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SYSTEMS OF GENERALIZED EQUILIBRIA WITH CONSTRAINTS OF VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS

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This paper is dedicated to Professor Wataru Takahashi on the occasions of his 70th birthday.

Abstract. In this paper, we introduce and analyze one iterative algorithm by hybrid shrinking projection method for finding a solution of the system of generalized equilibrium problems with constraints of several problems: a generalized mixed equilibrium problem, finitely many variational inclusions, and the common fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense and infinitely many nonexpansive mappings in a real Hilbert space. We prove strong convergence theorem for the iterative algorithm under suitable conditions.

1. Introduction and formulations

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping V is called strongly positive on H if there exists a constant $\bar{\gamma} \in (0,1]$ such that

$$\langle Vx, x \rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.1}$$

A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, S is called a nonexpansive mapping if $L = 1$ and A is called a contraction if $L \in [0, 1)$.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. Peng and Yao [15] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of GMEP (1.2) by $\mathrm{GMEP}(\Theta, \varphi, A)$. The GMEP (1.2) is very general in the sense that it includes, as special cases, optimization

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problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others.

Throughout this paper, we assume as in [15] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (H5), where

(H1) $\Theta(x, x) = 0$ for all $x \in C$;

(H2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(H3) Θ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x, y);$$

(H4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(H5) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0.$$

Given a positive number $r > 0$. Let $S_r^{(\Theta,\varphi)} : H \to C$ is the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$S_r^{(\Theta,\varphi)}(x) := \{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z - y\rangle \geq 0, \ \forall z \in C\}.$$

In particular, whenever $K(x) = \frac{1}{2}\|x\|^2, \forall x \in H$, $S_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{(\Theta,\varphi)}$.

Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions, and $A_1, A_2 : C \to H$ be two nonlinear mappings. Consider the following system of generalized equilibrium problems (SGEP): find $(x^*, y^*) \in C \times C$ such that

$$\text{(1.3)} \qquad \begin{cases} \Theta_1(x^*, x) + \langle A_1 y^*, x - x^*\rangle + \dfrac{1}{\nu_1}\langle x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \Theta_2(y^*, y) + \langle A_2 x^*, y - y^*\rangle + \dfrac{1}{\nu_2}\langle y^* - x^*, y - y^*\rangle \geq 0, & \forall y \in C, \end{cases}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. It is introduced and studied in [4]. When $\Theta_1 \equiv \Theta_2 \equiv 0$, the SGEP reduces to a system of variational inequalities, which is considered and studied in [3]. It is worth to mention that the system of variational inequalities is a tool to solve the Nash equilibrium problem for noncooperative games.

In 2010, Ceng and Yao [4] transformed the SGEP into a fixed point problem in the following way.

Proposition 1.1 (see [4]). *Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions (H1)-(H4) and let $A_k : C \to H$ be ζ_k-inverse-strongly monotone for $k = 1, 2$. Let $\nu_k \in (0, 2\zeta_k)$ for $k = 1, 2$. Then, $(x^*, y^*) \in C \times C$ is a solution of SGEP (1.3) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by $G = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1) T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)$ where $y^* = T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)x^*$. Here, we denote the fixed point set of G by* SGEP(G).

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n on H as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \cdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \cdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{1.4}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\lambda_n, \lambda_{n-1}, ..., \lambda_1$.

In 2011, for the case where $C = H$, Yao, Liou and Yao [19] proposed the following hybrid iterative algorithm

$$\begin{cases} \Theta(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), z - y_n\rangle \geq 0, \\ \hfill z \in H, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n(I + \mu V))W_n y_n, \\ \hfill \forall n \geq 1, \end{cases} \tag{1.5}$$

where $f : H \to H$ be a contraction, $K : H \to \mathbf{R}$ is differentiable and strongly convex, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $x_0, u \in H$ are given, for finding a common element of the set $\mathrm{MEP}(\Theta, \varphi)$ and the fixed point set $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ of an infinite family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ on H. They proved the strong convergence of the sequence generated by the hybrid iterative algorithm (1.5) to a point $x^* \in \Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{MEP}(\Theta, \varphi)$ under some appropriate conditions. This point x^* also solves the following optimization problem:

$$\min_{x \in \Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x) \tag{OP0}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

Let $f : H \to H$ be a contraction and V be a strongly positive bounded linear operator on H. Assume that $\varphi : H \to \mathbf{R}$ is a lower semicontinuous and convex functional, that $\Theta, \Theta_1, \Theta_2 : H \times H \to \mathbf{R}$ satisfy conditions (H1)-(H4), and that $A, A_1, A_2 : H \to H$ are inverse-strongly monotone. Let the mapping G be defined as in Proposition 1.1. Very recently, Ceng, Ansari and Schaible [2] introduced the following hybrid extragradient-like iterative algorithm

$$(1.6)\qquad \begin{cases} z_n = S_{r_n}^{(\Theta,\varphi)}(x_n - r_n A x_n), \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n \\ \qquad + ((1-\beta_n)I - \alpha_n(I + \mu V))W_n G z_n, \quad \forall n \geq 0, \end{cases}$$

for finding a common solution of GMEP (1.2), SGEP (1.3) and the fixed point problem of an infinite family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ on H, where $\{r_n\} \subset (0,\infty)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\nu_k \in (0, 2\zeta_k), k = 1,2$, and $x_0, u \in H$ are given. The authors proved the strong convergence of the sequence generated by the hybrid iterative algorithm (1.6) to a point $x^* \in \Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta,\varphi,A) \cap \mathrm{SGEP}(G)$ under some suitable conditions. This point x^* also solves the following optimization problem:

$$(\mathrm{OP1})\qquad \min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x)$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

On the other hand, let B be a single-valued mapping of C into H and R be a set-valued mapping with domain $D(R) = C$. Consider the following variational inclusion [9]: find a point $x \in C$ such that

$$(1.7)\qquad 0 \in Bx + Rx.$$

We denote by $\mathrm{I}(B,R)$ the solution set of the variational inclusion (1.7). It is known that problem (1.7) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, etc. Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H,$$

where λ is a positive number.

In 2011, for the case where $C = H$, Yao, Cho and Liou [17] introduced and analyzed the following iterative algorithms for finding an element of the intersection $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta,\varphi,A) \cap \mathrm{I}(B,R)$ of the solution set of the GMEP (1.2), the solution set of the variational inclusion (1.7) and the fixed point set of a countable family $\{T_n\}_{n=1}^{\infty}$ of nonexpansive mappings: for arbitrarily given $x_1 \in H$, let the sequence $\{x_n\}$ be generated by

$$(1.8)\qquad \begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle y - u_n, Ax_n\rangle \\ \qquad + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n \\ \qquad + [(1-\beta_n)I - \alpha_n V] W_n J_{R,\lambda}(u_n - \lambda B u_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0,1]$ and W_n is the W-mapping defined by (1.4). It is proven that under appropriate conditions the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(\gamma f(x^*) + (I - V)x^*)$ is a unique solution of the VIP:

$$\langle(\gamma f - V)x^*, y - x^*\rangle \le 0, \quad \forall y \in \Omega. \tag{1.9}$$

Next, we creall some concepts. Let C be a nonempty subset of a normed space X. A mapping $S : C \to C$ is called uniformly Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that

$$\|S^n x - S^n y\| \le \mathcal{L}\|x - y\|, \quad \forall n \ge 1,\ \forall x, y \in C.$$

Recently, Kim and Xu [11] introduced the concept of asymptotically k-strict pseudocontractive mappings in a Hilbert space as below:

Definition 1.2. *Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exists a constant $k \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that*

$$\|S^n x - S^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + k\|x - S^n x - (y - S^n y)\|^2, \quad \forall n \ge 1,\ \forall x, y \in C.$$

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup\left\{\frac{k+\sqrt{1+(1-k)\gamma_n}}{1+k} : n \ge 1\right\}$. Subsequently, Sahu, Xu and Yao [9] considered the concept of asymptotically k-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 1.3. *Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that*

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|S^n x - S^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - S^n x - (y - S^n y)\|^2) \le 0.$$

Put $c_n := \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 - k\|x - S^n x - (y - S^n y)\|^2)\}$. Then $c_n \ge 0$ $(\forall n \ge 1)$, $c_n \to 0$ $(n \to \infty)$ and (1.4) reduces to the relation

$$\begin{aligned}\|S^n x - S^n y\|^2 \le& (1 + \gamma_n)\|x - y\|^2 + k\|x - S^n x - (y - S^n y)\|^2 \\ &+ c_n, \quad \forall n \ge 1,\ \forall x, y \in C.\end{aligned} \tag{1.10}$$

Whenever $c_n = 0$ for all $n \ge 1$ in (1.10), then S is an asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$. In 2009, Sahu, Xu and Yao [16] derived the weak and strong convergence of the modified Mann iteration processes for an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. More precisely, they first established one weak convergence theorem for the following iterative scheme

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n x_n, \quad \forall n \ge 1, \end{cases}$$

where $0 < \delta \leq \alpha_n \leq 1 - k - \delta$, $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$; and then obtained another strong convergence theorem for the following iterative scheme

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ y_n = (1-\alpha_n)x_n + \alpha_n S^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \ \forall n \geq 1, \end{cases}$$

where $0 < \delta \leq \alpha_n \leq 1-k$, $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in \mathrm{Fix}(S)\} < \infty$.

Motivated and inspired by the above results, we aim to introduce and analyze an iterative algorithm by hybrid shrinking projection method for finding a solution of the system of generalized equilibrium problems with constraints of several problems: a generalized mixed equilibrium problem, finitely many variational inclusions, and the common fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense and infinitely many nonexpansive mappings in a real Hilbert space. Strong convergence theorem for the iterative algorithm will be established under mild conditions.

We remark that some more recent and related results can be found, e.g., in [1, 6].

2. Preliminaries and tools

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. We use the notations $x_n \rightharpoonup x$ and $x_n \to x$ to indicate the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x, respectively. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Definition 2.1. *A mapping $A : C \to H$ is called*

(i) *monotone if*

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(ii) *η-strongly monotone if there exists a constant $\eta > 0$ such that*

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C;$$

(iii) *ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that*

$$\langle Ax - Ay, x - y \rangle \geq \zeta \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 2.2. *A differentiable function $K : H \to \mathbf{R}$ is called:*

(i) *convex, if*

$$K(y) - K(x) \geq \langle K'(x), y - x\rangle, \quad \forall x, y \in H,$$

where $K'(x)$ is the Frechet derivative of K at x;

(ii) *strongly convex, if there exists a constant $\sigma > 0$ such that*

$$K(y) - K(x) - \langle K'(x), y - x\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in H.$$

It is easy to see that if $K : H \to \mathbf{R}$ is a differentiable strongly convex function with constant $\sigma > 0$ then $K' : H \to H$ is strongly monotone with constant $\sigma > 0$.

The metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are listed in the following proposition.

Proposition 2.3. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leq 0,\ \forall y \in C$;

(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2,\ \forall y \in H$. *(This implies that P_C is nonexpansive and monotone.)*

By using the technique of [7], we can readily obtain the following elementary result where MEP(Θ, φ) is the solution set of the mixed equilibrium problem [2].

Proposition 2.4 (see [2, Lemma 1 and Proposition 1])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying the conditions (H1)-(H4). Assume that*

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,*

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle K'(z) - K'(x), y_x - z\rangle < 0.$$

Then the following hold:

(a) *for each $x \in H$, $S_r^{(\Theta,\varphi)}(x) \neq \emptyset$;*

(b) *$S_r^{(\Theta,\varphi)}$ is single-valued;*

(c) *$S_r^{(\Theta,\varphi)}$ is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ and*

$$\langle K'(x_1) - K'(x_2), u_1 - u_2\rangle \leq \langle K'(u_1) - K'(u_2), u_1 - u_2\rangle, \quad \forall (x_1, x_2) \in H \times H,$$

where $u_i = S_r^{(\Theta,\varphi)}(x_i)$ for $i = 1, 2$;

(d) *for all $s, t > 0$ and $x \in H$*

$$\langle K'(S_s^{(\Theta,\varphi)}x) - K'(S_t^{(\Theta,\varphi)}x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\rangle$$

$$\leq \frac{s-t}{s}\langle K'(S_s^{(\Theta,\varphi)}x) - K'(x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\rangle;$$

(e) $\mathrm{Fix}(S_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta,\varphi)$;
(f) $\mathrm{MEP}(\Theta,\varphi)$ *is closed and convex.*

Remark 2.5. *In Proposition 2.4, whenever* $\Theta : C\times C \to \mathbf{R}$ *is a bifunction satisfying the conditions (H1)-(H4) and* $K(x) = \frac{1}{2}\|x\|^2, \forall x \in H$, *we have for any* $x, y \in H$,

$$\|S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y\|^2 \leq \langle S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y, x-y\rangle$$

($S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive) and*

$$\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\| \leq \frac{|s-t|}{s}\|S_s^{(\Theta,\varphi)}x - x\|, \quad \forall s,t>0, x\in H.$$

If, in addition, $\varphi \equiv 0$, *then* $T_r^{(\Theta,\varphi)}$ *is rewritten as* T_r^{Θ}; *see* [4, Lemma 2.1] *for more details.*

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.6. *Let* X *be a real inner product space. Then there holds the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y\in X.$$

Lemma 2.7. *Let* H *be a real Hilbert space. Then the following hold:*
(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x,y\in H$;
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x,y\in H$ *and* $\lambda,\mu\in[0,1]$ *with* $\lambda+\mu = 1$;
(c) *If* $\{x_n\}$ *is a sequence in* H *such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x-y\|^2, \quad \forall y\in H.$$

We have the following crucial lemmas concerning the W-mappings defined by (1.4).

Lemma 2.8 (see [13, Lemma 3.2])**.** *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive self-mappings on* H *such that* $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ *and let* $\{\lambda_n\}$ *be a sequence in* $(0,b]$ *for some* $b\in(0,1)$. *Then, for every* $x\in H$ *and* $k\geq 1$ *the limit* $\lim_{n\to\infty} U_{n,k}x$ *exists, where* $U_{n,k}$ *is defined by (1.4).*

Lemma 2.9 (see [13, Lemma 3.3])**.** *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive self-mappings on* H *such that* $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$, *and let* $\{\lambda_n\}$ *be a sequence in* $(0,b]$ *for some* $b\in(0,1)$. *Then,* $\mathrm{Fix}(W) = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Lemma 2.10 (see [8, Demiclosedness principle])**.** *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* T *be a nonexpansive self-mapping on* C. *Then* $I-T$ *is demiclosed. That is, whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x\in C$ *and the sequence* $\{(I-T)x_n\}$ *strongly converges to some* y, *it follows that* $(I-T)x = y$. *Here* I *is the identity operator of* H.

Lemma 2.11 ([16, Lemma 2.5]). *Let H be a real Hilbert space. Given a nonempty closed convex subset of H and points $x, y, z \in H$ and given also a real number $a \in \mathbf{R}$, the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Assume that $R : D(R) \subset H \to 2^H$ is a maximal monotone mapping. Let $\lambda > 0$. In [9], there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 2.12. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive, i.e.,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y\rangle \geq \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H.$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 2.13 (see [5]). *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (1.7) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu).$$

Lemma 2.14 (see [21]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous and single-valued mapping. Then for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.15 (see [5]). *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

Lemma 2.16 (see [20]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbf{R} \cup +\infty$ be a proper lower semicontinuous differentiable convex function. If x^* is a solution the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x),$$

then,

$$\langle g'(x), x - x^*\rangle \geq 0, \quad \forall x \in C.$$

In particular, if x^* solves (OP), then

$$\langle u + (\gamma f - (I + \mu V))x^*, x - x^*\rangle \leq 0.$$

Lemma 2.17 (see [16, Lemma 2.6]). *Let C be a nonempty subset of a Hilbert space H and $S : C \to C$ be an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|S^n x - S^n y\| \leq \frac{1}{1-k}(k\|x-y\| + \sqrt{(1+(1-k)\gamma_n)\|x-y\|^2 + (1-k)c_n})$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 2.18 ([16, Lemma 2.7]). *Let C be a nonempty subset of a Hilbert space H and $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - S^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.19 (Demiclosedness principle [16, Proposition 3.1]). *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be a continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I - S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - S^m x_n\| = 0$, then $(I - S)x = 0$.*

Lemma 2.20 (see [16, Proposition 3.2]). *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be a continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that* $\mathrm{Fix}(S) \neq \emptyset$. *Then* $\mathrm{Fix}(S)$ *is closed and convex.*

Lemma 2.21 ([14, p. 80]). *Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^\infty$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Recall that a Banach space X is said to satisfy the Opial condition [19] if for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x.$$

It is well known in [19] that every Hilbert space H satisfies the Opial condition.

Lemma 2.22 (see [10, Proposition 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - p\|^2 \leq (1+\lambda_n)\|x_n - p\|^2 + \delta_n, \quad \forall p \in C, \ n \geq 1,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^\infty \lambda_n < \infty$ and $\sum_{n=1}^\infty \delta_n < \infty$. Then $\{P_C x_n\}$ converges strongly in C.

Lemma 2.23 (see [12]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \text{for all } n,$$

then $x_n \to q$ *as* $n \to \infty$.

3. Strong convergence theorem

In this section, we will prove the strong convergence of an iterative algorithm by hybrid shrinking projection method for finding a solution of the system of generalized equilibrium problems with constraints of several problems: a generalized mixed equilibrium problem, finitely many variational inclusions, and the common fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense and infinitely many nonexpansive mappings in a real Hilbert space. This iterative algorithm is based on the extragradient method, viscosity approximation method, Mann-type iterative method and shrinking projection method.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* N *be an integer. Let* $\Theta, \Theta_1, \Theta_2$ *be three bifunctions from* $C \times C$ *to* $\mathbf{R}$ *satisfying (H1)-(H4) and* $\varphi : C \to \mathbf{R}$ *be a lower semicontinuous and convex functional. Let* $R_i : C \to 2^H$ *be a maximal monotone mapping and let* $A, A_k : H \to H$ *and* $B_i : C \to H$ *be* ζ*-inverse strongly monotone,* ζ_k*-inverse strongly monotone and* η_i*-inverse strongly monotone, respectively, where* $k \in \{1, 2\}$ *and* $i \in \{1, 2, ..., N\}$. *Let* $S : C \to C$ *be a uniformly continuous asymptotically* k*-strict pseudocontractive mapping in the intermediate sense for some* $0 \le k < 1$ *with sequence* $\{\gamma_n\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} \gamma_n = 0$ *and* $\{c_n\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} c_n = 0$. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive mappings on* H *and* $\{\lambda_n\}$ *be a sequence in* $(0, b]$ *for some* $b \in (0, 1)$. *Let* V *be a* $\bar{\gamma}$*-strongly positive bounded linear operator and* $f : H \to H$ *be an* l*-Lipschitzian mapping with* $\gamma l < (1 + \mu)\bar{\gamma}$. *Let* W_n *be the* W*-mapping defined by (1.4). Assume that* $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \cap_{i=1}^{N} \mathrm{I}(B_i, R_i) \cap \mathrm{Fix}(S)$ *is nonempty and bounded where* G *is defined as in Proposition 1.1. Let* $\{r_n\}$ *be a sequence in* $[0, 2\zeta]$ *and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\delta_n\}$ *be sequences in* $(0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $k \le \delta_n \le d < 1$. *Pick any* $x_0 \in H$ *and set* $C_1 = H$, $x_1 = P_{C_1} x_0$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$
(3.1)\qquad \begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n} B_N) J_{R_{N-1},\lambda_{N-1,n}}(I - \lambda_{N-1,n} B_{N-1}) \cdots \\ \qquad J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n} B_1) u_n, \\ k_n = \delta_n z_n + (1 - \delta_n) S^n z_n, \\ y_n = \alpha_n (u + \gamma f(x_n)) + \beta_n k_n + [(1 - \beta_n) I - \alpha_n (I + \mu V)] W_n G k_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases}
$$

where $\theta_n = (\alpha_n + \gamma_n)\Gamma_n \varrho + c_n \varrho$, $\Gamma_n = \sup\{\|x_n - p\|^2 + \|u + (\gamma f - (I + \mu V))p\|^2 : p \in \Omega\} < \infty$, *and* $\varrho = \frac{1}{1 - \sup_{n\ge1} \alpha_n} < \infty$. *Assume that the following conditions are satisfied:*

(i) $K : H \to \mathbf{R}$ *is strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that the function* $x \mapsto \langle y - x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y \in H$;

(ii) *for each* $x \in H$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$;

(iv) $\nu_k \in (0, 2\zeta_k), k \in \{1, 2\}$ *and* $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i),\ \forall i \in \{1, 2, ..., N\}$.

Suppose that $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive. Then the following statements hold:*

(I) $\{x_n\}$ *converges strongly to* $x^* = P_\Omega x_0$;

(II) $\{x_n\}$ *converges strongly to* $x^* = P_\Omega x_0$ *which solves the following optimization problem provided* $\gamma_n + c_n = o(\alpha_n)$ *and* $\|x_n - y_n\| = o(\alpha_n)$:

$$\text{(OP2)} \qquad \min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)$$

where $h : H \to \mathbf{R}$ *is the potential function of* γf.

Proof. Since $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)(1 + \mu\|V\|)^{-1}$. Since V is a $\bar{\gamma}$-strongly positive bounded linear operator on H, we know that

$$\|V\| = \sup\{\langle Vu, u\rangle : u \in H, \|u\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1 - \beta_n)I - \alpha_n(I + \mu V))u, u\rangle &= 1 - \beta_n - \alpha_n - \alpha_n\mu\langle Vu, u\rangle \\ &\geq 1 - \beta_n - \alpha_n - \alpha_n\mu\|V\| \\ &\geq 0,\end{aligned}$$

that is, $(1 - \beta_n)I - \alpha_n(I + \mu V)$ is positive. It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \alpha_n(I + \mu V)\| &= \sup\{\langle((1 - \beta_n)I - \alpha_n(I + \mu V))u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n - \alpha_n\mu\langle Vu, u\rangle : u \in H, \|u\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n - \alpha_n\mu\bar{\gamma}.\end{aligned}$$

Put

$$\Lambda^i = J_{R_i,\mu_i}(I - \mu_i B_i)J_{R_{i-1},\mu_{i-1}}(I - \mu_{i-1}B_{i-1}) \cdots J_{R_1,\mu_1}(I - \mu_1 B_1)$$

for all $i \in \{1, 2, ..., N\}$, and $\Lambda^0 = I$, where I is the identity mapping on H. Then we have that $z_n = \Lambda^N u_n$.

We divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is well defined. It is obvious that C_n is closed and convex. As the defining inequality in C_n is equivalent to the inequality

$$\langle 2(x_n - y_n), z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \theta_n,$$

by Lemma **??**, we know that C_n is convex for every $n \geq 1$.

First of all, let us show that $\Omega \subset C_n$ for all $n \geq 1$. Suppose that $\Omega \subset C_n$ for some $n \geq 1$. Take $p \in \Omega$ arbitrarily. Since $p = S_{r_n}^{(\Theta,\varphi)}(p - r_n Ap)$, A is ζ-inverse strongly monotone and $0 \leq r_n \leq 2\zeta$, we have, for any $n \geq 1$,

$$\begin{aligned}
\|u_n - p\|^2 &= \|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\|^2 \\
&\leq \|(I - r_n A)x_n - (I - r_n A)p\|^2 \\
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n\zeta\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{3.2}$$

Since $p = J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)p$, $\Lambda_n^i p = p$ and B_i is η_i-inverse strongly monotone, where $\lambda_{i,n} \in (0, 2\eta_i)$, $i \in \{1, 2, ..., N\}$, by Lemma 2.12 we deduce that for each $n \geq 1$

$$\begin{aligned}
\|z_n - p\| &= \|J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)\Lambda_n^{N-1}u_n - J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)\Lambda_n^{N-1}p\| \\
&\leq \|(I - \lambda_{N,n}B_N)\Lambda_n^{N-1}u_n - (I - \lambda_{N,n}B_N)\Lambda_n^{N-1}p\| \\
&\leq \|\Lambda_n^{N-1}u_n - \Lambda_n^{N-1}p\| \\
&\quad \vdots \\
&\leq \|\Lambda_n^0 x_n - \Lambda_n^0 p\| \\
&= \|u_n - p\|.
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.4}$$

By Lemma 2.7 (b), we deduce from (3.1) and (3.4) that

$$\begin{aligned}
\|k_n - p\|^2 &= \|\delta_n(z_n - p) + (1 - \delta_n)(S^n z_n - p)\|^2 \\
&= \delta_n\|z_n - p\|^2 + (1 - \delta_n)\|S^n z_n - p\|^2 - \delta_n(1 - \delta_n)\|z_n - S^n z_n\|^2 \\
&\leq \delta_n\|z_n - p\|^2 + (1 - \delta_n)[(1 + \gamma_n)\|z_n - p\|^2 + k\|z_n - S^n z_n\|^2 + c_n] \\
&\quad - \delta_n(1 - \delta_n)\|z_n - S^n z_n\|^2 \\
&= [1 + \gamma_n(1 - \delta_n)]\|z_n - p\|^2 + (1 - \delta_n)(k - \delta_n)\|z_n - S^n z_n\|^2 \\
&\quad + (1 - \delta_n)c_n \\
&\leq (1 + \gamma_n)\|z_n - p\|^2 + (1 - \delta_n)(k - \delta_n)\|z_n - S^n z_n\|^2 + c_n \\
&\leq (1 + \gamma_n)\|z_n - p\|^2 + c_n.
\end{aligned} \tag{3.5}$$

Since $p = Gp = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p$, A_k is ζ_k-inverse-strongly monotone for $k = 1, 2$, and $0 \leq \nu_k \leq 2\zeta_k$ for $k = 1, 2$, we deduce that, for any $n \geq 1$,

$$\begin{aligned}
\|Gk_n - p\|^2 &= \|T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)k_n - T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\|^2 \\
&\leq \|(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)k_n - (I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|[T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)k_n - T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)p] \\
&\quad - \nu_1[A_1 T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)k_n - A_1 T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)p]\|^2 \\
&\le \|T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)k_n - T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)p\|^2 \\
&\quad + \nu_1(\nu_1 - 2\zeta_1)\|A_1 T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)k_n - A_1 T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)p\|^2 \\
&\le \|T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)k_n - T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)p\|^2 \\
&\le \|(I-\nu_2 A_2)k_n - (I-\nu_2 A_2)p\|^2 \\
&= \|(k_n - p) - \nu_2(A_2 k_n - A_2 p)\|^2 \\
&\le \|k_n - p\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|A_2 k_n - A_2 p\|^2 \\
&\le \|k_n - p\|^2.
\end{aligned} \tag{3.6}
$$

(This shows that G is nonexpansive.) Set $\bar{V} = I + \mu V$. By Lemma 2.6 we deduce from (3.1), (3.4)-(3.6) and $\gamma l < (1+\mu)\bar{\gamma}$ that

$$
\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(u + \gamma f(x_n)) + \beta_n k_n + [(1-\beta_n)I - \alpha_n(I + \mu V)]W_n G k_n - p\|^2 \\
&= \|\alpha_n(u + \gamma f(x_n) - \bar{V}p) + \beta_n(k_n - p) + ((1-\beta_n)I - \alpha_n \bar{V})W_n G k_n \\
&\quad - ((1-\beta_n)I - \alpha_n \bar{V})W_n G p\|^2 \\
&= \|\alpha_n \gamma(f(x_n) - f(p)) + \beta_n(k_n - p) + ((1-\beta_n)I - \alpha_n \bar{V})W_n G k_n \\
&\quad - ((1-\beta_n)I - \alpha_n \bar{V})W_n G p + \alpha_n(u + \gamma f(p) - \bar{V}p)\|^2 \\
&\le \|\alpha_n \gamma(f(x_n) - f(p)) + \beta_n(k_n - p) \\
&\quad + ((1-\beta_n)I - \alpha_n \bar{V})(W_n G k_n - W_n G p)\|^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(p) - \bar{V}p, y_n - p\rangle \\
&\le [\alpha_n \gamma\|f(x_n) - f(p)\| + \beta_n\|k_n - p\| \\
&\quad + \|(1-\beta_n)I - \alpha_n \bar{V}\|\|W_n G k_n - W_n G p\|]^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(p) - \bar{V}p, y_n - p\rangle \\
&\le [\alpha_n \gamma l\|x_n - p\| + \beta_n\|k_n - p\| \\
&\quad + (1 - \beta_n - \alpha_n - \alpha_n \mu\bar{\gamma})\|G k_n - G p\|]^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(p) - \bar{V}p, y_n - p\rangle \\
&\le [\alpha_n \gamma l\|x_n - p\| + \beta_n\|k_n - p\| + (1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|]^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(p) - \bar{V}p, y_n - p\rangle \\
&\le [\alpha_n(1+\mu)\bar{\gamma}\|x_n - p\| + (1 - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|]^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(p) - \bar{V}p, y_n - p\rangle \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + (1 - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|^2 \\
&\quad + 2\alpha_n\|u + \gamma f(p) - \bar{V}p\|\|y_n - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + (1 - \alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|z_n - p\|^2 + c_n) \\
&\quad + \alpha_n(\|u + (\gamma f - \bar{V})p\|^2 + \|y_n - p\|^2)
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2 + (1-\alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|x_n-p\|^2 + c_n) \\
&\quad + \alpha_n(\|u+(\gamma f-\bar{V})p\|^2 + \|y_n-p\|^2) \\
&= \|x_n-p\|^2 + (1-\alpha_n(1+\mu)\bar{\gamma})\gamma_n\|x_n-p\|^2 + (1-\alpha_n(1+\mu)\bar{\gamma})c_n \\
&\quad + \alpha_n(\|u+(\gamma f-\bar{V})p\|^2 + \|y_n-p\|^2) \\
&\le (1+\gamma_n)\|x_n-p\|^2 + c_n + \alpha_n(\|u+(\gamma f-\bar{V})p\|^2 + \|y_n-p\|^2),
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
\|y_n-p\|^2 &\le \frac{1+\gamma_n}{1-\alpha_n}\|x_n-p\|^2 + \frac{\alpha_n}{1-\alpha_n}\|u+(\gamma f-\bar{V})p\|^2 + \frac{1}{1-\alpha_n}c_n \\
&= (1+\frac{\alpha_n+\gamma_n}{1-\alpha_n})\|x_n-p\|^2 + \frac{\alpha_n}{1-\alpha_n}\|u+(\gamma f-\bar{V})p\|^2 + \frac{1}{1-\alpha_n}c_n \\
&\le (1+\frac{\alpha_n+\gamma_n}{1-\alpha_n})\|x_n-p\|^2 + \frac{\alpha_n+\gamma_n}{1-\alpha_n}\|u+(\gamma f-\bar{V})p\|^2 + \frac{1}{1-\alpha_n}c_n \\
&= \|x_n-p\|^2 + \frac{\alpha_n+\gamma_n}{1-\alpha_n}(\|x_n-p\|^2 + \|u+(\gamma f-\bar{V})p\|^2) + \frac{1}{1-\alpha_n}c_n \qquad (3.7) \\
&\le \|x_n-p\|^2 + (\alpha_n+\gamma_n)\varrho(\|x_n-p\|^2 + \|u+(\gamma f-\bar{V})p\|^2) + \varrho c_n \\
&\le \|x_n-p\|^2 + (\alpha_n+\gamma_n)\Gamma_n\varrho + c_n\varrho \\
&= \|x_n-p\|^2 + \theta_n,
\end{aligned}
$$

where $\theta_n = (\alpha_n+\gamma_n)\Gamma_n\varrho + c_n\varrho$, $\Gamma_n = \sup\{\|x_n-p\|^2 + \|u+(\gamma f-\bar{V})p\|^2 : p \in \Omega\} < \infty$, and $\varrho = \frac{1}{1-\sup_{n\ge1}\alpha_n} < \infty$ (due to $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$). Hence $p \in C_{n+1}$. This implies that $\Omega \subset C_n$ for all $n \ge 1$. Therefore, $\{x_n\}$ is well defined.

Step 2. We prove that $\|x_n - k_n\| \to 0$ as $n \to \infty$.

Indeed, let $x^* = P_\Omega x_0$. From $x_n = P_{C_n}x_0$ and $x^* \in \Omega \subset C_n$, we obtain

$$\|x_n - x_0\| \le \|x^* - x_0\|. \qquad (3.8)$$

This implies that $\{x_n\}$ is bounded and hence $\{u_n\}, \{z_n\}, \{k_n\}$ and $\{y_n\}$ are also bounded. Since $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n}x_0$, we have

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \quad \forall n \ge 1.$$

Therefore $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From $x_n = P_{C_n}x_0$, $x_{n+1} \in C_{n+1} \subset C_n$, by Proposition 2.3 (ii) we obtain

$$\|x_{n+1} - x_n\|^2 \le \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2,$$

which implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (3.9)$$

It follows from $x_{n+1} \in C_{n+1}$ that $\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n$ and hence

$$
\begin{aligned}
\|x_n - y_n\|^2 &\le 2(\|x_n - x_{n+1}\|^2 + \|x_{n+1} - y_n\|^2) \\
&\le 2(\|x_n - x_{n+1}\|^2 + \|x_n - x_{n+1}\|^2 + \theta_n) \\
&= 2(2\|x_n - x_{n+1}\|^2 + \theta_n).
\end{aligned}
$$

From (3.9) and $\lim_{n\to\infty}\theta_n = 0$, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.10}$$

Also, utilizing Lemmas 2.6 and 2.7 (b) we obtain from (3.1), and (3.4)-(3.6) that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(u + \gamma f(x_n) - \bar{V}W_nGk_n) + \beta_n(k_n - p) + (1-\beta_n)(W_nGk_n - p)\|^2 \\
&\le \|\beta_n(k_n - p) + (1-\beta_n)(W_nGk_n - p)\|^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(x_n) - \bar{V}W_nGk_n, y_n - p\rangle \\
&= \beta_n\|k_n - p\|^2 + (1-\beta_n)\|W_nGk_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\quad + 2\alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\|\|y_n - p\| \\
&\le \beta_n\|k_n - p\|^2 + (1-\beta_n)\|k_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\quad + 2\alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\|\|y_n - p\| \\
&= \|k_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\quad + 2\alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\|\|y_n - p\| \\
&\le (1+\gamma_n)\|z_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\quad + 2\alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\|\|y_n - p\| \\
&\le (1+\gamma_n)\|x_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\quad + 2\alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\|\|y_n - p\|,
\end{aligned}$$

which leads to

$$\begin{aligned}
\beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\|u \\
&\quad + \gamma f(x_n) - \bar{V}W_nGk_n\|\|y_n - p\| \\
&\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \gamma_n\|x_n - p\|^2 + c_n \\
&\quad + 2\alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\|\|y_n - p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}c_n = 0$, it follows from (3.10) and condition (iii) that

$$\lim_{n\to\infty}\|k_n - W_nGk_n\| = 0. \tag{3.11}$$

Note that

$$y_n - k_n = \alpha_n(u + \gamma f(x_n) - \bar{V}W_nGk_n) + (1-\beta_n)(W_nGk_n - k_n),$$

which yields

$$\begin{aligned}
\|x_n - k_n\| &\le \|x_n - y_n\| + \|y_n - k_n\| \\
&\le \|x_n - y_n\| + \|\alpha_n(u + \gamma f(x_n) - \bar{V}W_nGk_n) + (1-\beta_n)(W_nGk_n - k_n)\| \\
&\le \|x_n - y_n\| + \alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\| + (1-\beta_n)\|W_nGk_n - k_n\| \\
&\le \|x_n - y_n\| + \alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\| + \|W_nGk_n - k_n\|.
\end{aligned}$$

So, from (3.10), (3.11) and $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\lim_{n\to\infty}\|x_n - k_n\| = 0. \tag{3.12}$$

Step 3. We prove that $\|x_n - u_n\| \to 0,\ \|u_n - z_n\| \to 0,\ \|k_n - Gk_n\| \to 0,\ \|k_n - Wk_n\| \to 0$ and $\|z_n - Sz_n\| \to 0$ as $n \to \infty$.

Indeed, from (3.4) and (3.5) it follows that

$$\begin{aligned} \|k_n - p\|^2 &\le [1 + \gamma_n(1-\delta_n)]\|z_n - p\|^2 \\ &\quad + (1-\delta_n)(k-\delta_n)\|z_n - S^n z_n\|^2 + (1-\delta_n)c_n \\ &\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\ &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned} \tag{3.13}$$

Next let us show that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.14}$$

For $p \in \Omega$, we find that

$$\begin{aligned} \|u_n - p\|^2 &= \|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\|^2 \\ &\le \|(I - r_n A)x_n - (I - r_n A)p\|^2 \\ &= \|x_n - p - r_n(Ax_n - Ap)\|^2 \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2. \end{aligned} \tag{3.15}$$

Combining (3.3), (3.13) and (3.15), we obtain

$$\begin{aligned} \|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2 + \gamma_n\|x_n - p\|^2 + c_n, \end{aligned}$$

which immediately implies that

$$\begin{aligned} r_n(2\zeta - r_n)\|Ax_n - Ap\|^2 &\le \|x_n - p\|^2 - \|k_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - k_n\|(\|x_n - p\| + \|k_n - p\|) + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned}$$

Since $\lim_{n\to\infty}\gamma_n = 0,\ \lim_{n\to\infty} c_n = 0$ and $\{x_n\}$ and $\{k_n\}$ are bounded sequences, it follows from (3.12) and condition (iii) that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.16}$$

Furthermore, from the firm nonexpansivity of $S_{r_n}^{(\Theta,\varphi)}$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\|^2 \\ &\le \langle (I - r_n A)x_n - (I - r_n A)p, u_n - p\rangle \\ &= \frac{1}{2}[\|(I - r_n A)x_n - (I - r_n A)p\|^2 + \|u_n - p\|^2 \\ &\quad - \|(I - r_n A)x_n - (I - r_n A)p - (u_n - p)\|^2] \\ &\le \frac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Ax_n - Ap)\|^2] \\ &= \frac{1}{2}[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Ax_n - Ap, x_n - u_n\rangle \\ &\quad - r_n^2\|Ax_n - Ap\|^2], \end{aligned}$$

which leads to

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\|. \tag{3.17}$$

From (3.13) and (3.17), we have

$$\begin{aligned} \|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| \\ &\quad +\gamma_n\|x_n - p\|^2 + c_n, \end{aligned}$$

which hence implies that

$$\begin{aligned} \|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|k_n - p\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| \\ &\quad + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - k_n\|(\|x_n - p\| + \|k_n - p\|) + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| \\ &\quad + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned}$$

Since $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$ and $\{x_n\}, \{u_n\}$ and $\{k_n\}$ are bounded sequences, it follows from (3.12) and (3.16) that (3.14) holds.

Next we show that $\lim_{n\to\infty} \|B_i\Lambda_n^i u_n - B_i p\| = 0$, $i = 1, 2, ..., N$. Observe that

$$\begin{aligned} \|\Lambda_n^i u_n - p\|^2 &= \|J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)p\|^2 \\ &\le \|(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)p\|^2 \\ &\le \|\Lambda_n^{i-1}u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2 \\ &\le \|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2 \\ &\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2. \end{aligned} \tag{3.18}$$

Combining (3.13) and (3.18), we have

$$\begin{aligned} \|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|\Lambda_n^i u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2 \\ &\quad + \gamma_n\|x_n - p\|^2 + c_n, \end{aligned}$$

which yields

$$\begin{aligned} \lambda_{i,n}(2\eta_i - \lambda_{i,n})\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2 &\le \|x_n - p\|^2 - \|k_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\le \|x_n - k_n\|(\|x_n - p\| + \|k_n - p\|) \\ &\quad + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned}$$

From $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i), i \in \{1, 2, ..., N\}$, $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$ and (3.12), we obtain

$$\lim_{n\to\infty} \|B_i\Lambda_n^{i-1}u_n - B_ip\| = 0, \quad i = 1, 2, ..., N. \tag{3.19}$$

By Lemma 2.12 and Lemma 2.7 (a), we obtain

$$\begin{aligned} \|\Lambda_n^i u_n - p\|^2 &= \|J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)p\|^2 \\ &\le \langle (I - \lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)p, \Lambda_n^i u_n - p\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(\|(I-\lambda_{i,n}B_i)\varLambda_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p\|^2 + \|\varLambda_n^i u_n - p\|^2 \\
&\quad - \|(I-\lambda_{i,n}B_i)\varLambda_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p - (\varLambda_n^i u_n - p)\|^2) \\
&\le \frac{1}{2}(\|\varLambda_n^{i-1}u_n - p\|^2 + \|\varLambda_n^i u_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n - \lambda_{i,n}(B_i\varLambda_n^{i-1}u_n - B_i p)\|^2) \\
&\le \frac{1}{2}(\|u_n - p\|^2 + \|\varLambda_n^i u_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n - \lambda_{i,n}(B_i\varLambda_n^{i-1}u_n - B_i p)\|^2) \\
&\le \frac{1}{2}(\|x_n - p\|^2 + \|\varLambda_n^i u_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n - \lambda_{i,n}(B_i\varLambda_n^{i-1}u_n - B_i p)\|^2),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\varLambda_n^i u_n - p\|^2 &\le \|x_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n - \lambda_{i,n}(B_i\varLambda_n^{i-1}u_n - B_i p)\|^2 \\
&= \|x_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|^2 - \lambda_{i,n}^2\|B_i\varLambda_n^{i-1}u_n - B_i p\|^2 \\
&\quad + 2\lambda_{i,n}\langle \varLambda_n^{i-1}u_n - \varLambda_n^i u_n, B_i\varLambda_n^{i-1}u_n - B_i p\rangle \\
&\le \|x_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|^2 \\
&\quad + 2\lambda_{i,n}\|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|\|B_i\varLambda_n^{i-1}u_n - B_i p\|.
\end{aligned} \tag{3.20}
$$

Combining (3.13) and (3.20) we get

$$
\begin{aligned}
\|k_n - p\|^2 &\le \|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\le \|\varLambda_n^i u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\le \|x_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|^2 \\
&\quad + 2\lambda_{i,n}\|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|\|B_i\varLambda_n^{i-1}u_n - B_i p\| + \gamma_n\|x_n - p\|^2 + c_n,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|^2 &\le \|x_n - p\|^2 - \|k_n - p\|^2 \\
&\quad + 2\lambda_{i,n}\|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|\|B_i\varLambda_n^{i-1}u_n - B_i p\| \\
&\quad + \gamma_n\|x_n - p\|^2 + c_n \\
&\le \|x_n - k_n\|(\|x_n - p\| + \|k_n - p\|) \\
&\quad + 2\lambda_{i,n}\|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|\|B_i\varLambda_n^{i-1}u_n - B_i p\| \\
&\quad + \gamma_n\|x_n - p\|^2 + c_n.
\end{aligned}
$$

Since $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$ and $\{x_n\}, \{u_n\}$ and $\{k_n\}$ are bounded sequences, we obtain from (3.12) and (3.19) that

$$
\lim_{n\to\infty}\|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\| = 0, \quad i = 1, 2, ..., N. \tag{3.21}
$$

From (3.21) we get

$$
\begin{aligned}
\|u_n - z_n\| &= \|\varLambda_n^0 u_n - \varLambda_n^N u_n\| \\
&\le \|\varLambda_n^0 u_n - \varLambda_n^1 u_n\| + \|\varLambda_n^1 u_n - \varLambda_n^2 u_n\| + \cdots + \|\varLambda_n^{N-1}u_n - \varLambda_n^N u_n\| \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.22}
$$

By (3.14) and (3.22), we have

$$\begin{aligned}\|x_n - z_n\| &\leq \|x_n - u_n\| + \|u_n - z_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.23}$$

From (3.9) and (3.23), we have

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.24}$$

By (3.12) and (3.23), we get

$$\begin{aligned}\|k_n - z_n\| &\leq \|k_n - x_n\| + \|x_n - z_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.25}$$

We observe that

$$k_n - z_n = (1 - \delta_n)(S^n z_n - z_n).$$

From $\delta_n \leq d < 1$ and (3.25), we have

$$\lim_{n\to\infty} \|S^n z_n - z_n\| = 0. \tag{3.26}$$

We note that

$$\begin{aligned}\|S^n z_n - S^{n+1} z_n\| \leq{}& \|S^n z_n - z_n\| + \|z_n - z_{n+1}\| \\ &+ \|z_{n+1} - S^{n+1} z_{n+1}\| + \|S^{n+1} z_{n+1} - S^{n+1} z_n\|.\end{aligned}$$

From (3.24), (3.26) and Lemma 2.17, we obtain

$$\lim_{n\to\infty} \|S^n z_n - S^{n+1} z_n\| = 0. \tag{3.27}$$

On the other hand, we note that

$$\|z_n - Sz_n\| \leq \|z_n - S^n z_n\| + \|S^n z_n - S^{n+1} z_n\| + \|S^{n+1} z_n - Sz_n\|.$$

From (3.26), (3.27) and the uniform continuity of S, we have

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{3.28}$$

On the other hand, for simplicity, we write $\tilde{p} = T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p$, $v_n = T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)k_n$ and $\tilde{v}_n = Gk_n = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)v_n$ for all $n \geq 1$. Then

$$p = Gp = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)\tilde{p} = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p.$$

We now show that $\lim_{n\to\infty} \|Gk_n - k_n\| = 0$, i.e., $\lim_{n\to\infty} \|\tilde{v}_n - k_n\| = 0$. As a matter of fact, for $p \in \Omega$, it follows from (3.1), (3.4)-(3.6) and (3.13) that

$$\begin{aligned}\|y_n - p\|^2 ={}& \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) + ((1 - \beta_n)I - \alpha_n\bar{V})(W_n Gk_n - p) \\ &+ \alpha_n(u + \gamma f(p) - \bar{V}p)\|^2 \\ \leq{}& \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) + ((1 - \beta_n)I - \alpha_n\bar{V})(W_n Gk_n - p)\|^2 \\ &+ 2\alpha_n\langle u + \gamma f(p) - \bar{V}p, y_n - p\rangle \\ \leq{}& [\alpha_n\gamma\|f(x_n) - f(p)\| + \beta_n\|k_n - p\| + \|(1 - \beta_n)I - \alpha_n\bar{V}\|\|W_n Gk_n - p\|]^2 \\ &+ 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\ \leq{}& [\alpha_n\gamma l\|x_n - p\| + \beta_n\|k_n - p\| + (1 - \beta_n - \alpha_n - \alpha_n\mu\bar{\gamma})\|Gk_n - p\|]^2 \qquad (3.29)\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\le [\alpha_n(1+\mu)\bar{\gamma}\|x_n - p\| + \beta_n\|k_n - p\| + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})\|Gk_n - p\|]^2 \\
&\quad + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})\|\tilde{v}_n - p\|^2 \\
&\quad + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})[\|v_n - \tilde{p}\|^2 \\
&\quad + \nu_1(\nu_1 - 2\zeta_1)\|A_1v_n - A_1\tilde{p}\|^2] + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})[\|k_n - p\|^2 \\
&\quad + \nu_2(\nu_2 - 2\zeta_2)\|A_2k_n - A_2p\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|A_1v_n - A_1\tilde{p}\|^2] \\
&\quad + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&= \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + (1 - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|^2 \\
&\quad + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\quad + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})[\nu_2(\nu_2 - 2\zeta_2)\|A_2k_n - A_2p\|^2 \\
&\qquad + \nu_1(\nu_1 - 2\zeta_1)\|A_1v_n - A_1\tilde{p}\|^2] \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + (1 - \alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|z_n - p\|^2 + c_n) \\
&\quad + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\quad + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})[\nu_2(\nu_2 - 2\zeta_2)\|A_2k_n - A_2p\|^2 \\
&\qquad + \nu_1(\nu_1 - 2\zeta_1)\|A_1v_n - A_1\tilde{p}\|^2] \\
&\le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + (1 - \alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|x_n - p\|^2 + c_n) \\
&\quad + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\quad + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})[\nu_2(\nu_2 - 2\zeta_2)\|A_2k_n - A_2p\|^2 \\
&\qquad + \nu_1(\nu_1 - 2\zeta_1)\|A_1v_n - A_1\tilde{p}\|^2] \\
&= \|x_n - p\|^2 + (1 - \alpha_n(1+\mu)\bar{\gamma})(\gamma_n\|x_n - p\|^2 + c_n) \\
&\quad + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\quad + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})[\nu_2(\nu_2 - 2\zeta_2)\|A_2k_n - A_2p\|^2 \\
&\quad + \nu_1(\nu_1 - 2\zeta_1)\|A_1v_n - A_1\tilde{p}\|^2] \\
&\le \|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\quad + (1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})[\nu_2(\nu_2 - 2\zeta_2)\|A_2k_n - A_2p\|^2 \\
&\qquad + \nu_1(\nu_1 - 2\zeta_1)\|A_1v_n - A_1\tilde{p}\|^2],
\end{aligned}
$$

which immediately implies that

$$
\begin{aligned}
(1-\beta_n - \alpha_n(1+\mu)\bar{\gamma})&[\nu_2(2\zeta_2 - \nu_2)\|A_2k_n - A_2p\|^2 \\
&+ \nu_1(2\zeta_1 - \nu_1)\|A_1v_n - A_1\tilde{p}\|^2] \\
&\le \|x_n - p\|^2 - \|y_n - p\|^2 + \gamma_n\|x_n - p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ c_n + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\| \\
&\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \gamma_n\|x_n - p\|^2 \\
&\quad + c_n + 2\alpha_n\|u + (\gamma f - \bar{V})p\|\|y_n - p\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$, $\limsup_{n\to\infty}\beta_n < 1$ and $\{x_n\}$ and $\{y_n\}$ are bounded sequences, we conclude from (3.10) and condition (iv) that

$$
\lim_{n\to\infty}\|A_2k_n - A_2p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|A_1v_n - A_1\tilde{p}\| = 0. \tag{3.30}
$$

Also, in terms of the firm nonexpansivity of $T_{\nu_k}^{\Theta_k}$ and the ζ_k-inverse strong monotonicity of A_k for $k = 1, 2$, we obtain from $\nu_k \in (0, 2\zeta_k)$, $k \in \{1, 2\}$ and (3.6) that

$$
\begin{aligned}
\|v_n - \tilde{p}\|^2 &= \|T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)k_n - T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)p\|^2 \\
&\leq \langle (I - \nu_2A_2)k_n - (I - \nu_2A_2)p, v_n - \tilde{p}\rangle \\
&= \frac{1}{2}[\|(I - \nu_2A_2)k_n - (I - \nu_2A_2)p\|^2 + \|v_n - \tilde{p}\|^2 \\
&\quad - \|(I - \nu_2A_2)k_n - (I - \nu_2A_2)p - (v_n - \tilde{p})\|^2] \\
&\leq \frac{1}{2}[\|k_n - p\|^2 + \|v_n - \tilde{p}\|^2 - \|(k_n - v_n) - \nu_2(A_2k_n - A_2p) - (p - \tilde{p})\|^2] \\
&= \frac{1}{2}[\|k_n - p\|^2 + \|v_n - \tilde{p}\|^2 - \|(k_n - v_n) - (p - \tilde{p})\|^2 \\
&\quad + 2\nu_2\langle (k_n - v_n) - (p - \tilde{p}), A_2k_n - A_2p\rangle - \nu_2^2\|A_2k_n - A_2p\|^2],
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tilde{v}_n - p\|^2 &= \|T_{\nu_1}^{\Theta_1}(I - \nu_1A_1)v_n - T_{\nu_1}^{\Theta_1}(I - \nu_1A_1)\tilde{p}\|^2 \\
&\leq \langle (I - \nu_1A_1)v_n - (I - \nu_1A_1)\tilde{p}, \tilde{v}_n - p\rangle \\
&= \frac{1}{2}[\|(I - \nu_1A_1)v_n - (I - \nu_1A_1)\tilde{p}\|^2 + \|\tilde{v}_n - p\|^2 \\
&\quad - \|(I - \nu_1A_1)v_n - (I - \nu_1A_1)\tilde{p} - (\tilde{v}_n - p)\|^2] \\
&\leq \frac{1}{2}[\|v_n - \tilde{p}\|^2 + \|\tilde{v}_n - p\|^2 - \|(v_n - \tilde{v}_n) + (p - \tilde{p})\|^2 \\
&\quad + 2\nu_1\langle A_1v_n - A_1\tilde{p}, (v_n - \tilde{v}_n) + (p - \tilde{p})\rangle - \nu_1^2\|A_1v_n - A_1\tilde{p}\|^2] \\
&\leq \frac{1}{2}[\|k_n - p\|^2 + \|\tilde{v}_n - p\|^2 - \|(v_n - \tilde{v}_n) + (p - \tilde{p})\|^2 \\
&\quad + 2\nu_1\langle A_1v_n - A_1\tilde{p}, (v_n - \tilde{v}_n) + (p - \tilde{p})\rangle].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|v_n - \tilde{p}\|^2 \leq{}& \|k_n - p\|^2 - \|(k_n - v_n) - (p - \tilde{p})\|^2 \\
&+ 2\nu_2\langle (k_n - v_n) - (p - \tilde{p}), A_2k_n - A_2p\rangle - \nu_2^2\|A_2k_n - A_2p\|^2,
\end{aligned} \tag{3.31}
$$

and

$$
\begin{aligned}
\|\tilde{v}_n - p\|^2 \leq{}& \|k_n - p\|^2 - \|(v_n - \tilde{v}_n) + (p - \tilde{p})\|^2 \\
&+ 2\nu_1\|A_1v_n - A_1\tilde{p}\|\|(v_n - \tilde{v}_n) + (p - \tilde{p})\|.
\end{aligned} \tag{3.32}
$$

Consequently, from (3.29) and (3.31) it follows that

$$\begin{aligned}\|y_n-p\|^2 &\le \alpha_n(1+\mu)\bar{\gamma}\|k_n-p\|^2+\beta_n\|k_n-p\|^2+(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|\tilde{v}_n-p\|^2\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\ &\le \alpha_n(1+\mu)\bar{\gamma}\|k_n-p\|^2+\beta_n\|k_n-p\|^2+(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|v_n-\tilde{p}\|^2\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\ &\le \alpha_n(1+\mu)\bar{\gamma}\|k_n-p\|^2+\beta_n\|k_n-p\|^2\\ &\quad+(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})[\|k_n-p\|^2-\|(k_n-v_n)-(p-\tilde{p})\|^2\\ &\qquad+2\nu_2\langle(k_n-v_n)-(p-\tilde{p}),A_2k_n-A_2p\rangle-\nu_2^2\|A_2k_n-A_2p\|^2]\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\ &\le \|k_n-p\|^2-(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|(k_n-v_n)-(p-\tilde{p})\|^2\\ &\quad+2\nu_2\|(k_n-v_n)-(p-\tilde{p})\|\|A_2k_n-A_2p\|\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|,\end{aligned}$$

which hence leads to

$$\begin{aligned}&(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|(k_n-v_n)-(p-\tilde{p})\|^2\\ &\le \|k_n-p\|^2-\|y_n-p\|^2+2\nu_2\|(k_n-v_n)-(p-\tilde{p})\|\|A_2k_n-A_2p\|\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\ &\le \|k_n-y_n\|(\|k_n-p\|+\|y_n-p\|)+2\nu_2\|(k_n-v_n)-(p-\tilde{p})\|\|A_2k_n-A_2p\|\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\ &\le (\|k_n-x_n\|+\|x_n-y_n\|)(\|k_n-p\|+\|y_n-p\|)\\ &\quad+2\nu_2\|(k_n-v_n)-(p-\tilde{p})\|\|A_2k_n-A_2p\|\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\limsup_{n\to\infty}\beta_n<1$ and $\{k_n\},\{v_n\}$ and $\{y_n\}$ are bounded sequences, we conclude from (3.10), (3.12) and (3.30) that

$$\lim_{n\to\infty}\|(k_n-v_n)-(p-\tilde{p})\|=0. \tag{3.33}$$

Furthermore, from (3.29) and (3.32) it follows that

$$\begin{aligned}\|y_n-p\|^2 &\le \alpha_n(1+\mu)\bar{\gamma}\|k_n-p\|^2+\beta_n\|k_n-p\|^2+(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|\tilde{v}_n-p\|^2\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\ &\le \alpha_n(1+\mu)\bar{\gamma}\|k_n-p\|^2+\beta_n\|k_n-p\|^2+(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})[\|k_n-p\|^2\\ &\quad-\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|^2+2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|]\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\ &\le \|k_n-p\|^2-(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|^2\\ &\quad+2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|\\ &\quad+2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|,\end{aligned}$$

which hence yields

$$\begin{aligned}
&(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma})\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|^2\\
&\le \|k_n-p\|^2-\|y_n-p\|^2+2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|\\
&\quad +2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\
&\le \|k_n-y_n\|(\|k_n-p\|+\|y_n-p\|)+2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|\\
&\quad +2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|\\
&\le (\|k_n-x_n\|+\|x_n-y_n\|)(\|k_n-p\|+\|y_n-p\|)\\
&\quad +2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|\\
&\quad +2\alpha_n\|u+(\gamma f-\bar{V})p\|\|y_n-p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\limsup_{n\to\infty}\beta_n<1$ and $\{k_n\},\{v_n\},\{y_n\}$ and $\{\tilde{v}_n\}$ are bounded sequences, we conclude from (3.10), (3.12) and (3.30) that

$$\lim_{n\to\infty}\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|=0. \tag{3.34}$$

Note that

$$\|k_n-\tilde{v}_n\|\le\|(k_n-v_n)-(p-\tilde{p})\|+\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|.$$

Hence from (3.33) and (3.34) we get

$$\lim_{n\to\infty}\|k_n-\tilde{v}_n\|=\lim_{n\to\infty}\|k_n-Gk_n\|=0, \tag{3.35}$$

which together with (3.11) and (3.35), implies that

$$\begin{aligned}
\|k_n-W_nk_n\| &\le \|k_n-W_nGk_n\|+\|W_nGk_n-W_nk_n\|\\
&\le \|k_n-W_nGk_n\|+\|Gk_n-k_n\|\\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.35}$$

Also, observe that

$$\|k_n-Wk_n\|\le\|k_n-W_nk_n\|+\|W_nk_n-Wk_n\|.$$

From (3.36), [18, Remark 3.2] and the boundedness of $\{k_n\}$ we immediately obtain

$$\lim_{n\to\infty}\|k_n-Wk_n\|=0.$$

Step 4. We prove that $x_n\to x^*=P_\Omega x_0$ as $n\to\infty$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to some w. From (3.12), (3.14), (3.21) and (3.23), we have that $k_{n_i}\rightharpoonup w$, $u_{n_i}\rightharpoonup w$, $\Lambda_{n_i}^m u_{n_i}\rightharpoonup w$ and $z_{n_i}\rightharpoonup w$, where $m\in\{1,2,...,N\}$. Since S is uniformly continuous, by (3.28) we get $\lim_{n\to\infty}\|z_n-S^mz_n\|=0$ for any $m\ge 1$. Hence from Lemma 2.19, we obtain $w\in\mathrm{Fix}(S)$. In the meantime, utilizing Lemma 2.10, we deduce from $k_{n_i}\rightharpoonup w$, (3.35) and (3.37) that $w\in\mathrm{SGEP}(G)$ and $w\in\mathrm{Fix}(W)=\cap_{n=1}^\infty\mathrm{Fix}(T_n)$ (due to Lemma 2.9). Next, we prove that $w\in\cap_{m=1}^N\mathrm{I}(B_m,R_m)$. As a matter of fact, since B_m is η_m-inverse strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 2.15 that R_m+B_m is maximal monotone. Let $(v,g)\in G(R_m+B_m)$, i.e., $g-B_mv\in R_mv$.

Again, since $\Lambda_n^m u_n = J_{R_m,\lambda_{m,n}}(I-\lambda_{m,n}B_m)\Lambda_n^{m-1}u_n, n \ge 1, m \in \{1,2,...,N\}$, we have

$$\Lambda_n^{m-1}u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n \in (I+\lambda_{m,n}R_m)\Lambda_n^m u_n,$$

that is,

$$\frac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n) \in R_m\Lambda_n^m u_n.$$

In terms of the monotonicity of R_m, we get

$$\langle v-\Lambda_n^m u_n, g - B_m v - \frac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n)\rangle \ge 0$$

and hence

$$\begin{aligned}\langle v-\Lambda_n^m u_n, g\rangle &\ge \Big\langle v-\Lambda_n^m u_n, B_m v + \frac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n)\Big\rangle \\ &= \Big\langle v-\Lambda_n^m u_n, B_m v - B_m\Lambda_n^m u_n + B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n \\ &\quad + \frac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n)\Big\rangle \\ &\ge \langle v-\Lambda_n^m u_n, B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n\rangle \\ &\quad + \Big\langle v-\Lambda_n^m u_n, \frac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n)\Big\rangle.\end{aligned}$$

In particular,

$$\begin{aligned}\langle v-\Lambda_{n_i}^m u_{n_i}, g\rangle \ge \langle v-\Lambda_{n_i}^m u_{n_i}, B_m\Lambda_{n_i}^m u_{n_i} - B_m\Lambda_{n_i}^{m-1}u_{n_i}\rangle \\ + \Big\langle v-\Lambda_{n_i}^m u_{n_i}, \frac{1}{\lambda_{m,n_i}}(\Lambda_{n_i}^{m-1}u_{n_i} - \Lambda_{n_i}^m u_{n_i})\Big\rangle.\end{aligned}$$

Since $\|\Lambda_n^m u_n - \Lambda_n^{m-1}u_n\| \to 0$ (due to (3.21)) and $\|B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n\| \to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$ and condition (iv) that

$$\lim_{i\to\infty}\langle v-\Lambda_{n_i}^m u_{n_i}, g\rangle = \langle v-w, g\rangle \ge 0.$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m+B_m)w$, i.e., $w \in \mathrm{I}(B_m,R_m)$. Therefore, $w \in \cap_{m=1}^N \mathrm{I}(B_m,R_m)$.

Next, we show that $w \in \mathrm{GMEP}(\Theta,\varphi,A)$. In fact, from $u_n = S_{r_n}^{(\Theta,\varphi)}(I-r_nA)x_n$, we know that

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\langle K'(u_n)-K'(x_n), y-u_n\rangle \ge 0, \quad \forall y \in C.$$

From (H2) it follows that

$$\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\langle K'(u_n)-K'(x_n), y-u_n\rangle \ge \Theta(y,u_n), \quad \forall y \in C.$$

Replacing n by n_i, we have

$$\begin{aligned}(3.36)\quad \varphi(y)-\varphi(u_{n_i})+\langle Ax_{n_i}, y-u_{n_i}\rangle + \Big\langle \frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}}, y-u_{n_i}\Big\rangle \\ \ge \Theta(y,u_{n_i}), \quad \forall y \in C.\end{aligned}$$

Put $u_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Then, from (3.38) we have

$$\begin{aligned}\langle u_t - u_{n_i}, Au_t\rangle &\geq \langle u_t - u_{n_i}, Au_t\rangle - \varphi(u_t) + \varphi(u_{n_i}) - \langle u_t - u_{n_i}, Ax_{n_i}\rangle \\ &\quad - \left\langle \frac{K'(u_{n_i}) - K'(x_{n_i})}{r_{n_i}}, u_t - u_{n_i}\right\rangle + \Theta(u_t, u_{n_i}) \\ &\geq \langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle + \langle u_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\ &\quad - \varphi(u_t) + \varphi(u_{n_i}) - \left\langle \frac{K'(u_{n_i}) - K'(x_{n_i})}{r_{n_i}}, u_t - u_{n_i}\right\rangle + \Theta(u_t, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$, we deduce from the Lipschitz continuity of A and K' that $\|Au_{n_i} - Ax_{n_i}\| \to 0$ and $\|K'(u_{n_i}) - K'(x_{n_i})\| \to 0$ as $i \to \infty$. Further, from the monotonicity of A, we have $\langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle \geq 0$. So, from (H4), the weakly lower semicontinuity of φ, $\frac{K'(u_{n_i}) - K'(x_{n_i})}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, we have

$$\langle u_t - w, Au_t\rangle \geq -\varphi(u_t) + \varphi(w) + \Theta(u_t, w), \quad \text{as } i \to \infty. \tag{3.37}$$

From (H1), (H4) and (3.39) we also have

$$\begin{aligned}0 &= \Theta(u_t, u_t) + \varphi(u_t) - \varphi(u_t) \\ &\leq t\Theta(u_t, y) + (1-t)\Theta(u_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(u_t) \\ &= t[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)] + (1-t)[\Theta(u_t, w) + \varphi(w) - \varphi(w) - \varphi(u_t)] \\ &\leq t[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)] + (1-t)\langle u_t - w, Au_t\rangle \\ &= t[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)] + (1-t)t\langle y - w, Au_t\rangle,\end{aligned}$$

and hence

$$0 \leq \Theta(u_t, y) + \varphi(y) - \varphi(u_t) + (1-t)\langle y - w, Au_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta(w, y) + \varphi(y) - \varphi(w) + \langle Aw, y - w\rangle.$$

This implies that $w \in \mathrm{GMEP}(\Theta, \varphi, A)$. Consequently, $w \in \Omega = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \cap_{i=1}^{N} \mathrm{I}(B_i, R_i) \cap \mathrm{Fix}(S)$. This shows that $\omega_w(x_n) \subset \Omega$. From (3.8) and Lemma 2.23, we infer that $x_n \to x^* = P_\Omega x_0$ as $n \to \infty$.

Finally, assume additionally that $\gamma_n + c_n = o(\alpha_n)$ and $\|x_n - y_n\| = o(\alpha_n)$. It is clear that

$$\langle(\bar{V} - \gamma f)x - (\bar{V} - \gamma f)y, x - y\rangle \geq ((1+\mu)\bar{\gamma} - \gamma l)\|x - y\|^2, \quad \forall x, y \in H.$$

So, we know from $0 \leq \gamma l < (1+\mu)\bar{\gamma}$ that $\bar{V} - \gamma f$ is $((1+\mu)\bar{\gamma} - \gamma l)$-strongly monotone. In the meantime, it is easy to see that $\bar{V} - \gamma f$ is $(\|\bar{V}\| + \gamma l)$-Lipschitzian with constant $\|\bar{V}\| + \gamma l > 0$. Thus, there exists a unique solution p in Ω to the VIP

$$\langle u + (\gamma f - \bar{V})p, v - p\rangle \leq 0, \quad \forall v \in \Omega.$$

Consequently, we deduce from (3.10) and $x_n \to x^* = P_\Omega x_0$ $(n \to \infty)$ that

$$\begin{aligned}\limsup_{n\to\infty}\langle u + (\gamma f - \bar{V})p, y_n - p\rangle &= \limsup_{n\to\infty}(\langle u + (\gamma f - \bar{V})p, x_n - p\rangle \\ &\quad + \langle u + (\gamma f - \bar{V})p, y_n - x_n\rangle) \\ &= \limsup_{n\to\infty}\langle u + (\gamma f - \bar{V})p, x_n - p\rangle \\ &= \langle u + (\gamma f - \bar{V})p, x^* - p\rangle \leq 0.\end{aligned} \tag{3.38}$$

Furthermore, by Lemma 2.6 we conclude from (3.1) and (3.4)-(3.6) that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(u + \gamma(f(x_n) - \bar{V}p)) + \beta_n(k_n - p) \\
&\qquad + ((1-\beta_n)I - \alpha_n\bar{V})(W_nGk_n - p)\|^2 \\
&= \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) \\
&\qquad + ((1-\beta_n)I - \alpha_n\bar{V})(W_nGk_n - p) + \alpha_n(u + \gamma f(p) - \bar{V}p)\|^2 \\
&\le \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(k_n - p) \\
&\qquad + ((1-\beta_n)I - \alpha_n\bar{V})(W_nGk_n - p)\|^2 \\
&\quad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&\le [\alpha_n\gamma\|f(x_n) - f(p)\| + \beta_n\|k_n - p\| \\
&\qquad + \|(1-\beta_n)I - \alpha_n\bar{V}\|\|W_nGk_n - p\|]^2 \\
&\quad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&\le [\alpha_n\gamma l\|x_n - p\| + \beta_n\|k_n - p\| + (1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|]^2 \\
&\quad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&= [\alpha_n(1+\mu)\bar{\gamma}\frac{\gamma l}{(1+\mu)\bar{\gamma}}\|x_n - p\| + \beta_n\|k_n - p\| \\
&\qquad + (1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|]^2 \\
&\quad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&= [\alpha_n(1+\mu)\bar{\gamma}\frac{\gamma l}{(1+\mu)\bar{\gamma}}\|x_n - p\| + (1 - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|]^2 \\
&\quad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&\le \alpha_n(1+\mu)\bar{\gamma}\frac{(\gamma l)^2}{(1+\mu)^2\bar{\gamma}^2}\|x_n - p\|^2 \\
&\quad + (1 - \alpha_n(1+\mu)\bar{\gamma})\|k_n - p\|^2 + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&\le \alpha_n(1+\mu)\bar{\gamma}\frac{(\gamma l)^2}{(1+\mu)^2\bar{\gamma}^2}\|x_n - p\|^2 \\
&\quad + (1 - \alpha_n(1+\mu)\bar{\gamma})((1+\gamma_n)\|z_n - p\|^2 + c_n) \\
&\quad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&\le \alpha_n\frac{(\gamma l)^2}{(1+\mu)\bar{\gamma}}\|x_n - p\|^2 + (1 - \alpha_n(1+\mu)\bar{\gamma})\|x_n - p\|^2 \\
&\quad + (1 - \alpha_n(1+\mu)\bar{\gamma})(\gamma_n\|x_n - p\|^2 + c_n) \\
&\quad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle \\
&\le (1 - \alpha_n\frac{(1+\mu)^2\bar{\gamma}^2 - (\gamma l)^2}{(1+\mu)\bar{\gamma}})\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n\langle u + (\gamma f - \bar{V})p, y_n - p\rangle,
\end{aligned}$$

and hence

$$\begin{aligned}\frac{(1+\mu)^2\bar{\gamma}^2-(\gamma l)^2}{(1+\mu)\bar{\gamma}}\|x_n-p\|^2 &\le \frac{\|x_n-p\|^2-\|y_n-p\|^2}{\alpha_n}+\frac{\gamma_n\|x_n-p\|^2+c_n}{\alpha_n}\\ &\quad+2\langle u+(\gamma f-\bar{V})p, y_n-p\rangle\\ &\le \frac{\|x_n-y_n\|}{\alpha_n}(\|x_n-p\|+\|y_n-p\|)\\ &\quad+\frac{\gamma_n+c_n}{\alpha_n}(\|x_n-p\|^2+1)\\ &\quad+2\langle u+(\gamma f-\bar{V})p, y_n-p\rangle.\end{aligned}$$

Since $\gamma_n+c_n=o(\alpha_n)$, $\|x_n-y_n\|=o(\alpha_n)$ and $x_n\to x^*=P_\Omega x_0$, we infer from (3.40) and $0\le\gamma l<(1+\mu)\bar{\gamma}$ that as $n\to\infty$

$$\frac{(1+\mu)^2\bar{\gamma}^2-(\gamma l)^2}{(1+\mu)\bar{\gamma}}\|x^*-p\|^2\le 0.$$

That is, $p=x^*=P_\Omega x_0$. By Lemma 2.16, we infer that x^* also solves the following optimization problem:

$$\text{(OP2)}\qquad \min_{x\in\Omega}\frac{\mu}{2}\langle Vx,x\rangle+\frac{1}{2}\|x-u\|^2-h(x)$$

where $h: H\to\mathbf{R}$ is the potential function of γf. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ,Θ_1,Θ_2 be three bifunctions from $C\times C$ to $\mathbf{R}$ satisfying (H1)-(H4) and $\varphi: C\to\mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i: C\to 2^H$ be a maximal monotone mapping and let $A,A_k: H\to H$ and $B_i: C\to H$ be ζ-inverse strongly monotone, ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, for $k=1,2$ and $i=1,2$. Let $S: C\to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0\le k<1$ with sequence $\{\gamma_n\}\subset[0,\infty)$ such that $\lim_{n\to\infty}\gamma_n=0$ and $\{c_n\}\subset[0,\infty)$ such that $\lim_{n\to\infty}c_n=0$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b\in(0,1)$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator and $f: H\to H$ be an l-Lipschitzian mapping with $\gamma l<(1+\mu)\bar{\gamma}$. Let W_n be the W-mapping defined by (1.4). Assume that $\Omega:=\cap_{n=1}^\infty\mathrm{Fix}(T_n)\cap\mathrm{GMEP}(\Theta,\varphi,A)\cap\mathrm{SGEP}(G)\cap\mathrm{I}(B_2,R_2)\cap\mathrm{I}(B_1,R_1)\cap\mathrm{Fix}(S)$ is nonempty and bounded where G is defined as in Proposition 1.1. Let $\{r_n\}$ be a sequence in $[0,2\zeta]$ and $\{\alpha_n\},\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n=0$ and $k\le\delta_n\le d<1$. Pick any $x_0\in H$ and set $C_1=H$, $x_1=P_{C_1}x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\text{(3.39)}\qquad \begin{cases} u_n=S_{r_n}^{(\Theta,\varphi)}(I-r_nA)x_n,\\ z_n=J_{R_2,\lambda_{2,n}}(I-\lambda_{2,n}B_2)J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1)u_n,\\ k_n=\delta_nz_n+(1-\delta_n)S^nz_n,\\ y_n=\alpha_n(u+\gamma f(x_n))+\beta_nk_n+[(1-\beta_n)I-\alpha_n(I+\mu V)]W_nGk_n,\\ C_{n+1}=\{z\in C_n:\|y_n-z\|^2\le\|x_n-z\|^2+\theta_n\},\\ x_{n+1}=P_{C_{n+1}}x_0,\quad \forall n\ge 0,\end{cases}$$

where $\theta_n = (\alpha_n + \gamma_n)\varGamma_n\varrho + c_n\varrho$, $\varGamma_n = \sup\{\|x_n - p\|^2 + \|u + (\gamma f - (I + \mu V))p\|^2 : p \in \varOmega\} < \infty$, *and* $\varrho = \frac{1}{1-\sup_{n\geq 1}\alpha_n} < \infty$. *Assume that the following conditions are satisfied:*

(i) $K : H \to \mathbf{R}$ *is strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that the function* $x \mapsto \langle y - x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y \in H$*;*

(ii) *for each* $x \in H$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$,

$$\varTheta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$*;*

(iv) $\nu_k \in (0, 2\zeta_k)$ *and* $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *for* $k = 1, 2$ *and* $i = 1, 2$.

Suppose that $S_r^{(\varTheta,\varphi)}$ *is firmly nonexpansive. Then the following statements hold:*

(I) $\{x_n\}$ *converges strongly to* $x^* = P_\varOmega x_0$*;*

(II) $\{x_n\}$ *converges strongly to* $x^* = P_\varOmega x_0$ *which solves the following optimization problem provided* $\gamma_n + c_n = o(\alpha_n)$ *and* $\|x_n - y_n\| = o(\alpha_n)$*:*

$$\text{(OP3)} \qquad \min_{x\in\varOmega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)$$

where $h : H \to \mathbf{R}$ *is the potential function of* γf.

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\varTheta, \varTheta_1, \varTheta_2$ *be three bifunctions from* $C \times C$ *to* $\mathbf{R}$ *satisfying (H1)-(H4) and* $\varphi : C \to \mathbf{R}$ *be a lower semicontinuous and convex functional. Let* $R : C \to 2^H$ *be a maximal monotone mapping and let* $A, A_k : H \to H$ *and* $B : C \to H$ *be* ζ*-inverse strongly monotone,* ζ_k*-inverse strongly monotone and* η*-inverse strongly monotone, respectively, for* $k = 1, 2$. *Let* $S : C \to C$ *be a uniformly continuous asymptotically* k*-strict pseudocontractive mapping in the intermediate sense for some* $0 \leq k < 1$ *with sequence* $\{\gamma_n\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} \gamma_n = 0$ *and* $\{c_n\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} c_n = 0$. *Let* V *be a* $\bar{\gamma}$*-strongly positive bounded linear operator and* $f : H \to H$ *be an* l*-Lipschitzian mapping with* $\gamma l < (1 + \mu)\bar{\gamma}$. *Assume that* $\varOmega :=$ $\mathrm{GMEP}(\varTheta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \mathrm{I}(B, R) \cap \mathrm{Fix}(S)$ *is nonempty and bounded where* G *is defined as in Proposition 1.1. Let* $\{r_n\}$ *be a sequence in* $[0, 2\zeta]$ *and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\delta_n\}$ *be sequences in* $(0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $k \leq \delta_n \leq d < 1$. *Pick any* $x_0 \in H$ *and set* $C_1 = H$, $x_1 = P_{C_1}x_0$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\text{(3.40)} \qquad \begin{cases} u_n = S_{r_n}^{(\varTheta,\varphi)}(I - r_nA)x_n, \\ z_n = J_{R,\rho_n}(I - \rho_nB)u_n, \\ k_n = \delta_nz_n + (1 - \delta_n)S^nz_n, \\ y_n = \alpha_n(u + \gamma f(x_n)) + \beta_nk_n + [(1 - \beta_n)I - \alpha_n(I + \mu V)]Gk_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases}$$

where $\theta_n = (\alpha_n + \gamma_n)\Gamma_n\varrho + c_n\varrho$, $\Gamma_n = \sup\{\|x_n - p\|^2 + \|u + (\gamma f - (I + \mu V))p\|^2 : p \in \Omega\} < \infty$, *and* $\varrho = \frac{1}{1-\sup_{n\geq 1}\alpha_n} < \infty$. *Assume that the following conditions are satisfied:*

(i) $K : H \to \mathbf{R}$ *is strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that the function* $x \mapsto \langle y - x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y \in H$*;*

(ii) *for each* $x \in H$*, there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$*,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$*;*

(iv) $\nu_k \in (0, 2\zeta_k)$ *and* $\{\rho_n\} \subset [a, b] \subset (0, 2\eta)$ *for* $k = 1, 2$.

Suppose that $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive. Then the following statements hold:*

(I) $\{x_n\}$ *converges strongly to* $x^* = P_\Omega x_0$*;*

(II) $\{x_n\}$ *converges strongly to* $x^* = P_\Omega x_0$ *which solves the following optimization problem provided* $\gamma_n + c_n = o(\alpha_n)$ *and* $\|x_n - y_n\| = o(\alpha_n)$*:*

$$\text{(OP4)} \qquad \min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)$$

where $h : H \to \mathbf{R}$ *is the potential function of* γf.

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