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FENCHEL-LAGRANGE DUALITY FOR DC PROGRAMS WITH COMPOSITE FUNCTIONS

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This paper was dedicated to Wataru Takahashi in honor of his 70th birthday.

Abstract. In this paper, we present some duality results for a DC programming problem (P) involving a composite function. To this end, by using the standard convexification technique, we first introduce a Fenchel-Lagrange dual problem for (P). Then, under a closedness qualification condition, we prove the existence of the strong duality for (P). The strong duality result is then applied to obtain an extended Farkas lemma and an alternative type theorem for (P). Moreover, as applications of these results, a composed convex optimization problem, a DC optimization problem, and a convex optimization problem with a linear operator are examined at the end of this paper.

1. Introduction

The dual method is an elegant and powerful ones to deal with an optimization problem since many primal problems can be studied via their dual problems. As we know, a challenge in convex analysis has been to give sufficient conditions which guarantee the strong duality, i.e. the situation when the values of the primal problem and the dual problem coincide, and the dual problem has at least an optimal solution. In last decades, the convex optimization problems have received extensive attentions and various regularity conditions have been proposed and applied to establish the duality results; see [1, 2, 3, 8, 14, 15, 17, 20, 28] and the references therein.

Recently, the DC programming problem (that is the objective function and/or the constraint in the problem is difference of two convex functions) has received much attention. And various dual schemes are proposed for different kinds of DC optimization problems; see [7, 11, 12, 13, 16, 21, 22, 23, 24, 25, 27] and the references therein. Here, we specially mention the works on duality defined via convexification techniques due to [7, 13, 16]. By using the interiority condition, Boţ et al. [7] investigated Fenchel-Lagrange duality results and extended Farkas lemmas for DC programming with DC objective functions and finitely many DC inequality constraints. By using the properties of the epigraph of the conjugate functions,

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Fang et al. [13] investigated some equivalent statements for weak, strong and stable Lagrange dualities and extended Farkas lemmas for DC infinite programs with DC constraint. Fang et al. [16] introduced some new constraint qualifications and obtained some complete characterizations of stable and total Fenchel dualities for DC optimization problems in locally convex spaces.

On the other hand, composed convex programming has also received considerable attention since it offers a unified framework for treating different kinds of optimization problems and many optimization problems generated practical fields like location and transports or economics and finance involve composed convex functions. Because of this important property, many important results have been established for composed convex optimization problems under various conditions in the last decades; see [4, 5, 6, 9, 10, 18, 19] and the references therein. Here we just mention some work of [4, 5, 19]. In [4], Boţ et al. obtained some equivalent statements for the formulae of the conjugate function of the sum of a convex function and a composite convex function in separated locally convex spaces. In [5], Boţ et al. obtained two generalized Moreau-Rockafellar-type results for the sum of a convex function and a composite convex function in separated locally convex spaces. By using the properties of the epigraph of the conjugated functions, Li et al. [19] obtained some necessary and sufficient conditions for the stable strong and total dualities of a composed convex optimization problem.

Motivated by the works mentioned above, in this paper, we consider an optimization problem which contains constrained DC optimization problem, convex optimization problem with a linear operator and composed convex optimization problem as special cases. The purpose of this paper is to establish duality results for this optimization problem. We make three key contributions in this research. First, we establish a Fenchel-Lagrange dual problem for this optimization problem which provides a new generalization of the celebrated Fenchel-Lagrange dual problem for convex and DC optimization problems [2, 4, 5, 6, 21, 25]. Then, we obtain Fenchel-Lagrange duality results and extended Farkas lemmas for this optimization problem by using a closedness qualification condition. As an application of these results, we obtain sufficient conditions for an alternative type theorem. Moreover, the results obtained here underline the connections that exist between Farkas lemmas and alternative type theorems and, on the other hand, the duality.

The paper is organized as follows. In Sect. 2, we recall some notions and give some preliminary results. In Sect. 3, we introduce an generalized optimization problem and construct its dual problem. Then, we prove the weak duality and strong duality. By using the duality assertions, we also obtain some extended Farkas lemmas for this problem. In Sect. 4, we give some special cases of our general results, which have been treated in the previous papers.

2. Mathematical preliminaries

Throughout this paper, let X and Y be two real locally convex Hausdorff topological vector spaces with their dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. Let $K \subseteq Y$ be a nonempty closed convex cone which defined the partial order " $\leq_K$" of Y, namely:

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in K, \text{ for any } y_1, y_2 \in Y.$$

We attach an element $\infty_Y \notin Y$ which is a greatest element with respect to " $\leq_K$" and let $Y^\bullet := Y \cup \{\infty_Y\}$. Then, for any $y \in Y^\bullet$, one has $y \leq_K \infty_Y$ and we define the following operations on $Y^\bullet$:

$$y + (\infty_Y) = (\infty_Y) + y = \infty_Y \text{ and } t(\infty_Y) = \infty_Y, \text{ for any } y \in Y \text{ and } t \geq 0.$$

Moreover, let D be a set in X, the interior (resp. closure, convex hull, convex cone hull) of D is denoted by int D (resp. cl D, co D, cone D). Thus if $W \subseteq X^*$, then cl W denotes the weak* closure of W. We shall adopt the convention that cone $D = \{0\}$ when D is an empty set. Let $D^* := \{x^* \in X^* : \langle x^*, x\rangle \geq 0, \forall x \in X\}$ be the dual cone of D. The indicator function $\delta_D : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ of X is defined by

$$\delta_D(x) := \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{if } x \notin D. \end{cases}$$

Let $f : X \to \overline{\mathbb{R}}$ be an extended real valued function. The effective domain and the epigraph are defined by

$$\text{dom } f := \{x \in X : f(x) < +\infty\}$$

and

$$\text{epi } f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\},$$

respectively. f is said to be proper, iff its effective domain is nonempty and $f(x) > -\infty$. The conjugate function $f^* : X^* \to \overline{\mathbb{R}}$ of f is defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}.$$

Let $A : X \to Y$ be a linear continuous mapping. The adjoint mapping $A^* : Y^* \to X^*$ of A is defined by

$$\langle A^* y^*, x\rangle := \langle y^*, Ax\rangle, \text{ for any } (x, y^*) \in X \times Y^*.$$

The infimal function $Af : Y \to \overline{\mathbb{R}}$ of f through A is defined by

$$Af(y) := \inf \{f(x) : x \in X, Ax = y\}, \text{ for any } y \in Y.$$

By convention, if $\{f(x) : x \in X, Ax = y\}$ is empty, then $Af(y) = \emptyset$.

Moreover, let $G : X \to Y^\bullet$ be an extended vector valued function. The domain and the K-epigraph of G are defined by

$$\text{dom } G := \{x \in X : G(x) \in Y\},$$

and

$$\text{epi}_K\, G := \{(x, y) \in X \times Y : y \in G(x) + K\},$$

respectively. G is said to be proper, iff dom $G \neq \emptyset$. G is said to be a K-convex function, iff for any $x, y \in X$ and $t \in [0, 1]$, we have

$$G(tx + (1 - t)y) \leq_K tG(x) + (1 - t)G(y).$$

For any subset $W \subseteq Y$, we denote

$$G^{-1}(W) := \{x \in X : \text{ there exists } y \in W \text{ such that } G(x) = y\}.$$

Moreover, let $\lambda \in K^*$. The function $(\lambda G) : X \to \overline{\mathbb{R}}$ is defined by

$$(\lambda G)(x) := \begin{cases} \langle \lambda, G(x) \rangle, & \text{if } x \in \text{dom } G, \\ +\infty, & \text{otherwise.} \end{cases}$$

We say that G is star K-lower semicontinuous, iff (λG) is lower semicontinuous, for any $\lambda \in K^*$.

Now, let us recall the following results which will be used in the following section.

Lemma 2.1 ([2]). *Let $f_1, f_2 : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that* dom $f_1 \cap$ dom $f_2 \neq \emptyset$. *Then, the following relation holds*

$$\text{epi } (f_1 + f_2)^* = \text{cl } (\text{epi } f_1^* + \text{epi } f_2^*),$$

where the closure is taken in the product topology of $(X^*, \tau) \times \mathbb{R}$, for any locally convex topology τ on X^* giving X as dual.

Lemma 2.2 ([26]). *Let $f_1, f_2 : X \to \overline{\mathbb{R}}$ be two proper, convex and lower semicontinuous functions. Then*

$$\inf_{x \in X} \{f_1(x) - f_2(x)\} = \inf_{x^* \in X^*} \{f_2^*(x^*) - f_1^*(x^*)\}.$$

3. Main results

In this paper, we deal with a new class of DC programming involving a composite function given in the following form:

$$(P) \ \inf_{x \in X} \Big\{ f(x) + g \circ G(x) - h(x) \Big\},$$

where $f, h : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ are three proper, convex and lower semicontinuous functions, and $G : X \to Y^\bullet$ is a proper, K-convex and star K-lower semicontinuous function. Let $G(\text{dom } f \cap \text{dom } h) \cap \text{dom } g \neq \emptyset$. Moreover, we assume that g is a K-increasing function, that is,

$$\text{for any } x, y \in Y \text{ such that } x \leq_K y, \text{ we have } g(x) \leq g(y).$$

Now, we first construct the dual problems of (P), and then present the duality assertions. By using the duality assertions, we also investigated some extended Farkas lemmas for (P). In order to introduce the dual scheme for (P), we need the following lemma.

Lemma 3.1 ([2]). *For any x feasible to the problem (P), we have*

$$h(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - h^*(x^*)\}.$$

Proof. Since h is proper convex, and lower semicontinuous function, we have

$$h(x) = h^{**}(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - h^*(x^*)\}.$$

This completes the proof. □

Since h is lower semicontinuous, the standard convexification technique can be applied. Then, by Lemma 3.1, the problem (P) can be rewritten as

$$\inf_{x \in X} \Big\{ f(x) + g \circ G(x) - \sup_{x^* \in X^*} \{\langle x^*, x \rangle - h^*(x^*)\} \Big\},$$

which is equivalent to

$$(P)\quad \inf_{x^*\in X^*}\inf_{x\in X}\Big\{f(x)+g\circ G(x)+h^*(x^*)-\langle x^*,x\rangle\Big\}.$$

Note that for any $x^*\in X^*$, the inner infimum of the last formula

$$(P^{x^*})\quad \inf_{x\in X}\Big\{f(x)+g\circ G(x)+h^*(x^*)-\langle x^*,x\rangle\Big\}$$

is a composed convex optimization problem, and its Fenchel-Lagrange dual problem is

$$(D^{x^*})\quad \sup_{\lambda\in K^*,u^*\in X^*}\Big\{h^*(x^*)-f^*(u^*)-g^*(\lambda)-(\lambda G)^*(x^*-u^*)\Big\}.$$

Thus, this reformulation motivates us to define the following dual problem of (P):

$$(D)\quad \inf_{x^*\in X^*}\sup_{\lambda\in K^*,u^*\in X^*}\Big\{h^*(x^*)-f^*(u^*)-g^*(\lambda)-(\lambda G)^*(x^*-u^*)\Big\}.$$

Here and throughout this paper, following Boţ [2], Fang et al. [16] and Zălinescu [28], we adapt the convention that

$$(+\infty)-(+\infty)=(-\infty)-(-\infty)=(+\infty)+(-\infty)=(-\infty)+(+\infty)=+\infty,$$

$$0\cdot(+\infty)=+\infty \text{ and } 0\cdot(-\infty)=0.$$

Now, we will study the weak and strong dualities between (P) and (D). For the optimization problem (P), we denote by $val(P)$ its optimal objective value, and this notation is extended to the optimization problems that we use in this paper.

Definition 3.2. We say that

(i) the weak duality between (P) and (D) holds, iff $val(P)\geq val(D)$.
(ii) the strong duality between (P) and (D) holds, iff $val(P)=val(D)$, and for any $x^*\in X^*$ satisfying $val(D^{x^*})=val(D)$, the dual problem (D^{x^*}) has an optimal solution.

Remark 3.3. It is easy to see that the strong duality between (P) and (D) holds if and only if

$$val(P)=\inf_{x^*\in X^*}\max_{\lambda\in K^*,u^*\in X^*}\Big\{h^*(x^*)-f^*(u^*)-g^*(\lambda)-(\lambda G)^*(x^*-u^*)\Big\}.$$

Theorem 3.4. *The weak duality between (P) and (D) is fulfilled, namely, $val(P)\geq val(D)$.*

Proof. Let $x\in\mathcal{A}:=\operatorname{dom} h^*\cap\operatorname{dom} f^*\cap G^{-1}(\operatorname{dom} g^*)$. For any $x\in\mathcal{A}$, $\lambda\in K^*$, and $x^*,\ u^*\in X^*$, by the definition of conjugate functions, one has

$$\begin{aligned}
&\ h^*(x^*)-f^*(u^*)-g^*(\lambda)-(\lambda G)^*(x^*-u^*)\\
\leq&\ h^*(x^*)-\langle u^*,x\rangle+f(x)-\langle\lambda,G(x)\rangle+g(G(x))-\langle x^*-u^*,x\rangle+(\lambda G)(x)\\
=&\ h^*(x^*)-\langle x^*,x\rangle+f(x)+g(G(x)).
\end{aligned}$$

Thus, for any $x \in \mathcal{A}$, we have

$$\begin{aligned}
& \inf_{x^* \in X^*} \sup_{\lambda \in K^*, u^* \in X^*} \Big\{ h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \Big\} \\
\leq \; & \inf_{x^* \in X^*} \{ h^*(x^*) - \langle x^*, x \rangle \} + f(x) + g(G(x)) \\
= \; & -h^{**}(x) + f(x) + g(G(x)) \\
= \; & f(x) + g(G(x)) - h(x),
\end{aligned}$$

which means that $val(P) \geq val(D)$, and the proof is complete. □

In order to obtain the strong duality assertions between (P) and (D), we introduce the following closedness condition.

Definition 3.5 ([5])**.** The problem (P) is said to satisfy the closedness qualification condition (CQC), if the set

$$\text{(CQC) epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{ epi } (\lambda G)^* + (0, g^*(\lambda)) \Big),$$

is weak* closed in the space $X^* \times \mathbb{R}$.

The next lemma provides several characterizations of the closedness qualification condition (CQC). Moreover, the condition will be crucial in the sequel and it also deserves some attention for its independent interest.

Lemma 3.6 ([5])**.** *The condition (CQC) holds if and only if for any $x^* \in X^*$,*

$$(f + g \circ G)^*(x^*) = \min_{\lambda \in K^*, u^* \in X^*} \Big\{ f^*(u^*) + g^*(\lambda) + (\lambda G)^*(x^* - u^*) \Big\}.$$

Theorem 3.7. *If the condition (CQC) is fulfilled, then, the strong duality between (P) and (D) holds, namely, $val(P) = val(D)$, and for any $x^* \in X^*$ satisfying $val(D^{x^*}) = val(D)$, the dual problem (D^{x^*}) has an optimal solution.*

Proof. It follows from Lemma 2.2 that

$$\inf_{x \in X} \Big\{ f(x) + g \circ G(x) - h(x) \Big\} = \inf_{x^* \in X^*} \{ h^*(x^*) - (f + g \circ G)^*(x^*) \}. \tag{3.1}$$

By Lemma 3.6 and the condition (CQC), we get

$$(f + g \circ G)^*(x^*) = \min_{\lambda \in K^*, u^* \in X^*} \Big\{ f^*(u^*) + g^*(\lambda) + (\lambda G)^*(x^* - u^*) \Big\}.$$

This equality and (3.1) lead to

$$\begin{aligned}
val(P) &= \inf_{x \in X} \Big\{ f(x) + g \circ G(x) - h(x) \Big\} \\
&= \inf_{x^* \in X^*} \Big\{ h^*(x^*) - \min_{\lambda \in K^*, u^* \in X^*} \Big\{ f^*(u^*) + g^*(\lambda) + (\lambda G)^*(x^* - u^*) \Big\} \Big\} \\
&= \inf_{x^* \in X^*} \max_{\lambda \in K^*, u^* \in X^*} \Big\{ h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \Big\}.
\end{aligned}$$

Thus, by Remark 3.3, the strong duality between (P) and (D) holds, and the proof is complete. □

By using the duality assertions presented in the previous theorems, we can obtain the following extended Farkas lemmas.

Theorem 3.8. *If the condition* (CQC) *is satisfied, then, for any* $\alpha \in \mathbb{R}$, *the following statements are equivalent:*

(i) $f(x) + g \circ G(x) - h(x) \geq \alpha$, $\forall x \in X$.

(ii) $(0, -\alpha) + \text{epi } h^* \subseteq \text{epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{epi } (\lambda G)^* + (0, g^*(\lambda))\Big)$.

(iii) *For any* $x^* \in X^*$, *there exist* $\lambda \in K^*$ *and* $u^* \in X^*$, *such that*

$$h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \geq \alpha.$$

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds. Then, for any $x \in X$, $(f + g \circ G)(x) \geq h(x) + \alpha$. This follows that $(h + \alpha)^* \geq (f + g \circ G)^*$. In turn, this gives that

$$(0, -\alpha) + \text{epi } h^* = \text{epi } (h + \alpha)^* \subseteq \text{epi } (f + g \circ G)^*.$$

Moreover, since the condition (CQC) is satisfied, it follows from [5, Corollary 3.6] that

$$\begin{aligned} \text{epi } (f + g \circ G)^* &= \text{cl } \left(\text{epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{epi } (\lambda G)^* + (0, g^*(\lambda))\Big)\right) \\ &= \text{epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{epi } (\lambda G)^* + (0, g^*(\lambda))\Big). \end{aligned}$$

Thus,

$$(0, -\alpha) + \text{epi } h^* \subseteq \text{epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{epi } (\lambda G)^* + (0, g^*(\lambda))\Big),$$

and (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. As $(x^*, h^*(x^*)) \in \text{epi } h^*$, by **(ii)**, we have

$$(x^*, h^*(x^*) - \alpha) \in \text{epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{epi } (\lambda G)^* + (0, g^*(\lambda))\Big).$$

Then, there exist $\lambda \in \text{ dom } g^* \subseteq K^*$, $(u^*, \alpha_1) \in \text{epi } f^*$, $(v^*, \alpha_2) \in \text{epi } (\lambda G)^*$ such that

$$(x^*, h^*(x^*) - \alpha) = (u^*, \alpha_1) + (v^*, \alpha_2) + (0, g^*(\lambda)),$$

which means that

$$x^* = u^* + v^* \tag{3.2}$$

and

$$h^*(x^*) - \alpha = \alpha_1 + \alpha_2 + g^*(\lambda). \tag{3.3}$$

Since $f^*(u^*) \leq \alpha_1$ and $(\lambda G)^*(v^*) \leq \alpha_2$, it follows from (3.2) and (3.3) that

$$\begin{aligned} h^*(x^*) - \alpha &\geq f^*(u^*) + (\lambda G)^*(v^*) + g^*(\lambda) \\ &= f^*(u^*) + (\lambda G)^*(x^* - u^*) + g^*(\lambda). \end{aligned}$$

Thus,

$$h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \geq \alpha,$$

and (iii) holds.

(iii) $\Rightarrow$ (i). Suppose that (iii) holds. Then, for any $x^* \in X^*$, there exist $\lambda \in S^*$ and $u^* \in X^*$ such that

$$h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \geq \alpha,$$

which implies that

$$\sup_{\lambda \in K^*, u^* \in X^*} \Big\{ h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \Big\} \geq \alpha.$$

Therefore, it comes that

$$\inf_{x^* \in X^*} \sup_{\lambda \in K^*, u^* \in X^*} \Big\{ h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \Big\} \geq \alpha.$$

Then,

$$val(D) \geq \alpha.$$

By Theorem 3.4, we obtain that

$$val(P) \geq \alpha,$$

and the proof is complete. □

The previous result can be reformulated as a theorem of the alternative in the following way.

Corollary 3.9. *Suppose that the condition (CQC) holds. Then, for any $\alpha \in \mathbb{R}$, precisely one of the following statements is true*

(i) *There exists $x \in X$, such that $f(x) + g \circ G(x) - h(x) < \alpha$.*

(ii) *For any $x^* \in X^*$, there exist $\lambda \in K^*$ and $u^* \in X^*$, such that*

$$h^*(x^*) - f^*(u^*) - g^*(\lambda) - (\lambda G)^*(x^* - u^*) \geq \alpha.$$

4. The special cases

In this section, we will give some special cases of our general results, which have been treated in the previous papers.

4.1. **A composed convex optimization problem.**

When $h(x) = 0$, (P) becomes the following composed convex optimization problem:

$$(P_1) \qquad \inf_{x \in X} \Big\{ f(x) + g \circ G(x) \Big\}.$$

Since

$$h^*(x^*) = \begin{cases} 0, & \text{if } x^* = 0, \\ +\infty, & \text{if } x^* \neq 0, \end{cases}$$

the dual problem of (P_1) is

$$(D_1) \quad \sup_{\lambda \in K^*, u^* \in X^*} \Big\{ -f^*(u^*) - g^*(\lambda) - (\lambda G)^*(-u^*) \Big\}.$$

As some consequences of the results which have been treated in Section 3, we obtain the following results for (P_1). In this particular case, the following results coincide with the results obtained in [2, 5].

Theorem 4.1. *The weak duality between (P_1) and (D_1) is fulfilled, namely $val(P_1) \geq val(D_1)$.*

Theorem 4.2. *If the condition (CQC) is fulfilled, then $val(P_1) = val(D_1)$, and the dual problem (D_1) has an optimal solution.*

Theorem 4.3. *If the condition (CQC) is satisfied, then, for any $\alpha \in \mathbb{R}$, the following statements are equivalent:*

(i) $f(x) + g \circ G(x) \geq \alpha$, $\forall x \in X$.

(ii) $(0, -\alpha) \subseteq \text{epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{epi } (\lambda G)^* + (0, g^*(\lambda)) \Big)$.

(iii) *There exist $\lambda \in K^*$ and $u^* \in X^*$, such that*

$$-f^*(u^*) - g^*(\lambda) - (\lambda G)^*(-u^*) \geq \alpha.$$

The previous result can be reformulated as a theorem of the alternative in the following way.

Corollary 4.4. *Suppose that the condition (CQC) holds. Then, for any $\alpha \in \mathbb{R}$, precisely one of the following statements is true*

(i) *There exists $x \in X$, such that $f(x) + g \circ G(x) < \alpha$.*

(ii) *There exist $\lambda \in K^*$ and $u^* \in X^*$, such that*

$$-f^*(u^*) - g^*(\lambda) - (\lambda G)^*(-u^*) \geq \alpha.$$

4.2. **A constrained DC optimization problem.** In this subsection, let $C \subseteq X$ be a nonempty closed convex set and $\phi : X \to \overline{\mathbb{R}}$ be a proper, convex, lower semicontinuous function. Now, we intend to apply our results to the case when $f = \phi + \delta_C$ and $g = \delta_{\{-K\}}$. Obviously, g is a proper, convex, lower semicontinuous and K-increasing function, while the feasibility condition becomes $G(\text{dom } f) \cap (-K) \neq \emptyset$. Then, (P) becomes the following DC optimization problem:

$$(P_2) \qquad \inf_{\substack{x \in C, \\ G(x) \in -K}} \{\phi(x) - h(x)\}.$$

Since $g^* = \delta_{K^*}$, we obtain that dom $g^* = K^*$. Then, the condition (CQC) becomes

$$(\text{CQC})_1 \text{ epi } (\phi + \delta_C)^* + \bigcup_{\lambda \in K^*} \text{ epi } (\lambda G)^* \text{ is weak}^* \text{ closed in the space } X^* \times \mathbb{R}.$$

Moreover, the dual problem of (P_2) is

$$(D_2) \inf_{x^* \in X^*} \max_{\lambda \in K^*, u^* \in X^*} \Big\{ h^*(x^*) - (\phi + \delta_C)^*(u^*) - (\lambda G)^*(x^* - u^*) \Big\}.$$

It is worth mentioning that the following results were recently obtained in [2, 11, 12, 23, 25] under some similar closedness qualification conditions.

Theorem 4.5. *The weak duality between (P_2) and (D_2) is fulfilled, namely $val(P_2) \geq val(D_2)$.*

Theorem 4.6. *If the condition $(CQC)_1$ is fulfilled, then $val(P_2) = val(D_2)$.*

Theorem 4.7. *If the condition $(CQC)_1$ is satisfied, then, for any $\alpha \in \mathbb{R}$, the following statements are equivalent:*

(i) $x \in C, G(x) \in -K \Rightarrow \phi(x) - h(x) \geq \alpha$.

(ii) $(0, -\alpha) + \text{epi } h^* \subseteq \text{epi } (\phi + \delta_C)^* + \bigcup_{\lambda \in K^*} \text{ epi } (\lambda G)^*$.

(iii) *For any $x^* \in X^*$, there exist $\lambda \in K^*$ and $u^* \in X^*$, such that*

$$h^*(x^*) - (\phi + \delta_C)^*(u^*) - (\lambda G)^*(x^* - u^*) \geq \alpha.$$

The previous result can be reformulated as a theorem of the alternative in the following way.

Corollary 4.8. *Suppose that the condition $(CQC)_1$ holds. Then, for any $\alpha \in \mathbb{R}$, precisely one of the following statements is true*

(i) *There exists $x \in C, G(x) \in -K$, such that $f(x) - h(x) < \alpha$.*

(ii) *For any $x^* \in X^*$, there exist $\lambda \in K^*$ and $u^* \in X^*$, such that*

$$h^*(x^*) - (\phi + \delta_C)^*(u^*) - (\lambda G)^*(x^* - u^*) \geq \alpha.$$

4.3. **A convex optimization problem with a linear operator.** Let $h \equiv 0$ and $G(x) = Ax$, for any $x \in X$, where $A : X \to Y$ is a linear continuous mapping. Taking $K = \{0\}$, one has that G is a K-convex function and $K^* = Y^*$. So, the problem (P) becomes

$$(P_3) \ \inf_{x \in X} \{f(x) + g(A(x))\}.$$

Since

$$(\lambda G)^*(-u^*) = \begin{cases} 0, & \text{if } A^*\lambda = -u^*, \\ +\infty, & \text{otherwise,} \end{cases}$$

the dual problems of (P_3) is

$$(D_3) \ \sup_{\lambda \in Y^*} \Big\{ - f^*(-A^*\lambda) - g^*(\lambda) \Big\}.$$

Moreover, since

$$\text{epi } (\lambda G)^* = \{(u^*, r) \in X^* \times \mathbb{R} : u^* = A^*\lambda, r \geq 0\},$$

we get

$$\text{epi } f^* + \bigcup_{\lambda \in \text{ dom } g^*} \Big(\text{epi } (\lambda G)^* + (0, g^*(\lambda)) \Big) = \text{epi } f^* + (A^* \times \text{id}_{\mathbb{R}}) \text{ epi } g^*,$$

where $\text{id}_{\mathbb{R}}$ denotes the identity mapping on $\mathbb{R}$. Thus, the condition (CQC) becomes in this special case:

$$(RC_A) \text{ epi } f^* + (A^* \times \text{id}_{\mathbb{R}}) \text{ epi } g^* \text{ is weak}^* \text{ closed in the space } X^* \times \mathbb{R}.$$

In this particular case, it is worth mentioning that the results given in Theorem 4.11 and Corollary 4.12 are new while the ones in Theorems 4.9 and 4.10 have been established recently in [2, 4, 5, 6].

Theorem 4.9. *The weak duality between (P_3) and (D_3) is fulfilled, namely $val(P_3) \geq val(D_3)$.*

Theorem 4.10. *If the condition (RC_A) is satisfied, then $val(P_3) = val(D_3)$, and the dual problem (D_3) has an optimal solution.*

Theorem 4.11. *If the condition (RC_A) is satisfied, then, for any $\alpha \in \mathbb{R}$, the following statements are equivalent:*

(i) $x \in X \Rightarrow f(x) + g(A(x)) \geq \alpha$.

(ii) $(0, -\alpha) \subseteq \text{epi } f^* + (A^* \times \text{id}_{\mathbb{R}}) \text{ epi } g^*$.

(iii) *There exists* $\lambda \in Y^*$*, such that* $-f^*(-A^*\lambda) - g^*(\lambda) \geq \alpha$.

The previous result can be reformulated as a theorem of the alternative in the following way.

Corollary 4.12. *Suppose that the condition* (RC_A) *holds. Then, for any* $\alpha \in \mathbb{R}$*, precisely one of the following statements is true*

(i) *There exists* $x \in X$*, such that* $f(x) + g(A(x))) < \alpha$.

(ii) *There exists* $\lambda \in Y^*$*, such that* $-f^*(-A^*\lambda) - g^*(\lambda) \geq \alpha$.

enumerate

5. Conclusions

In this paper, we investigate some duality results for a generalized optimization problem which contains convex and nonconvex constrained optimization problems as special cases. Our results generalize and rediscover some results given in the past in the literature. Moreover, the results obtained here underline the connections that exist between Farkas lemmas and alternative type theorems and, on the other hand, the duality.

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