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ON STRONG CONVERGENCE OF VISCOSITY TYPE METHOD USING AVERAGED TYPE MAPPINGS

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper we introduce a viscosity type algorithm to strongly approximate solutions of variational inequalities in the setting of Hilbert spaces. Moreover, depending on the hypothesis on the coefficients involved on the scheme, these solutions are common fixed points of a nonexpansive mapping and of a L-hybrid mapping or fixed points of one of them.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, which induces the norm $\| \cdot \|$.
Let C be a nonempty, closed and convex subset of H. Let T be a nonlinear mapping of C into itself; we denote with $Fix(T)$ the set of fixed points of T, that is, $Fix(T) = \{z \in C : Tz = z\}$.
In this paper, to approximate solutions of variational inequalities, we introduce a viscosity type iterative method involving a nonexpansive mapping and a L-hybrid mapping.
K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi [2] first introduced the class of L-hybrid mappings in Hilbert spaces. Let $T : H \to H$ be a mapping and $L \geq 0$ a nonnegative number. T is said L-hybrid, signified as $T \in \mathcal{H}_L$, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in H, \tag{1.1}$$

or equivalently

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\left(1 - \frac{L}{2}\right)\langle x - Tx, y - Ty\rangle. \tag{1.2}$$

Notice that for particular choices of L we obtain several important classes of nonlinear mappings. In fact

- $\mathcal{H}_0$ is the class of the nonexpansive mappings;
- $\mathcal{H}_2$ is the class of the nonspreading mappings;
- $\mathcal{H}_1$ is the class of the hybrid mappings.

Moreover

- if $Fix(T) \neq \emptyset$, each L-hybrid mapping is quasi-nonexpansive mapping (see, [2]), i.e.

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C \quad \text{and} \quad p \in Fix(T);$$

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- the set of fixed points of a quasi-nonexpansive mapping is closed and convex (see, [7]);
- if $T \in \mathcal{H}_L$, then $T_\delta := (1-\delta)I + \delta T$ belongs to $\mathcal{H}_{\frac{L}{\delta}}$ for $\delta > 0$ (see, [1]).

The problem of finding fixed points of a nonlinear mappings has been widely investigated by many authors.
Iemoto and Takahashi [8] approximated common fixed points of a nonexpansive mapping T and of a nonspreading mapping S in a Hilbert space. In particular, they obtained the weak convergence of the following iterative method based on Moudafi's iterative scheme [13]:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n S x_n + (1-\beta_n)T x_n], \end{cases}$$

for all $n \in \mathbb{N}$, where $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}} \subset [0,1]$ satisfies appropriate conditions.

In [5], inspired by Iemoto and Takahashi [8], we introduced an iterative method of Halpern's type to approximate strongly fixed points of a nonexpansive mapping T and a nonspreading mapping S. We obtained the following:

Theorem 1.1. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a nonspreading mapping such that $Fix(S) \cap Fix(T) \neq \emptyset$. Let T_δ and S_δ be the averaged type mappings. Suppose that $(\alpha_n)_{n\in\mathbb{N}}$ is a real sequence in $(0,1)$ satisfying the conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$,

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$, we define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n], \quad n \in \mathbb{N}. \end{cases}$$

Then, the following hold:

(i) *If $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $\overline{p} = P_{Fix(T)}u$ which is the unique solution in $Fix(T)$ of the variational inequality $\langle u-\overline{p}, x-\overline{p}\rangle \le 0$, for all $x \in Fix(T)$.*

(ii) *If $\sum_{n=1}^{\infty} \beta_n < \infty$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $\widehat{p} = P_{Fix(S)}u$ which is the unique solution in $Fix(S)$ of the variational inequality $\langle u-\widehat{p}, x-\widehat{p}\rangle \le 0$, for all $x \in Fix(S)$.*

(iii) *If $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $(x_n)_{n\in\mathbb{N}}$ strongly converges to $p_0 = P_{Fix(T)\cap Fix(S)}u$ which is the unique solution in $Fix(T) \cap Fix(S)$ of the variational inequality $\langle u-p_0, x-p_0\rangle \le 0$, for all $x \in Fix(T) \cap Fix(S)$.*

Let $D : H \to H$ be a nonlinear operator and let C be a nonempty closed and convex subset of H. The variational inequality is formulated as finding a point

$p \in C$ such that

$$\langle Dp, p - y\rangle \leq 0, \quad \forall y \in C, \tag{1.3}$$

Variational inequalities were initially studied by Stampacchia [16] and there after the problem of existence and uniqueness of solutions of (1.3) has been widely investigated by many authors in different disciplines as partial differential equations, optimal control, optimization, mechanics and finance. It is known that the problem (1.3) admits a unique solution if D is β-strongly monotone, i.e.

$$\langle Dx - Dy, x - y\rangle \geq \beta\|x - y\|^2 \quad \forall\, x, y \in H;$$

and if D is ρ-Lipschitz continuous operator, i.e.

$$\|Dx - Dy\| \leq \rho\|x - y\|, \quad \forall\, x, y \in H.$$

G. Marino and L. Muglia in [10] considered a viscosity type iterative method:

$$z_0 \in H,\ z_{n+1} = \alpha_n(I - \mu_n D)z_n + (1 - \alpha_n)W_n z_n$$

where W_n is an appropriate family of mappings and they proved the strong convergence to the unique solution of the variational inequality (1.3) on the set of common fixed points of a family of mappings.

Inspired by [10] and [5], in this paper, we introduce the following viscosity type algorithm

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n(I - \mu_n D)x_n + (1 - \alpha_n)[\beta_n T_\delta x_n + (1 - \beta_n)S_\delta x_n], \quad n \in \mathbb{N}, \end{cases}$$

where $T_\delta = (1 - \delta)I + \delta T$, $S_\delta = (1 - \delta)I + \delta S$, $\delta \in (0, 1)$ are the averaged type mappings with T is a nonexpansive mapping and S is a L-hybrid mapping and D is a β-strongly monotone and a ρ-Lipschitzian operator.

The introduced iterative scheme is very interesting because some well known iterative methods can be obtained by it.

For example, if $\mu_n = \mu$ and $D = \frac{I-u}{\mu}$, where u is the constant contraction, we obtain the algorithm proposed in [5].

Instead, if we consider $\mu_n = \mu$ and $D = \frac{I-f}{\mu}$, where f is a contraction, we have a viscosity method

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[\beta_n T_\delta x_n + (1 - \beta_n)S_\delta x_n]. \tag{1.4}$$

For $\beta_n = 1$ this scheme (1.4) was proposed by A. Moudafi in [12]. He proved the strong convergence of algorithm (1.4) to the unique solution $x^* \in C$ of the variational inequality

$$\langle (I - f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in C,$$

where f is a contraction, in Hilbert spaces.

In [18] Xu extended Moudafi's results in a uniformly smooth Banach space.

Moreover, for $D = (I - \gamma f)$, where f is a contraction with coefficient α and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$, we have

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n)[\beta_n T_\delta x_n + (1 - \beta_n)S_\delta x_n]$$

where $y_n = \mu_n \gamma f(x_n) + (1-\mu_n)x_n$.
If $A : H \to H$ is a strongly positive operator, i.e. there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

and f is a contraction, we can take $D = A - \gamma f$ that is a strongly monotone operator (see, [11]).
In [6], [11], [13], the authors consider iterative methods approximating a fixed point of nonexpansive mappings that is also the unique solution of the variational inequality problem

$$\langle (A-\gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in C. \tag{1.5}$$

(1.5) is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf, i.e., $h'(x) = \gamma f(x)$ for $x \in H$.
Depending on the choice of controll coefficients α_n and β_n, we prove the strong convergence of our iterative method to the unique solution of a variational inequality (1.3) on the set of common fixed points of T and S or on the fixed points of one of them. As in [5], the regularization with the averaged type mappings plays a crucial role for the strong convergence of our iterative method.

2. Preliminaries

To begin, we collect some Lemmas which we use in our proofs in the next section. In the sequel, we denote by H a real Hilbert space, by C a nonempty closed convex subset of H, by T a nonexpansive mapping and by S a L-hybrid mapping.

Lemma 2.1. *The following known results hold:*

1. $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, *for all* $x, y \in H$ *and for all* $t \in [0,1]$.
2. $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in H$.
3. $2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$, *for all* $x, y, z, w \in H$.

To prove our main Theorem, we need some fundamental properties of the involved mappings in the variational inequality.
Let $B_n := I - \mu_n D$ and $\tau := \frac{(2\beta - \mu\delta^2)}{2}$. It is known that B_n is a contraction [19], that is,

$$\|B_n x - B_n y\| \leq (1-\mu_n\tau)\|x-y\|, \forall x, y \in H.$$

The following result summarizes some significant properties of $I - T$ if T is a nonexpansive mapping ([3],[4]).

Lemma 2.2. *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be nonexpansive. Then:*

(1) *$I-T:C\to H$ is $\frac{1}{2}$-inverse strongly monotone, i.e.,*

$$\frac{1}{2}\|(I-T)x-(I-T)y\|^2 \le \langle x-y,(I-T)x-(I-T)y\rangle,$$

for all $x,y\in C$;
in particular, if $y\in Fix(T)\neq\emptyset$, we get,

$$\frac{1}{2}\|x-Tx\|^2 \le \langle x-y,x-Tx\rangle; \tag{2.1}$$

(2) *moreover, if $Fix(T)\neq\emptyset$, $I-T$ is demiclosed at 0, i.e. for every sequence $(x_n)_{n\in\mathbb{N}}$ weakly convergent to p such that $x_n-Tx_n\to 0$ as $n\to\infty$, it follows $p\in Fix(T)$.*

If T is a nonexpansive mapping of C into itself, Byrne [3] defined the averaged mapping as follows

$$T_\delta=(1-\delta)I+\delta T=I-\delta(I-T) \tag{2.2}$$

where $\delta\in(0,1)$.
Moreover, Byrne [3] and successively Moudafi [14], proved some properties of the averaged mappings; in particular, they showed that T_δ is a nonexpansive mapping.
In [5], we defined *the averaged type mapping* V_δ as in (2.2) for a nonlinear mapping $V:C\to C$; we notice that $Fix(V)=Fix(V_\delta)$.
It is easy to verify that if S is a L-hybrid mapping of C into itself and $Fix(S)\neq\emptyset$ the averaged type mapping S_δ is quasi-nonexpansive and consequently the set of fixed points of S_δ is closed and convex.
The following Lemma shows the demiclosedness of $I-S$ at 0.

Lemma 2.3 ([2])**.** *Let C be a nonempty, closed and convex subset of H. Let $S:C\to H$ be a L-hybrid mapping such that $Fix(S)\neq\emptyset$. Then $I-S$ is demiclosed at* 0.

In the next Lemma we prove a suitable property of $I-S$. We follow the same line in [8] but for completeness we report the proof of Lemma,

Lemma 2.4. *Let C be a nonempty, closed and convex subset of H. Let $S:C\to C$ be a L-hybrid mapping. Then*

$$\begin{aligned}&\|(I-S)x-(I-S)y\|^2\\ &\qquad\le \langle x-y,(I-S)x-(I-S)y\rangle+\frac{1}{2}\Bigg(\|x-Sx\|^2+\|y-Sy\|^2\\ &\qquad\qquad -2\left(1-\frac{L}{2}\right)\langle x-Sx,y-Sy\rangle\Bigg),\end{aligned}$$

for all $x,y\in C$.

Proof. Put $A=I-S$. For all $x,y\in C$ we have

$$\begin{aligned}\|Ax-Ay\|^2 &= \langle Ax-Ay,Ax-Ay\rangle\\ &= \langle (x-y)-(Sx-Sy),Ax-Ay\rangle\\ &= \langle x-y,Ax-Ay\rangle-\langle Sx-Sy,Ax-Ay\rangle.\end{aligned} \tag{2.3}$$

Using , we obtain

$$
\begin{aligned}
2\langle Sx - Sy, Ax - Ay\rangle &= 2\langle Sx - Sy, (x-y) - (Sx - Sy)\rangle \\
&= 2\langle Sx - Sy, x - y\rangle - 2\|Sx - Sy\|^2 \\
\text{(by Lemma (2.1))} \quad &\geq \|x - Sy\|^2 + \|y - Sx\|^2 - \|x - Sx\|^2 - \|y - Sy\|^2 \\
\text{(by (1.2))} \quad &- \left(\|x - Sy\|^2 + \|y - Sx\|^2 - 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle\right) \\
&= -\|x - Sx\|^2 - \|Sy - y\|^2 + 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle \\
&= -\|Ax\|^2 - \|Ay\|^2 + 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle. \qquad (2.4)
\end{aligned}
$$

So, from (2.3) and (2.4), we can conclude

$$
\begin{aligned}
&\|(I-S)x - (I-S)y\|^2 \\
&\quad \leq \langle x - y, (I-S)x - (I-S)y\rangle + \frac{1}{2}\Big(\|x - Sx\|^2 + \|y - Sy\|^2 \\
&\qquad\qquad - 2(1 - \frac{L}{2})\langle x - Sx, y - Sy\rangle\Big).
\end{aligned}
$$

□

Remark 2.5. If $p \in Fix(S) \neq \emptyset$, we have,

$$\|(I-S)x\|^2 \leq \langle x - p, (I-S)x\rangle + \frac{1}{2}\|(I-S)x\|^2,$$

in particular,

$$\langle x - p, (I-S)x\rangle \geq \frac{1}{2}\|(I-S)x\|^2. \qquad (2.5)$$

We prove that the averaged type mapping S_δ is quasi-firmly type nonexpansive mapping, i.e. if there exists $k \in (0, +\infty)$ such that

$$\|S_\delta x - p\|^2 \leq \|x - p\|^2 - k\|x - S_\delta x\|^2, \quad \forall x \in C, \quad \forall p \in Fix(S) \neq \emptyset. \qquad (2.6)$$

The proof is similar to that in [5] for the nonspreading mappings, but we include it for completeness.

Proposition 2.6. *Let C be a nonempty closed and convex subset of H and let $S : C \to C$ be a L-hybrid mapping such that $Fix(S)$ is nonempty. Then the averaged type mapping $S - \delta$*

$$S_\delta = (1-\delta)I + \delta S, \qquad (2.7)$$

is quasi-firmly type nonexpansive mapping with coefficient $k = (1-\delta) \in (0,1)$.

Proof

Proof. We obtain

$$
\begin{aligned}
\|S_\delta x - S_\delta y\|^2 &= \|(1-\delta)(x-y) + \delta(Sx - Sy)\|^2 \\
\text{(by Lemma 2.1)} \quad &= (1-\delta)\|x-y\|^2 + \delta\|Sx - Sy\|^2
\end{aligned}
$$

$$
\begin{aligned}
& - \ \delta\,(1-\delta)\,\|(x-Sx)-(y-Sy)\|^2 \\
\text{(by (1.1))} \ & \leq \ (1-\delta)\,\|x-y\|^2 + \delta\left[\|x-y\|^2 + L\,\langle x-Sx, y-Sy\rangle\right] \\
& - \ \delta\,(1-\delta)\,\|(x-Sx)-(y-Sy)\|^2 \\
& = \ \|x-y\|^2 + \frac{L}{\delta}\,\langle \delta\,(x-Sx)\,, \delta\,(y-Sy)\rangle \\
& - \ \frac{1-\delta}{\delta}\,\|\delta\,(x-Sx) - \delta\,(y-Sy)\|^2 \\
\text{(by (2.7))} \ & = \ \|x-y\|^2 + \frac{L}{\delta}\,\langle x-S_\delta x, y-S_\delta y\rangle \\
& - \ \frac{1-\delta}{\delta}\,\|(x-S_\delta x)-(y-S_\delta y)\|^2 \\
& \leq \ \|x-y\|^2 + \frac{L}{\delta}\,\langle x-S_\delta x, y-S_\delta y\rangle \\
& - \ (1-\delta)\,\|(x-S_\delta x)-(y-S_\delta y)\|^2 .
\end{aligned}
$$

In particular, we have

$$
\|S_\delta x - S_\delta y\|^2 \leq \|x-y\|^2 + \frac{L}{\delta}\langle x-S_\delta x, y-S_\delta y\rangle - (1-\delta)\|(x-S_\delta x)-(y-S_\delta y)\|^2. \tag{2.8}
$$

Observe that S_δ is $\frac{L}{\delta}$-hybrid. Moreover, choosing in (2.8) $y = p$, where $p \in Fix(S) = Fix(S_\delta)$ we obtain

$$
\|S_\delta x - p\|^2 \leq \|x-p\|^2 - (1-\delta)\,\|x-S_\delta x\|^2 . \tag{2.9}
$$

□

A pertinent tool for us is the well-known Lemma of Xu [17].

Lemma 2.7. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$
a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,
$$

where,

- $(\alpha_n)_{n\in\mathbb{N}} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
- $\limsup_{n\to\infty} \sigma_n \leq 0$*;*
- $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then,

$$
\lim_{n\to\infty} a_n = 0.
$$

Finally, the next result is of crucial importance for the techniques used in this paper.

Lemma 2.8. [9] *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $(\gamma_{n_j})_{j\in\mathbb{N}}$ of $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$, for all $j \in \mathbb{N}$. Consider the sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ defined by*

$$\tau(n) := \max\{k \le n : \quad \gamma_k < \gamma_{k+1}\}.$$

Then $(\tau(n))_{n\in\mathbb{N}}$ is a nondecreasing sequence for all $n \ge n_0$, satisfying

- $\lim_{n\to\infty} \tau(n) = \infty$;
- $\gamma_{\tau(n)} < \gamma_{\tau(n)+1}$, $\forall\, n \ge n_0$;
- $\gamma_n < \gamma_{\tau(n)+1}$, $\forall\, n \ge n_0$.

3. Main results

In all section, $(\beta_n)_{n\in\mathbb{N}} \subset [0,1]$ denotes a real sequence and $U_n : C \to C$ denotes the convex combination of T_δ and S_δ, i.e.

$$U_n = \beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n, \quad n \in \mathbb{N}.$$

Further we assume that

- $Fix(S) \cap Fix(T) \neq \emptyset$;
- $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1)$ a real sequence such that $\lim_{n\to\infty} \alpha_n = 0$;
- $O(1)$ is any bounded real sequence;
- $D : H \to H$ is a β-strongly monotone and ρ-Lipschitzian operator;
- $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $(0,\mu)$ such that $\mu < \frac{2\beta}{\rho^2}$.

We start with the two following Lemmas.

Lemma 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by*

$$x_{n+1} = \alpha_n(1-\mu_n D)x_n + (1-\alpha_n)U_n x_n,$$

where

$$U_n = \beta_n T_\delta + (1-\beta_n) S_\delta.$$

Then

1. *U_n is quasi-nonexpansive for all $n \in \mathbb{N}$.*
2. *$(x_n)_{n\in\mathbb{N}}, (Sx_n)_{n\in\mathbb{N}}, (Tx_n)_{n\in\mathbb{N}}, (S_\delta x_n)_{n\in\mathbb{N}}, (T_\delta x_n)_{n\in\mathbb{N}}, (U_n x_n)_{n\in\mathbb{N}}$ are bounded sequences.*

Proof. (1) To simplify the notation, we set

$$B_n := I - \mu_n D. \tag{3.1}$$

We observe that U_n is quasi-nonexpansive, for all $n \in \mathbb{N}$, since T_δ is a nonexpansive mapping and S_δ is a quasi-nonexpansive mapping.

(2) First we prove that $(x_n)_{n\in\mathbb{N}}$ is bounded.
Moreover, we recall that B_n is a contraction.
For $q \in Fix(T) \cap Fix(S)$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n(B_n x_n - q) + (1-\alpha_n)(U_n x_n - q)\| \\
&\le \alpha_n \|B_n x_n - q\| + (1-\alpha_n)\|U_n x_n - q\| \\
(U_n \text{ quasi-nonexpansive}) \quad &\le \alpha_n \|B_n x_n - q\| + (1-\alpha_n)\|x_n - q\|
\end{aligned}$$

$$\begin{aligned}
& \leq \alpha_n\|B_nx_n - B_nq\| + \alpha_n\|B_nq - q\| \\
& \quad +(1-\alpha_n)\|x_n - q\| \\
(B_n \text{ contraction}) \quad & \leq \alpha_n(1-\mu_n\tau)\|x_n - q\| + \alpha_n\|B_nq - q\| \\
& \quad +(1-\alpha_n)\|x_n - q\| \\
& = (1-\alpha_n\mu_n\tau)\|x_n - q\| + \alpha_n\|B_nq - q\| \\
(\text{by } (3.1)) \quad & = (1-\alpha_n\mu_n\tau)\|x_n - q\| + \alpha_n\mu_n\tau\frac{\|Dq\|}{\tau} \qquad (3.2)
\end{aligned}$$

Since

$$\|x_1 - q\| \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\},$$

and by induction we assume that

$$\|x_n - q\| \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\},$$

then

$$\begin{aligned}
\|x_{n+1} - q\| & \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} + \alpha_n\mu_n\tau\frac{\|Dq\|}{\tau} \\
& \leq \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} + \max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\} \\
& = 2\max\left\{\frac{\|Dq\|}{\tau}, \|x_1 - q\|\right\}.
\end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is bounded. Consequently, $(T_\delta x_n)_{n\in\mathbb{N}}$, $(S_\delta x_n)_{n\in\mathbb{N}}$, $(U_nx_n)_{n\in\mathbb{N}}$ and $(B_nx_n)_{n\in\mathbb{N}}$ are bounded as well.

□

Lemma 3.2. *Let C be a nonempty closed and convex subspace of H, let $D : H \to H$ be a β-strongly monotone and ρ-Lipschitzian operator.*

(i) *Let V be a nonlinear mapping from C into itself such that $I-V$ is demiclosed at 0 and $Fix(V) \neq \emptyset$. Consider a bounded sequence $(y_n)_{n\in\mathbb{N}} \subset C$ such that $\|y_n - Vy_n\| \to 0$, as $n \to \infty$, then:*

$$\limsup_{n\to\infty} \langle D\bar{p}, \bar{p} - y_n\rangle \leq 0,$$

where $\bar{p}$ is the unique point in $Fix(V)$ that satisfies the variational inequality

$$\langle D\bar{p}, \bar{p} - x\rangle \leq 0, \quad \forall x \in Fix(V). \qquad (3.3)$$

(ii) *Let V, W be a nonlinear mappings from C into itself such that $I - V$ and $I - W$ are demiclosed at 0 and $Fix(V) \cap Fix(W) \neq \emptyset$. Consider a bounded sequence $(y_n)_{n\in\mathbb{N}} \subset C$ such that $\|y_n - Vy_n\| \to 0$ and $\|y_n - Wy_n\| \to 0$, as $n \to \infty$, then:*

$$\limsup_{n\to\infty} \langle Dp_0, p_0 - y_n\rangle \leq 0,$$

where p_0 is the unique point in $Fix(V) \cap Fix(W)$ that satisfies the variational inequality

$$\langle Dp_0, p_0 - x\rangle \leq 0, \quad \forall x \in Fix(V) \cap Fix(W). \tag{3.4}$$

Proof. (i) Let $\bar{p}$ satisfying (3.3). Let $(y_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(y_n)_{n\in\mathbb{N}}$ for which

$$\limsup_{n\to\infty} \langle D\bar{p}, \bar{p} - y_n\rangle = \lim_{k\to\infty} \langle D\bar{p}, \bar{p} - y_{n_k}\rangle .$$

Select a subsequence $(y_{n_{k_j}})_{j\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ such that $y_{n_{k_j}} \rightharpoonup v$ (this is possible by boundedness of $(y_n)_{n\in\mathbb{N}}$). By the hypothesis $\|y_n - Vy_n\| \to 0$, as $n \to \infty$, and by demiclosedness of $I - V$ at 0 we have $v \in Fix(V)$, and

$$\limsup_{n\to\infty} \langle D\bar{p}, \bar{p} - y_n\rangle = \lim_{j\to\infty} \left\langle D\bar{p}, \bar{p} - y_{n_{k_j}}\right\rangle = \langle D\bar{p}, \bar{p} - v\rangle ,$$

so the claim follows by (3.3).

(ii) Select a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \langle Dp_0, p_0 - y_n\rangle = \lim_{k\to\infty} \langle Dp_0, p_0 - y_{n_k}\rangle ,$$

where p_0 satisfies (3.4). Now select a subsequence $(y_{n_{k_j}})_{j\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ such that $y_{n_{k_j}} \rightharpoonup w$. Then by demiclosedness of $I - V$ and $I - W$ at 0, and by the hypotheses $\|y_n - Vy_n\| \to 0$ and $\|y_n - Wy_n\| \to 0$, as $n \to \infty$, we obtain that $w = Vw = Ww$, i.e. $w \in Fix(V) \cap Fix(W)$. So,

$$\limsup_{n\to\infty} \langle Dp_0, p_0 - y_n\rangle = \lim_{j\to\infty} \left\langle Dp_0, p_0 - y_{n_{k_j}}\right\rangle = \langle Dp_0, p_0 - w\rangle ,$$

so the claim follows by (3.4). □

Now, we prove our main Result.

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a L-hybrid mapping such that $Fix(S) \cap Fix(T) \neq \emptyset$. Let T_δ and S_δ be the averaged type mappings, i.e.*

$$T_\delta = (1 - \delta)I + \delta T,\ S_\delta = (1 - \delta)I + \delta S,\ \delta \in (0, 1).$$

Let $D : H \to H$ be a β-strongly monotone and ρ-Lipschitzian operator. Suppose that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $(0, \mu)$, $\mu < \frac{2\beta}{\rho^2}$, and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $(0, 1)$, satisfying the conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$,

(2) $\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0$,

(3) $\sum_{n=1}^{\infty} \alpha_n\mu_n = \infty$.

If $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $[0, 1]$, we define a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n(I - \mu_n D)x_n + (1 - \alpha_n)[\beta_n T_\delta x_n + (1 - \beta_n)S_\delta x_n], \quad n \in \mathbb{N}. \end{cases}$$

Then, the following hold:

(i) *If* $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, *then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $\overline{p} \in Fix(T)$ *which is the unique solution in* $Fix(T)$ *of the variational inequality* $\langle D\overline{p}, \overline{p} - x\rangle \leq 0$, *for all* $x \in Fix(T)$.

(ii) *If* $\sum_{n=1}^{\infty}\beta_n < \infty$, *then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $\widehat{p} \in Fix(S)$ *which is the unique solution in* $Fix(S)$ *of the variational inequality* $\langle D\widehat{p}, \widehat{p} - x\rangle \leq 0$, *for all* $x \in Fix(S)$.

(iii) *If* $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, *then* $(x_n)_{n\in\mathbb{N}}$ *converges strongly to* $p_0 \in Fix(T) \cap Fix(S)$ *which is the unique solution in* $Fix(T) \cap Fix(S)$ *of the variational inequality* $\langle Dp_0, p_0 - x\rangle \leq 0$, *for all* $x \in Fix(T) \cap Fix(S)$.

Proof of (i). We rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1} = \alpha_n B_n x_n + (1-\alpha_n)T_\delta x_n + (1-\beta_n)E_n, \tag{3.5}$$

where $E_n = (1-\alpha_n)(S_\delta x_n - T_\delta x_n)$ is bounded, i.e. $\|E_n\| \leq O(1)$.
We begin to prove that $\lim_{n\to\infty}\|x_n - T_\delta x_n\| = 0$.
Let $\overline{p}$ the unique solution in $Fix(T) = Fix(T_\delta)$ of the variational inequality

$$\langle D\overline{p}, \overline{p} - x\rangle \leq 0, \quad \forall x \in Fix(T).$$

We have

$$\begin{aligned}
\|x_{n+1} - \overline{p}\|^2 &= \|\alpha_n B_n x_n + (1-\alpha_n)(1-\delta)x_n \\
&\quad +(1-\alpha_n)\delta T x_n + (1-\beta_n)E_n - \overline{p}\|^2 \\
&= \|[(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}] \\
&\quad +[\alpha_n(B_n x_n - x_n) + (1-\beta_n)E_n]\|^2 \\
(\text{ by Lemma 2.1}) \quad &\leq \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
&\quad +2\langle \alpha_n(B_n x_n - x_n) + (1-\beta_n)E_n, x_{n+1} - \overline{p}\rangle \\
&= \|(1-\alpha_n)\delta(Tx_n - x_n) + x_n - \overline{p}\|^2 \\
&\quad +2\alpha_n\langle B_n x_n - x_n, x_{n+1} - \overline{p}\rangle + 2(1-\beta_n)\langle E_n, x_{n+1} - \overline{p}\rangle \\
&\leq (1-\alpha_n)^2\delta^2\|Tx_n - x_n\|^2 + \|x_n - \overline{p}\|^2 \\
&\quad -2(1-\alpha_n)\delta\langle x_n - \overline{p}, x_n - Tx_n\rangle \\
&\quad +2\alpha_n\|B_n x_n - x_n\|\|x_{n+1} - \overline{p}\| + 2(1-\beta_n)\|E_n\|\|x_{n+1} - \overline{p}\| \\
(\text{by (2.1)}) \quad &\leq \|x_n - \overline{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Tx_n\|^2 \\
&\quad -(1-\alpha_n)\delta\|x_n - Tx_n\|^2 + \alpha_n O(1) + (1-\beta_n)O(1) \\
&= \|x_n - \overline{p}\|^2 - (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \\
&\quad +\alpha_n O(1) + (1-\beta_n)O(1)
\end{aligned}$$

and hence

$$\begin{aligned}
0 &\leq (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \\
&\leq \|x_n - \overline{p}\|^2 - \|x_{n+1} - \overline{p}\|^2 + \alpha_n O(1) + (1-\beta_n)O(1). \tag{3.6}
\end{aligned}$$

We turn our attention on the monotony of the sequence $(\|x_n - \overline{p}\|)_{n\in\mathbb{N}}$.
We consider the following two cases.

Case A. $\|x_{n+1} - \overline{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \overline{p}\| < \|x_{n_k+1} - \overline{p}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - \overline{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - \overline{p}\|^2$ exists. From (3.6), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \left((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Tx_n\|^2 \right) \\ &\leq \limsup_{n\to\infty} \Big(\|x_n - \overline{p}\|^2 - \|x_{n+1} - \overline{p}\|^2 \\ &\quad +\alpha_n O(1) + (1-\beta_n)O(1) \Big) = 0, \end{aligned}$$

so, we can conclude that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

and

$$\lim_{n\to\infty} \|x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \tag{3.7}$$

Since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and from $I - T$ is demiclosed at 0, we can use Lemma 3.2 (i), so we get

$$\limsup_{n\to\infty} \langle D\overline{p}, \overline{p} - x_n\rangle \leq 0. \tag{3.8}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\overline{p}$.
We compute

$$\begin{aligned} \|x_{n+1} - \overline{p}\|^2 &\leq \|\alpha_n(B_n x_n - \overline{p}) + (1-\alpha_n)(T_\delta x_n - \overline{p}) + (1-\beta_n)E_n\|^2 \\ \text{(by Lemma 2.1)} &\leq \|(1-\alpha_n)(T_\delta x_n - \overline{p})\|^2 \\ &\quad +2\langle (1-\beta_n)E_n + \alpha_n(B_n x_n - \overline{p}), x_{n+1} - \overline{p}\rangle \\ &\leq (1-\alpha_n)^2\|T_\delta x_n - \overline{p}\|^2 + 2\alpha_n\langle B_n x_n - \overline{p}, x_{n+1} - \overline{p}\rangle \\ &\quad +(1-\beta_n)O(1) \\ (T_\delta \text{ nonexpansive}) &\leq (1-\alpha_n)^2\|x_n - \overline{p}\|^2 + 2\alpha_n\langle B_n x_n - B_n\overline{p}, x_{n+1} - \overline{p}\rangle \\ &\quad +2\alpha_n\langle B_n\overline{p} - \overline{p}, x_{n+1} - \overline{p}\rangle + (1-\beta_n)O(1) \\ (B_n \text{ contraction}) &\leq (1-\alpha_n)^2\|x_n - \overline{p}\|^2 \\ &\quad +2\alpha_n(1-\mu_n\tau)\|x_n - \overline{p}\|\|x_{n+1} - \overline{p}\| \\ &\quad +2\alpha_n\langle B_n\overline{p} - \overline{p}, x_{n+1} - \overline{p}\rangle + (1-\beta_n)O(1) \\ (B_n := (1-\mu_n D)) &\leq (1-\alpha_n)^2\|x_n - \overline{p}\|^2 \end{aligned}$$

$$+\alpha_n(1-\mu_n\tau)\Big[\|x_n-\overline{p}\|^2+\|x_{n+1}-\overline{p}\|^2\Big]$$
$$-2\alpha_n\mu_n\langle D\overline{p},x_{n+1}-\overline{p}\rangle+(1-\beta_n)O(1),$$

Then it follows that

$$\begin{aligned}
\|x_{n+1}-\overline{p}\|^2 \leq\ & \frac{1-(1+\mu_n\tau)\alpha_n+\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}\|x_n-\overline{p}\|^2 \\
& -\frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle D\overline{p},x_{n+1}-\overline{p}\rangle \\
& +\frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1) \\
\leq\ & \frac{1-(1+\mu_n\tau)\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\|x_n-\overline{p}\|^2 \\
& -\frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle D\overline{p},x_{n+1}-\overline{p}\rangle \\
& +\frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1)+\frac{\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}O(1) \\
\leq\ & \left(1-\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\right)\|x_n-\overline{p}\|^2 \\
& +\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\left[-\frac{1}{\tau}\langle D\overline{p},x_{n+1}-\overline{p}\rangle+\frac{\alpha_n}{2\mu_n\tau}O(1)\right] \\
& +\frac{1-\beta_n}{1-(1-\mu_n\tau)\alpha_n}O(1)
\end{aligned}$$

Notice that by

$$\lim_{n\to\infty}\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}=0,$$

it follows that

$$0<\frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}<1, \quad definitively.$$

Moreover, using $\sum_{n=1}^{\infty}\alpha_n\mu_n=\infty$, $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, (3.8) and $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n}=0$, we can apply Lemma 2.7 and conclude that

$$\lim_{n\to\infty}\|x_{n+1}-\overline{p}\|=0.$$

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k}-\overline{p}\|<\|x_{n_k+1}-\overline{p}\| \text{ for all } k\in\mathbb{N}.$$

Then by Maingé Lemma 2.8 there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies

(1) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;
(2) $\lim_{n\to\infty}\tau(n)=\infty$;
(3) $\|x_{\tau(n)}-\overline{p}\|<\|x_{\tau(n)+1}-\overline{p}\|$;

(4) $\|x_n - \overline{p}\| < \|x_{\tau(n)+1} - \overline{p}\|$.

Consequently,

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}\Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\|\Big) \\
&\leq \limsup_{n\to\infty}\Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\|\Big) \\
&\leq \limsup_{n\to\infty}\Big(\|x_{n+1} - \overline{p}\| - \|x_n - \overline{p}\|\Big) \\
\text{(by (3.5))} \quad &= \limsup_{n\to\infty}\Big(\|\alpha_n(B_n x_n - T_\delta x_n) + T_\delta x_n - \overline{p} \\
&\qquad +(1-\beta_n)E_n\| - \|x_n - \overline{p}\|\Big) \\
(T_\delta \text{ nonexpansive}) \quad &\leq \limsup_{n\to\infty}\Big(\alpha_n O(1) + \|x_n - \overline{p}\| \\
&\qquad +(1-\beta_n)O(1) - \|x_n - \overline{p}\|\Big) = 0,
\end{aligned}$$

so

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1} - \overline{p}\| - \|x_{\tau(n)} - \overline{p}\|\Big) = 0. \tag{3.9}$$

By (3.6), we have

$$\begin{aligned}
0 &\leq (1-\alpha_{\tau(n)})\delta[1-\delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\
&\leq \|x_{\tau(n)} - \overline{p}\|^2 - \|x_{\tau(n)+1} - \overline{p}\|^2 + \alpha_n O(1) + (1-\beta_{\tau(n)})O(1),
\end{aligned}$$

from (3.9), $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$ we get

$$\lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0. \tag{3.10}$$

By (3.10), as in the Case A, we have

$$\limsup_{n\to\infty}\langle D\overline{p}, \overline{p} - x_{\tau(n)}\rangle \leq 0$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)} - \overline{p}\| = 0;$$

then, in the light of property (d) of Maingé Lemma 2.8 and (3.9) we conclude that

$$\lim_{n\to\infty}\|x_n - \overline{p}\| = 0.$$

□

Proof of (ii). Now, we rewrite the sequence $(x_{n+1})_{n\in\mathbb{N}}$ as

$$x_{n+1} = \alpha_n B_n x_n + (1-\alpha_n)S_\delta x_n + \beta_n E_n, \tag{3.11}$$

where $E_n = (1-\alpha_n)(T_\delta x_n - S_\delta x_n)$ is bounded, i.e. $\|E_n\| \le O(1)$.
We begin to prove that $\lim_{n\to\infty} \|x_n - S_\delta x_n\| = 0$.

Let $\widehat{p}$ the unique solution in $Fix(S) = Fix(S_\delta)$ of the variational inequality $\langle D\widehat{p}, \widehat{p} - x\rangle \le 0$, for all $x \in Fix(S)$. We have

$$\begin{aligned}
\|x_{n+1} - \widehat{p}\|^2 &= \|\alpha_n B_n x_n + (1-\alpha_n)(1-\delta)x_n \\
&\quad +(1-\alpha_n)\delta S x_n + \beta_n E_n - \widehat{p}\|^2 \\
&= \|[(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}] \\
&\quad +[\alpha_n(B_n x_n - x_n) + \beta_n E_n]\|^2 \\
\text{(by Lemma 2.1)} \quad &\le \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
&\quad +2\langle \alpha_n(B_n x_n - x_n) + \beta_n E_n, x_{n+1} - \widehat{p}\rangle \\
&\le \|(1-\alpha_n)\delta(Sx_n - x_n) + x_n - \widehat{p}\|^2 \\
&\quad +2\alpha_n\langle B_n x_n - x_n, x_{n+1} - \widehat{p}\rangle + 2\beta_n\langle E_n, x_n - \widehat{p}\rangle \\
&\le (1-\alpha_n)^2\delta^2\|Sx_n - x_n\|^2 + \|x_n - \widehat{p}\|^2 \\
&\quad -2(1-\alpha_n)\delta\langle x_n - \widehat{p}, x_n - Sx_n\rangle \\
&\quad +\alpha_n O(1) + \beta_n O(1) \\
\text{(by 2.5)} \quad &\le \|x_n - \widehat{p}\|^2 + (1-\alpha_n)^2\delta^2\|x_n - Sx_n\|^2 \\
&\quad -(1-\alpha_n)\delta\|x_n - Sx_n\|^2 + \alpha_n O(1) + \beta_n O(1) \\
&= \|x_n - \widehat{p}\|^2 - (1-\alpha_n)\delta[1 - \delta(1-\alpha_n)]\|x_n - Sx_n\|^2 \\
&\quad +\alpha_n O(1) + \beta_n O(1)
\end{aligned}$$

and hence

$$\begin{aligned}
0 &\le (1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2 \\
&\le \|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2 + \alpha_n O(1) + \beta_n O(1). \qquad (3.12)
\end{aligned}$$

Again, we turn our attention on the monotony of the sequence $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$. We consider the following two cases.

Case A. $\|x_{n+1} - \widehat{p}\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - \widehat{p}\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - \widehat{p}\|^2$ exists. From (3.6), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, we have

$$\begin{aligned}
0 &\le \limsup_{n\to\infty}\Big((1-\alpha_n)\delta[1-\delta(1-\alpha_n)]\|x_n - Sx_n\|^2\Big) \\
&\le \limsup_{n\to\infty}\Big(\|x_n - \widehat{p}\|^2 - \|x_{n+1} - \widehat{p}\|^2
\end{aligned}$$

$$+\alpha_n O(1) + \beta_n O(1)\Big) = 0,$$

hence

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \tag{3.13}$$

Since $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and from $I - S$ is demiclosed at 0, we can use Lemma 3.2 (i) and we have

$$\limsup_{n\to\infty} \langle D\widehat{p}, \widehat{p} - x_n\rangle \leq 0. \tag{3.14}$$

Finally, we can prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\widehat{p}$ as in the proof (i).

So, using $\sum_{n=1}^{\infty} \alpha_n\mu_n = \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, (3.14) and $\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0$, we can apply Lemma 2.7 and conclude that

$$\lim_{n\to\infty} \|x_{n+1} - \widehat{p}\| = 0.$$

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - \widehat{p}\| < \|x_{n_k+1} - \widehat{p}\| \text{ for all } k \in \mathbb{N}.$$

Then by Maingé Lemma there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies

(1) $(\tau(n))_{n\in\mathbb{N}}$ is nondecreasing;
(2) $\lim_{n\to\infty} \tau(n) = \infty$;
(3) $\|x_{\tau(n)} - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$;
(4) $\|x_n - \widehat{p}\| < \|x_{\tau(n)+1} - \widehat{p}\|$.

Consequently,

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty} \Big(\|x_{\tau(n)+1} - \widehat{p}\| - \|x_{\tau(n)} - \widehat{p}\|\Big) \\
&\leq \limsup_{n\to\infty} \Big(\|x_{\tau(n)+1} - \widehat{p}\| - \|x_{\tau(n)} - \widehat{p}\|\Big) \\
&\leq \limsup_{n\to\infty} \Big(\|x_{n+1} - \widehat{p}\| - \|x_n - \widehat{p}\|\Big) \\
\text{(by (3.11))} \quad &= \limsup_{n\to\infty} \Big(\|\alpha_n(B_n x_n - S_\delta x_n) + S_\delta x_n - \widehat{p} + \beta_n E_n\| \\
&\qquad -\|x_n - \widehat{p}\|\Big) \\
(S_\delta \text{ quasi-nonexpansive}) \quad &\leq \limsup_{n\to\infty} \Big(\alpha_n O(1) + \|x_n - \widehat{p}\| + \beta_n O(1) - \|x_n - \widehat{p}\|\Big) \\
&= 0,
\end{aligned}$$

so

$$\lim_{n\to\infty} \Big(\|x_{\tau(n)+1} - \widehat{p}\| - \|x_{\tau(n)} - \widehat{p}\|\Big) = 0. \tag{3.15}$$

By (3.12), we obtain

$$\begin{aligned} 0 &\le (1-\alpha_{\tau(n)})\delta[1-\delta(1-\alpha_{\tau(n)})]\|x_{\tau(n)} - Sx_{\tau(n)}\|^2 \\ &\le \|x_{\tau(n)} - \widehat{p}\|^2 - \|x_{\tau(n)+1} - \widehat{p}\|^2 + \alpha_n O(1) + \beta_{\tau(n)} O(1), \end{aligned}$$

from (3.15), $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \beta_n < \infty$ we get

$$\lim_{n\to\infty} \|x_{\tau(n)} - Sx_{\tau(n)}\| = 0. \tag{3.16}$$

By(3.16), as in the Case A, we get

$$\limsup_{n\to\infty} \langle D\widehat{p}, \widehat{p} - x_{\tau(n)}\rangle \le 0,$$

and

$$\lim_{n\to\infty} \|x_{\tau(n)} - \widehat{p}\| = 0,$$

then, from property (d) of Maingé Lemma and (3.15) it follows that

$$\lim_{n\to\infty} \|x_n - \widehat{p}\| = 0.$$

□

Proof of (iii). We recall that the sequence $(x_{n+1})_{n\in\mathbb{N}}$ is defined as

$$x_{n+1} = \alpha_n B_n x_n + (1-\alpha_n) U_n x_n, \tag{3.17}$$

where $U_n = \beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n$.
We first show that $\lim_{n\to\infty} \|x_n - T_\delta x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - S_\delta x_n\| = 0$.

Let $p_0 \in Fix(T) \cap Fix(S)$ is the unique solution of the variational inequality $\langle Dp_0, p_0 - x\rangle \le 0$, for all $x \in Fix(T) \cap Fix(S)$. We compute

$$\begin{aligned} \|U_n x_n - p_0\|^2 &= \|\beta_n(T_\delta x_n - p_0) + (1-\beta_n)(S_\delta x_n - p_0)\|^2 \\ (\text{ by Lemma 2.1}) \quad &= \beta_n\|T_\delta x_n - p_0\|^2 + (1-\beta_n)\|S_\delta x_n - p_0\|^2 \\ &\quad -\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\ (T_\delta \text{ nonexpansive and by (2.9)}) \quad &\le \beta_n\|x_n - p_0\|^2 + (1-\beta_n)\|x_n - p_0\|^2 \\ &\quad -(1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\ &\quad -\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \\ &= \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\ &\quad -\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2. \end{aligned}$$

So, we get

$$\|U_n x_n - p_0\|^2 \le \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 - \beta_n(1-\beta_n)\|T_\delta x_n - S - \delta x_n\|^2. \tag{3.18}$$

We have

$$\begin{aligned} \|x_{n+1} - p_0\|^2 &= \|U_n x_n - p_0 + \alpha_n(B_n x_n - U_n x_n)\|^2 \\ &\le \|U_n x_n - p_0\|^2 \\ &\quad +\alpha_n(\alpha_n\|B_n x_n - U_n x_n\|^2 + 2\|U_n x_n - p_0\|\|B_n x_n - U_n x_n\|) \end{aligned}$$

$$\begin{aligned}
& \leq \|U_n x_n - p_0\|^2 + \alpha_n O(1) \\
\text{(by (3.18))} \quad & \leq \|x_n - p_0\|^2 - (1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \\
& \quad -\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 + \alpha_n O(1),
\end{aligned} \tag{3.19}$$

From (3.19), we derive

$$(1-\beta_n)(1-\delta)\|x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \tag{3.20}$$

and

$$\beta_n(1-\beta_n)\|T_\delta x_n - S_\delta x_n\|^2 \leq \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n O(1). \tag{3.21}$$

Now, also we consider two cases.

Case A. $\|x_{n+1} - p_0\|$ is definitively nonincreasing.

Case B. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_k+1} - p_0\| \text{ for all } k \in \mathbb{N}.$$

Case A. Since $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ is definitively nonincreasing, $\lim_{n\to\infty} \|x_n - p_0\|^2$ exists. From (3.20), $\lim_{n\to\infty} \alpha_n = 0$ and since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ we conclude

$$\lim_{n\to\infty} \|x_n - S_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Sx_n\| = 0. \tag{3.22}$$

Furthermore, from (3.21) we have

$$\lim_{n\to\infty} \|S_\delta x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|Sx_n - Tx_n\| = 0; \tag{3.23}$$

since

$$\|x_n - Tx_n\| \leq \|x_n - Sx_n\| + \|Sx_n - Tx_n\|,$$

by (3.22) and (3.23) we obtain

$$\lim_{n\to\infty} \|x_n - T_\delta x_n\| = \lim_{n\to\infty} \delta\|x_n - Tx_n\| = 0. \tag{3.24}$$

By (3.24) and (3.22) and by the demiclosedness of $I - T$ at 0 and of $I - S$ at 0, we can conclude using Lemma 3.2 (*ii*)

$$\limsup_{n\to\infty} \langle Dp_0, p_0 - x_n\rangle \leq 0. \tag{3.25}$$

Finally, $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.

We compute

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &= \|\alpha_n(B_n x_n - p_0) + (1-\alpha_n)(U_n x_n - p_0)\|^2 \\
\text{(by Lemma 2.1)} \quad &\leq (1-\alpha_n)^2\|U_n x_n - p_0\|^2 \\
&\quad +2\alpha_n\langle B_n x_n - p_0, x_{n+1} - p_0\rangle \\
&= (1-\alpha_n)^2\|U_n x_n - p_0\|^2 \\
&\quad +2\alpha_n\langle B_n x_n - B_n p_0, x_{n+1} - p_0\rangle \\
&\quad +2\alpha_n\langle B_n p_0 - p_0, x_{n+1} - p_0\rangle \\
(U_n \text{ quasi-nonexpansive}) \quad &\leq (1-\alpha_n)^2\|x_n - p_0\|^2 \\
(B_n \text{ contraction}) \quad &\quad +2\alpha_n(1-\mu_n\tau)\|x_n - p_0\|\|x_{n+1} - p_0\|
\end{aligned}$$

$$\begin{aligned} & && +2\alpha_n\langle B_n p_0 - p_0, x_{n+1} - p_0\rangle \\ (B_n := (1-\mu_n D)) & \leq && (1-\alpha_n)^2\|x_n - p_0\|^2 \\ & && +\alpha_n(1-\mu_n\tau)(\|x_n - p_0\|^2 + \|x_{n+1} - p_0\|^2) \\ & && -2\alpha_n\mu_n\langle Dp_0, x_{n+1} - p_0\rangle \end{aligned}$$

Then, it follows that

$$\begin{aligned} \|x_{n+1} - p_0\|^2 & \leq \frac{1-(1+\mu_n\tau)\alpha_n + \alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}\|x_n - p_0\|^2 \\ & \quad - \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle Dp_0, x_{n+1} - p_0\rangle \\ & \leq \frac{1-(1+\mu_n\tau)\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\|x_n - p_0\|^2 \\ & \quad + \frac{\alpha_n^2}{1-(1-\mu_n\tau)\alpha_n}O(1) - \frac{2\alpha_n\mu_n}{1-(1-\mu_n\tau)\alpha_n}\langle Dp_0, x_{n+1} - p_0\rangle \\ & \leq \left(1 - \frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\right)\|x_n - p_0\|^2 \\ & \quad + \frac{2\mu_n\tau\alpha_n}{1-(1-\mu_n\tau)\alpha_n}\left[-\frac{1}{\tau}\langle Dp_0, x_{n+1} - p_0\rangle + \frac{\alpha_n}{2\mu_n\tau}O(1)\right]. \end{aligned} \tag{3.26}$$

Using $\sum_{n=1}^{\infty}\alpha_n\mu_n = \infty$, (3.25) and $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n} = 0$, we can apply Lemma 2.7 and conclude that

$$\lim_{n\to\infty}\|x_{n+1} - p_0\| = 0.$$

So, $(x_n)_{n\in\mathbb{N}}$ converges strongly to a fixed point of $Fix(T) \cap Fix(S)$.

Case B. $(\|x_n - p_0\|)_{n\in\mathbb{N}}$ does not be definitively nonincreasing. This means that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\|x_{n_k} - p_0\| < \|x_{n_k+1} - p_0\| \text{ for all } k \in \mathbb{N}.$$

Then by Maingé Lemma 2.8 there exists a sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ that it satisfies some properties defined previous.
Consequently,

$$\begin{aligned} 0 & \leq \liminf_{n\to\infty}\Big(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\Big) \\ & \leq \limsup_{n\to\infty}\Big(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\Big) \\ & \leq \limsup_{n\to\infty}\Big(\|x_{n+1} - p_0\| - \|x_n - p_0\|\Big) \\ (\text{by } (3.17)) & = \limsup_{n\to\infty}\Big(\|\alpha_n(B_n x_n - p_0) \\ & \qquad + (1-\alpha_n)(U_n x_n - p_0)\| - \|x_n - p_0\|\Big) \end{aligned}$$

$$(U_n \text{ quasi-nonexpansive}) \quad \leq \quad \limsup_{n\to\infty}\Big(\alpha_n O(1) + \|x_n - p_0\| - \|x_n - p_0\|\Big) = 0,$$

hence

$$\lim_{n\to\infty}\Big(\|x_{\tau(n)+1} - p_0\| - \|x_{\tau(n)} - p_0\|\Big) = 0. \tag{3.27}$$

By (3.20) we get

$$(1-\beta_{\tau(n)})(1-\delta)\|x_{\tau(n)}-S_\delta x_{\tau(n)}\|^2 \leq \|x_{\tau(n)}-p_0\|^2-\|x_{n+1}-p_0\|^2+\alpha_{\tau(n)}O(1), \tag{3.28}$$

and by(3.21) we have

$$\beta_{\tau(n)}(1-\beta_{\tau(n)})\|T_\delta x_{\tau(n)} - S_\delta x_{\tau(n)}\|^2 \leq \|x_{\tau(n)} - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_{\tau(n)}O(1), \tag{3.29}$$

As in the Case A., we get

(1) $\lim_{n\to\infty} \|x_{\tau(n)} - Sx_{\tau(n)}\| = 0,$
(2) $\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$

By (a) and (b), as in the Case A, we have

$$\limsup_{n\to\infty}\langle Dp_0, p_0 - x_{\tau(n)}\rangle \leq 0. \tag{3.30}$$

Finally, we prove that $(x_n)_{n\in\mathbb{N}}$ converges strongly to p_0.

As in the Case A., using $\sum_{n=1}^{\infty}\alpha_n\mu_n = \infty$, $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n} = 0$, and (3.30) we can apply Xu's Lemma 2.7 and we yield that

$$\lim_{n\to\infty}\|x_{\tau(n)} - \widehat{p}\| = 0,$$

then, from property (d) of Maingé Lemma and (3.27) we can derive

$$\lim_{n\to\infty}\|x_n - p_0\| = 0.$$

□

Example 3.4. The sequences

$$\alpha_n = \frac{1}{n^{\frac{2}{3}}}, \quad \mu_n = \frac{\beta}{\rho^2 n^{\frac{1}{3}}}, \quad \forall n \in \mathbb{N},$$

satisfy the conditions:

(1) $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $(0, 1)$,
(2) $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $(0, \mu)$, $\mu < \frac{2\beta}{\rho^2}$,
(3) $\lim_{n\to\infty}\alpha_n = 0,$
(4) $\lim_{n\to\infty}\frac{\alpha_n}{\mu_n} = 0,$
(5) $\sum_{n=1}^{\infty}\alpha_n\mu_n = \infty.$

Remark 3.5. We remark that (i) and (ii) of Theorem 3.3 actually hold for a wide class of nonlinear mappings. In fact, in (i) we can substitute a L-hybrid mapping S with a quasi-nonexpansive mapping because we use only the boundedness of $(S_\delta x_n)_{n\in\mathbb{N}}$.

For the same reason, in (ii) we can replace a nonexpansive mapping T with a quasi-nonexpansive mapping.

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