X, we will also denote the product space $A \times A \times \overset{(n)}{\cdots} \times A \subseteq X^n$ by A^n.

Let A and B be two non-empty subsets of a metric space (X, d). The distance between A and B is

$$d(A,B) = \inf\left(\{d(a,b) : a \in A, b \in B\}\right).$$

By $A_0^{d,B}$ and $B_0^{d,A}$ we denote the following sets:

$$A_0^{d,B} = \{x \in A \colon d(x,y) = d(A,B) \text{ for some } y \in B\} \qquad \text{and}$$
$$B_0^{d,A} = \{y \in B \colon d(x,y) = d(A,B) \text{ for some } x \in A\}\ .$$

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Although $A_0^{d,B}$ depends on the metric d and on the subset B, for simplicity, it is usual to denote it by A_0, and B_0 will stand for $B_0^{d,A}$. Notice that if $a \in A$ and $b \in B$ verify $d(a,b) = d(A,B)$, then $a \in A_0$ and $b \in B_0$. Therefore, A_0 is nonempty if, and only if, B_0 is nonempty. Therefore, if A_0 is nonempty, then A, B and B_0 are non-empty subsets of X. It is clear that if $A \cap B \neq \emptyset$, then A_0 is nonempty. In [17], the authors discussed sufficient conditions in order to guarantee the non-emptiness of A_0. In general, if A and B are closed subsets of a normed linear space such that $d(A,B) > 0$, then A_0 is contained in the boundary of A (see [25]).

Given a metric space (X,d), let define $d_n : X^n \times X^n \to [0,\infty)$, for all $(x_1,x_2,\ldots,x_n)$, $(y_1,y_2,\ldots,y_n) \in X^n$, by

$$d_n\left((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n)\right) = \max_{1\leq i\leq n} d(a_i,b_i).$$

It is well known that d_n is a metric on X^n. Furthermore, (X,d) is a complete metric space if, and only if, (X^n,d_n) is also a complete metric space.

Definition 2.1. Let A and B be two subsets of a metric space (X,d) such that A_0 is nonempty. We say that the pair (A,B) has the *P-property* if

$$\left.\begin{array}{c} a_1,a_2 \in A_0, \quad b_1,b_2 \in B_0 \\ d(a_1,b_1) = d(A,B) \\ d(a_2,b_2) = d(A,B) \end{array}\right\} \;\Rightarrow\; d(a_1,a_2) = d(b_1,b_2).$$

We will consider a weaker condition than the P-property as follows.

Definition 2.2. Let A and B be two subsets of a metric space (X,d) such that A_0 is nonempty. We say that the pair (A,B) has the *weak P-property* if

$$\left.\begin{array}{c} a_1,a_2 \in A_0, \quad b_1,b_2 \in B_0 \\ d(a_1,b_1) = d(A,B) \\ d(a_2,b_2) = d(A,B) \end{array}\right\} \;\Rightarrow\; d(a_1,a_2) \leq d(b_1,b_2).$$

Example 2.3. Let X be the plane $\mathbb{R}^2$ endowed with the Euclidean metric

$$d_E\left((x_1,y_1),(x_2,y_2)\right) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

for all $(x_1,y_1),(x_2,y_2) \in X$. Let A and B be the subsets

$$A = \{\,(x,1) : -1 \leq x \leq 1\,\} \quad \text{and} \quad B = \{\,(x,0) : |y| \geq 2\,\}.$$

Clearly, A and B are non-empty, closed subsets of X. Furthermore,

$$d_E(A,B) = \sqrt{2}, \quad A_0 = \{\,(-1,1),\,(1,1)\,\} \quad \text{and} \quad B_0 = \{\,(-2,0),\,(2,0)\,\}.$$

It can be easily checked that the pair (A,B) has the weak P-property, but it does not satisfy the P-property.

Example 2.4. Also in $\left(\mathbb{R}^2, d_E\right)$, let consider the subsets

$$A = \{\,(x,0) \in X : x \geq 0\,\},$$
$$B = \left\{\,(x,y) \in X : x \leq 0,\ x^2+y^2 = 1\,\right\} \cup \{\,(0,y) \in X : |y| \geq 1\,\}.$$

Clearly, A and B are non-bounded, closed subsets of X. Furthermore, $d_E(A,B) = 1$, $A_0 = \{(0,0)\}$ and

$$B_0 = \left\{\,(x,y) \in X : x \leq 0,\ x^2+y^2 = 1\,\right\}.$$

Again, the pair (A, B) has the weak P-property, but it does not satisfy the P-property.

Definition 2.5. We will say a point $x \in A$ is a *best proximity point of* T if $d(x, Tx) = d(A, B)$. If $A = B$, a best proximity point of T is a *fixed point of* T (that is, $Tx = x$).

Let Ψ denote the family of all non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = 0 \Leftrightarrow t = 0$, and let Ψ' the subset of all $\psi \in \Psi$ such that ψ is continuous and $\lim_{t\to\infty} \psi(t) = \infty$.

Definition 2.6 (Sankar Raj [26], Definition 2)**.** Let A and B be non-empty subsets of a metric space (X, d). A map $T : A \to B$ is said to be a *weakly contractive mapping* if there exists $\psi \in \Psi'$ (if A is bounded, the infinity condition can be omitted) such that

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \qquad \text{for all } x, y \in A.$$

In [26], the author proved the following result.

Theorem 2.7 (Sankar Raj [26], Theorem 3.1)**.** *Let (A, B) be a pair of two non-empty closed subsets of a complete metric space (X, d) such that A_0 is non-empty. Let $T : A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the* P*-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.*

In the original version, the author assumed the ψ is continuous, but, as we shall prove, this hypothesis is not necessary. Moreover, the previous result also holds under the weak P-property.

3. An extension of Raj's theorem

The following result proves that T has a best proximity point under weaker assumptions than the ones appearing in Theorem 2.7.

Theorem 3.1. *Let (A, B) be a pair of two non-empty closed subsets of a complete metric space (X, d) such that A_0 is non-empty. Let $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and suppose that there exists $\psi \in \Psi$ verifying*

$$\begin{aligned} &\textit{if} \quad x, y \in A_0 \quad \textit{and} \quad d(x, Tx) \leq d(x, y) + d(A, B), \\ &\qquad \textit{then} \quad d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)). \end{aligned} \tag{3.1}$$

Assume that T is continuous and the pair (A, B) has the weak P*-property. Then T has a unique best proximity point.*

Notice that if ψ is continuous (for instance, if $\psi \in \Psi'$), then all weakly contractive mappings in the sense of Definition 2.6 are continuous. Therefore, the previous result improves Theorem 2.7 in three senses: (1) the pair (A, B) must only verify the weak P-property; (2) our contractivity condition must be only verified by points which satisfy the antecedent condition "$d(x, Tx) \leq d(x, y) + d(A, B)$", but not over all points in A; (3) furthermore, this condition must be only verified by points in A_0. As a consequence, our result is clearly an improvement.

Proof. Part I. Existence. Starting from any $x_0 \in X$ and following a well known argument, there exists a sequence $\{x_n\}$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{for all } n \geq 0. \tag{3.2}$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then

$$d(A, B) = d(x_{n_0+1}, Tx_{n_0}) = d(x_{n_0}, Tx_{n_0}),$$

so x_{n_0} is a best proximity point of T. In the sequel, assume that $x_{n+1} \neq x_n$ for all $n \geq 0$. Using the weak P-property, for all $n.m \in \mathbb{N}$,

$$\left.\begin{array}{c} x_n, x_m \in A_0, \quad Tx_n, Tx_m \in B_0 \\ d(x_{n+1}, Tx_n) = d(A, B) \\ d(x_{m+1}, Tx_m) = d(A, B) \end{array}\right\} \Rightarrow d(x_{n+1}, x_{m+1}) \leq d(Tx_n, Tx_m). \tag{3.3}$$

Taking into account that

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + d(A, B),$$

the contractivity condition (3.1) guarantees that, for all $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) \leq d(Tx_n, Tx_{n+1}) &\leq d(x_n, x_{n+1}) - \psi(d(x_n, x_{n+1})) \\ &\leq d(x_n, x_{n+1}). \end{aligned} \tag{3.4}$$

Let $r \geq 0$ be such that $\{d(x_n, x_{n+1})\} \to r$. We shall prove that $r = 0$ by contradiction. If $r > 0$, letting $n \to \infty$ in (3.4), we deduce that $r \leq r - \lim_{n\to\infty} \psi(d(x_n, x_{n+1})) \leq r$, so

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0. \tag{3.5}$$

However, as ψ is non-decreasing, then $\psi(r) \leq \psi(d(x_n, x_{n+1}))$ for all n, which is a contradiction with (3.5) and the fact that $\psi(r) > 0$ because $\psi \in \Psi$ and $r > 0$. This contradiction ensures that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. On the contrary case, a well known argument using (3.6) shows that there exists $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that

$$k \leq m(k) < n(k), \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon_0 \leq d(x_{m(k)}, x_{n(k)}) \quad \text{for all } k, \tag{3.7}$$

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon_0. \tag{3.8}$$

By (3.6) and (3.8), there exists $n_0 \in \mathbb{N}$ such that

$$d(x_k, x_{k+1}) \leq \frac{\varepsilon_0}{2} \leq d(x_{m(k)}, x_{n(k)}) \quad \text{for all } k \geq n_0.$$

Therefore, for all $k \geq n_0$,

$$\begin{aligned} d(x_{m(k)}, Tx_{m(k)}) &\leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, Tx_{m(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)+1}) + d(A, B) \\ &\leq \frac{\varepsilon_0}{2} + d(A, B) \leq d(x_{m(k)}, x_{n(k)}) + d(A, B). \end{aligned}$$

Applying the contractivity condition (3.1) and (3.3), it follows that, for all $k \geq n_0$,

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) &\leq d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq d(x_{m(k)}, x_{n(k)}) - \psi(d(x_{m(k)}, x_{n(k)})) \\ &\leq d(x_{m(k)}, x_{n(k)}). \end{aligned}$$

Taking into account (3.8), we deduce that

$$\lim_{n\to\infty} \psi(d(x_{m(k)}, x_{n(k)})) = 0,$$

but this is a contradiction with the fact that

$$0 < \psi(\varepsilon_0/2) \leq \psi\left(d(x_{m(k)}, x_{n(k)})\right)$$

because $\psi \in \Psi$. Hence, $\{x_n\}$ is a Cauchy sequence.

As (X, d) is complete, there exists $x \in X$ such that $\{x_n\} \to x$. Since $\{x_n\} \subseteq A_0 \subseteq A$ and A is closed, then $x \in A$. Furthermore, as T is continuous, letting $n \to \infty$ in (3.2), we conclude that $d(x, Tx) = d(A, b)$, so x is a best proximity point of T.

Part II. Uniqueness. Let $x, y \in X$ be two best proximity points of T. Since $d(x, Tx) = d(y, Ty) = d(A, B)$, the weak P-property guarantees

$$d(x, y) \leq d(Tx, Ty).$$

On the other hand $d(x, Tx) = d(A, B) \leq d(x, y) + d(A, B)$, which implies that

$$d(x, y) \leq d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \leq d(x, y).$$

Hence $\psi(d(x, y)) = 0$, so $d(x, y) = 0$ and $x = y$. This completes the proof. □

One of the main advantages of the contractivity condition (3.1) is that it must be only verified for different points $x, y \in A_0$. When A_0 contains few points, then it can be easily verified, as in the following example.

Example 3.2. Let A and B be the subsets of the complete metric space $\left(\mathbb{R}^2, d_E\right)$ given in Example 2.3 and let $T : A \to B$ be the mapping

$$T(x, 1) = (3 - |x|, 0) \quad \text{for all } x \in [-1, 1].$$

Since $A_0 = \{(-1, 1), (1, 1)\}$ and $B_0 = \{(-2, 0), (2, 0)\}$, then $T(A_0) = \{(2, 0)\} \subseteq B_0$. If $x, y \in A_0$ are different points, then $x = (-1, 1)$ and $y = (1, 1)$, or viceversa. In any case, as $T(-1, 1) = T(1, 1)$, then the contractivity condition (3.1) trivially holds. As (A, B) has the weak P-property and T is continuous, then Theorem 3.1 guarantees that T has a unique best proximity point, which is the point $(1, 1)$.

If $d(A, B) = 0$, then $A_0 = B_0 = A \cap B$ is a closed, complete subset of (X, d). The condition $T(A_0) \subseteq B_0$ means that we can consider a self-mapping $T|_{A\cap B} : A \cap B \to A \cap B$. Therefore, in the following corollary, which corresponds to the case $d(A, B) = 0$, we do not consider subsets.

Corollary 3.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous mapping. Assume that there exists $\psi \in \Psi$ verifying*

$$\begin{aligned} &\textit{if} \quad x, y \in X \quad \textit{and} \quad d(x, Tx) \leq d(x, y), \\ &\qquad \textit{then} \quad d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)). \end{aligned}$$

Then T has a unique fixed point.

Proof. It follows from Theorem 3.1 using $A = B = X$ and taking into account that (X, X) has the weak P-property (in fact, it verifies the P-property). □

4. Weak P-property on product spaces

For simplicity, we will use the notation $\mathsf{A} = (a_1, a_2, \ldots, a_n) \in A^n$ and $\mathsf{B} = (b_1, b_2, \ldots, b_n) \in B^n$ to denote arbitrary points of A^n and B^n.

Lemma 4.1. *If A and B be non-empty subsets of a metric space (X, d), then the following statements hold.*

(1) $d_n(A^n, B^n) = d(A, B)$.

(2) *If* $\mathsf{A} = (a_1, a_2, \ldots, a_n) \in A^n$ *and* $\mathsf{B} = (b_1, b_2, \ldots, b_n) \in B^n$, *then*

$$d_n(\mathsf{A}, \mathsf{B}) = d_n(A^n, B^n) \quad \Leftrightarrow \quad [\ d(a_i, b_i) = d(A, B) \text{ for all } i \in \{1, 2, \ldots, n\}\ ].$$

(3) $(A^n)_0^{d_n, B^n} = (A_0^{d,B})^n$ *(that is,* $(A \times A \times \overset{(n)}{\cdots} \times A)_0 = A_0 \times A_0 \times \overset{(n)}{\cdots} \times A_0$*).*

(4) *In particular,* $A_0 \neq \varnothing$ *if, and only if,* $(A^n)_0 \neq \varnothing$.

Proof. **(1)** Let $\mathsf{A} = (a_1, a_2, \ldots, a_n) \in A^n$ and $\mathsf{B} = (b_1, b_2, \ldots, b_n) \in B^n$ be arbitrary. Since

$$d_n(\mathsf{A}, \mathsf{B}) = \max_{1 \le i \le n} d(a_i, b_i) \ge d(a_1, b_1) \ge d(A, B),$$

taking infimum on $\mathsf{A} \in A^n$ and $\mathsf{B} \in B^n$ we deduce that $d_n(A^n, B^n) \ge d(A, B)$. On the other hand, as

$$d(a, b) = d_n((a, a, \ldots, a), (b, b, \ldots, b)) \ge d_n(A^n, B^n)$$

for all $a \in A$ and all $b \in B$, we conclude the contrary inequality $d(A, B) \ge d_n(A^n, B^n)$.

(2) Assume that $d_n(\mathsf{A}, \mathsf{B}) = d_n(A^n, B^n)$. By item (1), we have that, for all $i \in \{1, 2, \ldots, n\}$,

$$d(A, B) \le d(a_i, b_i) \le \max_{1 \le j \le n} d(a_j, b_j) = d_n(A^n, B^n) = d(A, B).$$

Therefore, $d(a_i, b_i) = d(A, B)$ for all $i \in \{1, 2, \ldots, n\}$.

Conversely, assume that $d(a_i, b_i) = d(A, B)$ for all $i \in \{1, 2, \ldots, n\}$. Then

$$d_n(\mathsf{A}, \mathsf{B}) = \max_{1 \le i \le n} d(a_i, b_i) = d(A, B) = d_n(A^n, B^n).$$

(3) Assume that $\mathsf{A} = (a_1, a_2, \ldots, a_n) \in (A_0)^n = A_0 \times A_0 \times \overset{(n)}{\cdots} \times A_0$. Then there exist respective $b_1, b_2, \ldots, b_n \in B$ such that $d(a_i, b_i) = d(A, B)$ for all $i \in \{1, 2, \ldots, n\}$. In particular, by item 2, $d_n(\mathsf{A}, \mathsf{B}) = d_n(A^n, B^n)$, so $\mathsf{A} \in (A^n)_0$.

Conversely, assume that $\mathsf{A} \in (A^n)_0$. Then, there exists $\mathsf{B} \in B^n$ such that $d_n(\mathsf{A}, \mathsf{B}) = d_n(A^n, B^n)$. Also by item 2, it follows that $d(a_i, b_i) = d(A, B)$ for all $i \in \{1, 2, \ldots, n\}$, which means that $a_i \in A_0$ for all $i \in \{1, 2, \ldots, n\}$. This proves that $\mathsf{A} = (a_1, a_2, \ldots, a_n) \in A_0 \times A_0 \times \overset{(n)}{\cdots} \times A_0 = (A_0)^n$. □

Theorem 4.2. *If A and B are non-empty subsets of a metric space (X, d), then the pair (A, B) has the weak* P*-property on (X, d) if, and only if, the pair (A^n, B^n) has the weak* P*-property on (X^n, d_n).*

Proof. Assume that the pair (A,B) has the weak P-property on (X,d). Then $A_0 \neq \varnothing$ and $B_0 \neq \varnothing$. By item 3 of Lemma 4.1, $(A^n)_0 \neq \varnothing$ and $(B^n)_0 \neq \varnothing$. Assume that $\mathsf{A} = (a_1, a_2, \ldots, a_n), \mathsf{A}' = (a'_1, a'_2, \ldots, a'_n) \in (A^n)_0$ and $\mathsf{B} = (b_1, b_2, \ldots, b_n), \mathsf{B}' = (b'_1, b'_2, \ldots, b'_n) \in (B^n)_0$ are such that $d_n(\mathsf{A},\mathsf{B}) = d_n(\mathsf{A}',\mathsf{B}') = d_n(A^n, B^n)$. By item 2 of Lemma 4.1, we have that

$$d(a_i, b_i) = d(a'_i, b'_i) = d(A,B) \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Since $a_i, a'_i \in A_0$ and $b_i, b'_i \in B_0$ for all $i \in \{1, 2, \ldots, n\}$, and (A,B) has the weak P-property on (X,d), it follows that

$$d(a_i, a'_i) \leq d(b_i, b'_i) \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

In particular,

$$d_n(\mathsf{A},\mathsf{A}') \leq \max_{1 \leq i \leq n} d(a_i, a'_i) \leq \max_{1 \leq i \leq n} d(b_i, b'_i) \leq d_n(\mathsf{B},\mathsf{B}'),$$

which means that (A^n, B^n) has the weak P-property on (X^n, d_n).

Conversely, assume that (A^n, B^n) has the weak P-property on (X^n, d_n). Then $(A^n)_0 \neq \varnothing$ and $(B^n)_0 \neq \varnothing$. By item 3 of Lemma 4.1, $A_0 \neq \varnothing$ and $B_0 \neq \varnothing$. Let $a, a' \in A_0$ and $b, b' \in B_0$ be arbitrary points such that $d(a,b) = d(a',b') = d(A,B)$. Let define $\mathsf{A} = (a, a, \ldots, a), \mathsf{A}' = (a', a', \ldots, a') \in (A^n)_0$ and $\mathsf{B} = (b, b, \ldots, b), \mathsf{B}' = (b', b', \ldots, b') \in (B^n)_0$. Therefore $d_n(\mathsf{A},\mathsf{B}) = d_n(\mathsf{A}',\mathsf{B}') = d_n(A^n, B^n)$. As (A^n, B^n) has the weak P-property on (X^n, d_n), we deduce that $d_n(\mathsf{A},\mathsf{A}') \leq d_n(\mathsf{B},\mathsf{B}')$, which means that $d(a,a') \leq d(b,b')$. Hence, the pair (A,B) has the weak P-property on (X,d). □

The same proof can be followed point by point to show the following result.

Theorem 4.3. *If A and B are non-empty subsets of a metric space (X,d), then the pair (A,B) has the* P*-property on (X,d) if, and only if, the pair (A^n, B^n) has the* P*-property on (X^n, d_n).*

5. A multidimensional best proximity point theorem

The notion of fixed point was generalized to the coupled case by Guo and Lakshmikantham in [12] and, shortly after, Bhaskar and Lakshmikantham [11] introduced the mixed monotone property in order to guarantee existence and uniqueness of coupled fixed points. After that, Berinde and Borcut [7] presented the notion of tripled fixed point. Berinde and Borcut's definition has a disadvantage: the mixed monotone property forces to repeat its second variable. Quadruple case was introduced by Karapınar in [14]. The multidimensional case was not studied until the works of Berzig and Samet [8] (which did not solve the problem of how permuting the variables) and, especially, Roldán *et al.* See references in [20, 21, 23].

In the setting of best proximity theory, the notion of *coupled best proximity point* must be as follows.

Definition 5.1. Let A and B be non-empty subsets of a metric space (X,d) and let $F : A^2 \to B$ be a mapping. We will say that $(x,y) \in A^2$ is a *coupled best proximity point of F* if

$$d(x, F(x,y)) = d(y, F(y,x)) = d(A,B).$$

The more general notion of multidimensional fixed point was given by Roldán *et al.* in [21,23]. Following their idea, we could establish a notion of multidimensional best proximity point for a nonlinear operator. However, for simplicity, we present the following concept which was also used by other authors in the past. We must advise that the following definition is not compatible with the mixed monotone property when the dimension is odd (see [16]).

Definition 5.2. Let A and B be non-empty subsets of a metric space (X,d) and let $F : A^n \to B$ be a mapping. We will say that $(x_1, x_2, \ldots, x_n) \in A^n$ is a *multidimensional best proximity point of* F if

$$d(x_i, F(x_i, x_{i+1}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1})) = d(A,B) \text{ for all } i \in \{1,2,\ldots,n\}.$$

In order to guarantee existence and uniqueness of n-dimensional best proximity points, item 3 of Lemma 4.1 will play a crucial role. It guarantees that $(A \times A \times \overset{(n)}{\cdots} \times A)_0 = A_0 \times A_0 \times \overset{(n)}{\cdots} \times A_0$, that is, $(A^n)_0 = (A_0)^n$. Therefore, we can denote this set by A_0^n and we can describe it as:

$$A_0^n = \{\ (a_1, a_2, \ldots, a_n) \in A^n :\ \exists\, b_1, b_2, \ldots, b_n \in B \text{ such that } d(a_i, b_i) = d(A,B) \text{ for all } 1 \le i \le n\ \}.$$

Given two non-empty subsets A and B of a metric space (X,d) and a mapping $F : A^n \to B$, let consider the mapping $T_F^n : A^n \to B^n$ given by

$$T_F^n(x_1, x_2, \ldots, x_n) = \big(F(x_1, x_2, x_3, \ldots, x_{n-1}, x_n),\ F(x_2, x_3, x_4, \ldots, x_n, x_1), \ldots,\ F(x_n, x_1, x_2, \ldots, x_{n-2}, x_{n-1})\big) \tag{5.1}$$

for all $(x_1, x_2, \ldots, x_n) \in A^2$.

Lemma 5.3. *If A and B are non-empty subsets of a metric space (X,d) and $F : A^n \to B$ is a mapping such that $F(A_0^n) \subseteq B_0$, then $T_F^n(A_0^n) \subseteq B_0^n$.*

Proof. Let $(a_1, a_2, \ldots, a_n) \in A_0^n = (A_0)^n$ be arbitrary. Therefore $a_i \in A_0$ for all $i \in \{1,2,\ldots,n\}$. Hence, the points

$$(a_1, a_2, a_3, \ldots, a_{n-1}, a_n),\quad (a_2, a_3, a_4, \ldots, a_n, a_1),\ \ldots,\quad (a_n, a_1, a_2, \ldots, a_{n-2}, a_{n-1})$$

are all in $(A_0)^n$. By hypothesis, since $F(A_0^n) \subseteq B_0$, we deduce that the points

$$F(a_1, a_2, a_3, \ldots, a_{n-1}, a_n),\quad F(a_2, a_3, a_4, \ldots, a_n, a_1),\ \ldots,\quad F(a_n, a_1, a_2, \ldots, a_{n-2}, a_{n-1})$$

are all in B_0. Hence, $T_F^n(a_1, a_2, \ldots, a_n) \in (B_0)^n = B_0^n$. □

Our main result of this section is the following one.

Theorem 5.4. *Let A and B be closed subsets of a complete metric space (X,d) such that $A_0 \neq \varnothing$ and let $F : A^n \to B$ be a mapping such that the following properties hold.*

(a) $F(A_0^n) \subseteq B_0$.

(b) *F is continuous.*
(c) *(A, B) has the weak* P*-property.*
(d) *There exists $\psi \in \Psi$ such that*

$$d\left(F\left(x_1, x_2, \ldots, x_n\right), F\left(y_1, y_2, \ldots, y_n\right)\right) \leq \max_{1 \leq i \leq n} d(x_i, y_i) - \psi\left(\max_{1 \leq i \leq n} d(x_i, y_i)\right)$$

for all points $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in A_0$ verifying

$$\max_{1 \leq i \leq n} d(x_i, F(x_i, x_{i+1}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1})) \leq \max_{1 \leq i \leq n} d(x_i, y_i) + d(A, B).$$

Then the mapping F has a unique n-dimensional best proximity point.

Proof. Since $A_0 \neq \varnothing$, then A, B, A_0^n and B_0^n are non-empty sets by item 4 of Lemma 4.1. As A and B are non-empty closed subsets, then A^n and B^n are also non-empty closed subsets of (X^n, d_n). Consider the mapping $T_F^n : A^n \to B^n$ defined by (5.1). By hypothesis (a) and Lemma 5.3, we have that $T_F\left(A_0^n\right) \subseteq B_0^n$. As F is continuous, then T_F^n is also continuous. Furthermore, using Theorem 4.2, taking into account that (A, B) has the weak P-property, then (A^n, B^n) has the weak P-property on (X^n, d_n). In addition to this, assume that $\mathsf{X} = (x_1, x_2, \ldots, x_n), \mathsf{Y} = (y_1, y_2, \ldots, y_n) \in (A^n)_0 = A_0^n = (A_0)^n$ are points such that

$$d_n(\mathsf{X}, T_F^n\mathsf{X}) \leq d_n(\mathsf{X}, \mathsf{Y}) + d_n(A^n, B^n). \tag{5.2}$$

Since $\mathsf{X} = (x_1, x_2, \ldots, x_n) \in (A_0)^n$, then $x_i \in A_0$ and similarly $y_i \in A_0$ for all $i \in \{1, 2, \ldots, n\}$. By item 1 of Lemma 4.1, $d_n(A^n, B^n) = d(A, B)$. Hence, (5.2) means that

$$\max_{1 \leq i \leq n} d(x_i, F(x_i, x_{i+1}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1})) \leq \max_{1 \leq i \leq n} d(x_i, y_i) + d(A, B). \tag{5.3}$$

Using condition (d),

$$d\left(F\left(x_1, x_2, \ldots, x_n\right), F\left(y_1, y_2, \ldots, y_n\right)\right) \leq \max_{1 \leq i \leq n} d(x_i, y_i) - \psi\left(\max_{1 \leq i \leq n} d(x_i, y_i)\right).$$

Exactly the same argument can be applied to any points

$$\left(x_j, x_{j+1}, \ldots, x_n, x_1, x_2, \ldots, x_{j-1}\right), \quad \left(y_j, y_{j+1}, \ldots, y_n, y_1, y_2, \ldots, y_{j-1}\right) \in (A_0)^n = A_0^n$$

because condition (5.3) does not depend on the initial value j. Therefore, assumption (d) yields

$$\begin{aligned} &d\left(F\left(x_j, x_{j+1}, \ldots, x_n, x_1, \ldots, x_{j-1}\right), F\left(y_j, y_{j+1}, \ldots, y_n, y_1, \ldots, y_{j-1}\right)\right) \\ &\quad \leq \max_{1 \leq i \leq n} d(x_i, y_i) - \psi\left(\max_{1 \leq i \leq n} d(x_i, y_i)\right) \end{aligned}$$

for all $j \in \{1, 2, \ldots, n\}$ (notice that the second member is independent from j). Taking maximum on j, we deduce that

$$\begin{aligned} d_n(T_F^n\mathsf{X}, T_F^n\mathsf{Y}) = \max_{1 \leq j \leq n} d\Big(&F\left(x_j, x_{j+1}, \ldots, x_n, x_1, \ldots, x_{j-1}\right), \\ &F\left(y_j, y_{j+1}, \ldots, y_n, y_1, \ldots, y_{j-1}\right)\Big) \end{aligned}$$

$$\leq \max_{1\leq i\leq n} d(x_i, y_i) - \psi\left(\max_{1\leq i\leq n} d(x_i, y_i)\right)$$
$$= d_n(\mathsf{X}, \mathsf{Y}) - \psi\left(d_n(\mathsf{X}, \mathsf{Y})\right).$$

As a consequence, Theorem 3.1 guarantees that T_F^n has a unique best proximity point. In particular, there exists a unique $\mathsf{Z} = (z_1, z_2, \ldots, z_n) \in A^n$ such that

$$d_n(\mathsf{Z}, T_F\mathsf{Z}) = d_n(A^n, B^n).$$

Using item 2 of Lemma 4.1, $z_1, z_2, \ldots, z_n$ are points in A such that

$$d(z_i, F(z_i, z_{i+1}, \ldots, z_n, z_1, z_2, \ldots, z_{i-1})) = d(A, B)$$

for all $i \in \{1, 2, \ldots, n\}$, which means that Z is an n-dimensional best proximity point of F. To prove the uniqueness, assume that $\mathsf{W} = (\omega_1, \omega_2, \ldots, \omega_n) \in A^n$ is another n-dimensional best proximity point of F, that is,

$$d(\omega_i, F(\omega_i, \omega_{i+1}, \ldots, \omega_n, \omega_1, \omega_2, \ldots, \omega_{i-1})) = d(A, B)$$

for all $i \in \{1, 2, \ldots, n\}$. Again, using 2 of Lemma 4.1, we deduce that $d_n(\mathsf{W}, T_F\mathsf{W}) = d_n(A^n, B^n)$, which means that W is a best proximity point of T_F^n. As it is unique, we conclude that $\mathsf{W} = \mathsf{Z}$. Hence, Z is the unique n-dimensional best proximity point of F. □

Example 5.5. Let A and B be the subsets of the complete metric space $\left(\mathbb{R}^2, d_E\right)$ given in Example 2.4. Recall that (A, B) has the weak P-property. Given $n \in \mathbb{N}$, notice that

$$A^n = \left\{ \left((x_1, 0), (x_2, 0), \ldots, (x_n, 0)\right) \in \mathbb{R}^{2n} : x_1, x_2, \ldots, x_n \geq 0 \right\}.$$

Let us consider the mapping $F : A^n \to B$ defined by

$$F\left((x_1, 0), (x_2, 0), \ldots, (x_n, 0)\right)$$
$$= \left(-\left|\sin\left(x_1 + x_2 + \ldots + x_n\right)\right|, \cos\left(x_1 + x_2 + \ldots + x_n\right)\right)$$

for all $x_1, x_2, \ldots, x_n \geq 0$. Then

$$F\left(A^n\right) = \left\{ (x, y) \in X : x \leq 0,\ x^2 + y^2 = 1 \right\} = B_0.$$

In particular, $F\left(A_0^n\right) \subseteq B_0$. Furthermore, as $A_0 = \{ (0, 0) \}$, then $A_0^n = \{ (0, 0, \overset{(2n)}{\ldots}, 0) \}$. As a consequence, the contractivity condition (d) of Theorem 5.4 holds. Since F is continuous, the aforementioned theorem guarantees that F has a unique n-dimensional best proximity point, which is $(0, 0, \overset{(2n)}{\ldots}, 0) \in A_0^n$.

We can present different consequences modifying the contractivity condition (for instance, avoiding the antecedent condition or replacing A_0 by A).

Corollary 5.6. *Theorem 5.4 also holds if we replace condition (d) by one of the following assumptions.*

(d′) *There exists $\psi \in \Psi$ such that*

$$d\left(F\left(x_1, x_2, \ldots, x_n\right), F\left(y_1, y_2, \ldots, y_n\right)\right)$$
$$\leq \max_{1\leq i\leq n} d(x_i, y_i) - \psi\left(\max_{1\leq i\leq n} d(x_i, y_i)\right)$$

for all points $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in A_0$.

(d″) *There exists* $\psi \in \Psi$ *such that*

$$d\left(F\left(x_1, x_2, \ldots, x_n\right), F\left(y_1, y_2, \ldots, y_n\right)\right) \leq \max_{1 \leq i \leq n} d(x_i, y_i) - \psi\left(\max_{1 \leq i \leq n} d(x_i, y_i)\right)$$

for all points $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in A$.

(d‴) *There exists* $k \in [0, 1)$ *such that*

$$d\left(F\left(x_1, x_2, \ldots, x_n\right), F\left(y_1, y_2, \ldots, y_n\right)\right) \leq k \max_{1 \leq i \leq n} d(x_i, y_i)$$

for all points $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in A_0$.

Proof. Notice that (d″)⇒(d′)⇒(d), and item (d‴) is based on taking $\psi\left(t\right) = (1-k)t$ for all $t \geq 0$, which is in Ψ. □

For the sake of completeness, we particularize the previous results to the cases $n = 2$ and $n = 3$.

Corollary 5.7. *Let* A *and* B *be closed subsets of a complete metric space* (X, d) *such that* $A_0 \neq \varnothing$ *and let* $F : A^2 \to B$ *be a mapping such that the following properties hold.*

(a) $F(A_0^2) \subseteq B_0$.
(b) F *is continuous.*
(c) (A, B) *has the weak* P*-property.*
(d) *There exists* $\psi \in \Psi$ *such that*

$$d(F(x, y), F(u, v)) \leq \max\left\{d(x, u), d(y, v)\right\} - \psi\left(\max\left\{d(x, u), d(y, v)\right\}\right)$$

for all $x, y, u, v \in A_0$ *such that*

$$\max\left(d(x, F(x, y)), d(y, F(y, x))\right) \leq \max\left\{d(x, u), d(y, v)\right\} + d(A, B).$$

Then the mapping F *has a unique coupled best proximity point, that is, there exist unique* $x, y \in A$ *such that*

$$d(x, F(x, y)) = d(y, F(y, x)) = d\left(A, B\right).$$

Corollary 5.8. *Let* A *and* B *be closed subsets of a complete metric space* (X, d) *such that* $A_0 \neq \varnothing$ *and let* $F : A^3 \to B$ *be a mapping such that the following properties hold.*

(a) $F(A_0^3) \subseteq B_0$.
(b) F *is continuous.*
(c) (A, B) *has the weak* P*-property.*
(d) *There exists* $\psi \in \Psi$ *such that*

$$d(F(x, y, z), F(u, v, w)) \leq \max\left\{d(x, u), d(y, v), d\left(z, w\right)\right\} - \psi\left(\max\left\{d(x, u), d(y, v), d\left(z, w\right)\right\}\right)$$

for all $x, y, z, u, v, w \in A_0$ *such that*

$$\max\left(d(x, F(x, y, z)), d(y, F(y, z, x)), d(z, F(z, x, y))\right) \leq \max\left\{d(x, u), d(y, v), d\left(z, w\right)\right\} + d(A, B).$$

Then the mapping F has a unique tripled best proximity point, that is, there exist unique $x, y, z \in A$ such that

$$d(x, F(x, y, z)) = d(y, F(y, z, x)) = d(z, F(z, x, y)) = d(A, B).$$

In the next results, we avoid the antecedent condition, and we replace A_0 by A.

Corollary 5.9. *Let A and B be closed subsets of a complete metric space (X, d) such that $A_0 \neq \varnothing$ and let $F : A^2 \to B$ be a mapping such that the following properties hold.*

(a) $F(A_0^2) \subseteq B_0$.
(b) *F is continuous.*
(c) *(A, B) has the weak* P*-property.*
(d) *There exists $\psi \in \Psi$ such that*

$$d(F(x, y), F(u, v)) \leq \max\{d(x, u), d(y, v)\} - \psi(\max\{d(x, u), d(y, v)\})$$

for all $x, y, u, v \in A$.

Then the mapping F has a unique coupled best proximity point.

Corollary 5.10. *Let A and B be closed subsets of a complete metric space (X, d) such that $A_0 \neq \varnothing$ and let $F : A^2 \to B$ be a mapping such that the following properties hold.*

(a) $F(A_0^2) \subseteq B_0$.
(b) *F is continuous.*
(c) *(A, B) has the weak* P*-property.*
(d) *There exists $\psi \in \Psi$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq \max\{d(x, u), d(y, v), d(z, w)\} - \psi(\max\{d(x, u), d(y, v), d(z, w)\})$$

for all $x, y, z, u, v, w \in A$.

Then the mapping F has a unique tripled best proximity point.

Finally, we present a version of Theorem 5.4 in which $A = B = X$.

Corollary 5.11. *Let (X, d) be a complete metric space and let $F : X^n \to X$ be a continuous mapping. Assume that there exists $\psi \in \Psi$ such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \max_{1 \leq i \leq n} d(x_i, y_i) - \psi\left(\max_{1 \leq i \leq n} d(x_i, y_i)\right)$$

for all points $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ verifying

$$\max_{1 \leq i \leq n} d(x_i, F(x_i, x_{i+1}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1})) \leq \max_{1 \leq i \leq n} d(x_i, y_i).$$

Then F has a unique n-dimensional fixed point, that is, there exist unique $z_1, z_2, \ldots, z_n \in X$ such that

$$z_i = F(z_i, z_{i+1}, \ldots, z_n, z_1, z_2, \ldots, z_{i-1}) \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

We left to the reader to particularize the previous corollary to the cases $n = 2$ and $n = 3$.

References

[1] M. Abbas, W. Sintunavarat and P. Kumam, *Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces*, Fixed Point Theory Appl. **2012:31** (2012).

[2] R. P. Agarwal and E. Karapınar, *Remarks on some coupled fixed point theorems in G-metric spaces*, Fixed Point Theory Appl. **2013: 2** (2013).

[3] R. P. Agarwal, E. Karapınar, and A. Roldán, *Fixed point theorems in quasi-metric spaces and applications to multidimensional fixed points on G^*-metric spaces*, To appear in *J. Nonlinear Convex Anal.*

[4] A. Almeida, A. Roldán and K. Sadarangani, *On existence and uniqueness of best proximity points under a Popescu's type contractivity condition.* To appear in *J. Nonlinear Convex Anal.*

[5] S. A. Al-Mezel, H. H. Alsulami, E. Karapınar and A. Roldán, *Discussion on "Multidimensional Coincidence Points" via recent publications*, Abstr. Appl. Anal., Article ID **287492**, 13 pages (2014).

[6] H. Aydi, B. Damjanović, B. Samet and W. Shatanawi, *Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces*, Math. Comput. Modelling **54** (2011), 2443–2450.

[7] V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces* Nonlinear Anal. **74** (2011), 4889–4897.

[8] M. Berzig and B. Samet, *An extension of coupled fixed point's concept in higher dimension and applications*, Comput. Math. Appl. **63** (2012), 1319–1334.

[9] Y. J. Cho, A. Roldán, J. Martínez-Moreno, and C. Roldán, *Coupled Coincidence Point Theorems in (intuitionistic) fuzzy normed spaces*, J. Ineq. Appl., Article ID **2013:104** (2013).

[10] B. S. Choudhury and P. Maity, *Coupled fixed point results in generalized metric spaces*, Math. Comput. Modelling **54** (2011), 73–79.

[11] T. Gnana-Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal **65** (2006), 1379–1393.

[12] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. **11** (1987), 623–632.

[13] M. Jleli and B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appl. **2012:210** (2012)

[14] E. Karapınar, *Quadruple fixed point theorems for weak ϕ-contractions*, ISRN Mathematical Analysis, 2011, Article ID **989423**, 16 pages (2011).

[15] E. Karapınar and A. Roldán, *Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under (ψ,φ)-contractivity conditions*, Fixed Point Theory Appl. **2013:158** (2013).

[16] E. Karapınar and A. Roldán,*A note on "n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces"*, J. Ineq. Appl. **2013:567** (2013).

[17] W. A. Kirk, S. Reich and P. Veeramani,*Proximal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim. **24** (2003), 851–862.

[18] P. Kumam, E. Karapınar and A. Roldán, *G-metric spaces in any number of arguments and related fixed point theorems*, Fixed Point Theory Appl. **2014:13** (2014).

[19] A. Roldán and E. Karapınar, *Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under (ψ,ϕ)-contractivity conditions*, Fixed Point Theory Appl. **2013:158** (2013).

[20] A. Roldán, E. Karapınar and P. Kumam, *G-Metric spaces in any number of arguments and related fixed-point theorems*, Fixed Point Theory Appl. **2014:13** (2014).

[21] A. Roldán, J. Martínez-Moreno and C. Roldán, *Multidimensional fixed point theorems in partially ordered complete metric spaces*, J. Math. Anal. Appl. **396** (2012), 536–545.

[22] A. Roldán, J. Martínez-Moreno, C. Roldán and Y. J. Cho, *Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces*, Fuzzy Sets Syst., D.O.I. 10.1016/j.fss.2013.10.009

[23] A. Roldán, J. Martínez-Moreno, C. Roldán and E. Karapınar, *Multidimensional fixed point theorems in partially ordered complete partial metric spaces under (ψ,φ)-contractivity conditions*, Abstr. Appl. Anal., Article ID **634371** (2013)

[24] A. Roldán, J. Martínez-Moreno, C. Rol.dán and E. Karapınar, *Meir-Keeler type multidimensional fixed point theorems in partially ordered complete metric spaces*, Abstr. Appl. Anal., Article ID: 406026 (2013)

[25] S. Sadiq Basha, *Best proximity points: global optimal aproximate solutions*, J. Glob. Optim. **49** (2011), 15–21.

[26] V. Sankar Raj, *A best proximity theorem for weakly contractive non-self mappings*, Nonlinear Anal. **74** (2011), 4804–4808.

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