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ITERATIVE SCHEMES FOR GENERALIZED NONLINEAR COMPLEMENTARITY PROBLEMS ON ISOTONE PROJECTION CONES

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Dedicated to Wataru Takahashi on the occasion of his 70th birth day

Abstract. The aim of this paper is to consider generalized nonlinear complementarity problems on a closed convex cone in Hilbert spaces. First we present an equivalence of fixed point and generalized nonlinear complementarity problem and then two iterative schemes are introduced that converge to a solution of generalized nonlinear complementarity problem under certain conditions. We obtain convergence results by employing the concept of isotone projection cones on Hilbert spaces. Examples are given to support the newly defined notions and the main results.

1. Introduction

Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone and $T : K \rightsquigarrow H$ a set-valued mapping. The generalized nonlinear complementarity problem defined by K and T is to find an element $x^* \in K$ such that there exists $y^* \in Tx^* \cap K^*$ and $\langle x^*, y^* \rangle = 0$, where K^* is a dual cone of K. The generalized nonlinear complementarity problem defined by K and T is denoted by GSNCP$-(K,T)$. The complementarity theory derives its importance form the fact that it unifies problems in the fields such as, mathematical programming, game theory, the theory of equilibrium in a competitive economy, equilibrium of traffic flows, mechanics, engineering lubricant evaporation in the cavity of a cylindrical bearing, elasticity theory, fluid flow through a semiimperable membrane, maximizing oil production, computation of fixed point etc. For further details on complementarity problems and their generalizations, we refer [3–6, 8, 9] and the references therein.

In 1990, Isac and Németh [11] showed that the isotone project cone is a good instrument to study the iterative methods for solving complementarity problems. This method further investigated in [10–14, 17] and the references therein.

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There are several problems from nonlinear analysis, optimization and management which are closely related to a complementarity problem. The fixed point problem is one of them.

A fixed point problem defined by K and T is to find a point $x^* \in K$ such that $x^* \in Tx^*$.

Several equivalence relations between a complementarity problem and a fixed point problem can be found in [1, 2] and the references therein.

In this paper, we use isotone projection cone and fixed point formulation of generalized complementarity problem to propose iterative schemes to find the solutions of generalized complementarity problems.

The organization of this paper is as follows: In Section 2, we present some basic definitions and results needed later on. The concept of isotone projection cones [17] is the basic tool for the convergence of iterative algorithms to a solution of GSNCP$-(K, T)$ that follow in the subsequent sections. For the sake of completeness, we present the basic fact about the equivalence of solution of GSNCP$-(K, T)$ and the fixed point defined by $P_K \circ (I - T)$ and K employing Morea's theorem. This section also contains some new classes of monotone set-valued mappings. In Section 3, we propose two iterative schemes followed by the results dealing with the conditions under which the presented schemes are convergent to a solution of GSNCP$-(K, T)$. We also consider the iterative scheme with mapping defined by a positive scaling of T. As a particular case, we consider the problem of finding the zeros of the mapping T ($x \in K$ is a zero of T if $0 \in Tx$).

2. Preliminaries

Let $(H, \langle ., . \rangle)$ be a real Hilbert space. A subset K of H is called a closed convex cone if it is a closed convex set and for any $\lambda > 0$ and $x \in K$, $\lambda x \in K$. A closed convex cone K is called pointed if $K \cap (-K) = \{0\}$. If $K \subseteq H$ is a closed convex cone, then

$$K^* = \{y \in H : \langle x, y \rangle \geq 0 \text{ for all } x \in K\}$$

is called the dual cone of K and

$$K^\circ = \{y \in H : \langle x, y \rangle \leq 0 \text{ for all } x \in K\}$$

is called the polar of K.

A relation ρ on H is called (a) reflexive if $x\rho x$ for all $x \in H$; (b) transitive if $x\rho y$ and $y\rho z$ imply $x\rho z$; (c) antisymmetric if $x\rho y$ and $y\rho x$ imply $x = y$; (d) translation invariant if $x\rho y$ implies $(x+z)\rho(y+z)$ for any $z \in H$; (e) scale invariant if $x\rho y$ implies $(\lambda x)\rho(\lambda y)$ for any $\lambda > 0$; (f) continuous if for any two convergent sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ with $x_n\rho y_n$ for all $n \in \mathbb{N}$ we have $x^*\rho y^*$, where x^* and y^* are the limits of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, respectively (see for details [17]).

A relation ρ on H is called a preorder if it is reflexive and transitive. A preorder is called order if it is antisymmetric.

The relation ρ on H is a continuous, translation and scale invariant preorder if and only if it is induced by a closed convex cone $K \subseteq H$, that is, $\rho = \leq_K$, where $x \leq_K y$ if and only if $y - x \in K$. For simplicity, we denote " $\leq_K$ " by " $\leq$ ". The closed convex cone K written as

$$K = \{x \in H : 0 \leq x\}$$

is called the positive cone of the preorder " $\leq$ ". The triplet $(H, \langle .,. \rangle, K)$ is called an ordered vector space. If the closed convex cone K is pointed, then the preorder " $\leq$ " becomes an order [17].

A closed convex cone K is called regular if every decreasing sequence of elements in K is convergent. The ordered vector space $(H, \langle .,. \rangle, K)$ is called a vector lattice if for every $x, y \in H$, there exist

$$x \wedge y := \inf\{x, y\}$$

and

$$x \vee y := \sup\{x, y\}.$$

In this case, we say that the cone K is lattical and for each $x \in H$, we denote $x^+ = 0 \vee x$, $x^- = 0 \vee (-x)$ and $|x| = x \vee (-x)$. Then, $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Recall that the pointed closed convex cone $K \subseteq H$ is called an isotone projection cone (see [10–14]) if $y - x \in K$ implies that $P_K(y) - P_K(x) \in K$, where P_K is a metric projection onto K. By using the order relation defined by K, this condition can be written as $x \leq y$ implies that $P_K(x) \leq P_K(y)$ (see [14]).

Every isotone projection cone is lattical and regular [11]. A closed generating cone in $\mathbb{R}^n$ is an isotone projection cone if and only if it is polyhedral and correct. For more details on isotone projection cones, we refer to [14].

Theorem 2.1 (Moreau's Theorem). *Let $(H, \langle .,. \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, K^* the dual cone of K and K° be the polar cone of K. For $x, y, z \in H$, following the statements are equivalent:*

(a) *$z = x + y$, $x \in K$, $y \in K^\circ$ and $\langle x, y \rangle = 0$.*
(b) *$x = P_K z$ and $y = P_{K^\circ} z$.*

We establish the equivalence between a solution of GSNCP$-(K, T)$ and a fixed point of certain nonlinear mapping using Morea's Theorem 2.1.

Lemma 2.2. *Let $(H, \langle .,. \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, and $T : K \rightsquigarrow H$ a set-valued mapping. Then, x^* is a solution of GSNCP$-(K, T)$ if and only if x^* is a fixed point of $P_K \circ (I - T)$.*

Proof. Let x^* be a solution of GSNCP(K, T). Then, there exists $y^* \in Tx^* \cap K^*$ such that $\langle x^*, y^* \rangle = 0$. Let

$$z = x^* - y^* = x^* + y,$$

where $-y^* = y$. As $y^* \in K^*$, so $-y^* \in K^\circ$. By Morea's Theorem 2.1, we have

$$x^* = P_K z = P_K(x^* - y^*) \in P_K \circ (I - T)x^*.$$

Conversely, let $x^* \in P_K \circ (I - T)x^*$. Thus, there exists $y^* \in Tx^*$ such that

$$x^* = P_K(x^* - y^*) = P_K z,$$

where $z = x^* - y^* = x^* + y$. As $x^* = P_K z$ for all $c \in K$, we have

$$\langle x^* - z, x^* - c \rangle \leq 0.$$

In particular, if $c = 0$, then we get $\langle x^* - z, x^* \rangle \leq 0$, that is,

$$\langle y, x^* \rangle \geq 0. \tag{2.1}$$

If $c = 2x^*$, then we get $\langle x^* - z, -x^* \rangle \leq 0$, that is,

$$\langle y, x^* \rangle \leq 0. \tag{2.2}$$

Inequalities (2.1) and (2.2) give

$$\langle y, x^* \rangle = 0. \tag{2.3}$$

Now for an arbitrary $p \in K$, we have

$$\langle y, p \rangle = \langle y, p \rangle + \langle y, -x^* \rangle = \langle z - x^*, p - x^* \rangle \leq 0.$$

This implies that $y \in K^\circ$. Also, for any $q \in K^\circ$, we have

$$\langle z - y, q - y \rangle = \langle x^*, q \rangle - \langle x^*, y \rangle = \langle x^*, q \rangle \leq 0.$$

Hence, $y = P_{K^\circ} z$. Since $z = x^* + y$ and $x^* = P_K z$, by Moreau's Theorem 2.1, we have $\langle x^*, y \rangle = \langle x^*, y^* \rangle = 0$. Hence, x^* is a solution of GSNCP$-(K, T)$. □

Recently, Németh [17] considered the following recursion formula in connection with nonlinear complementarity problem of a single-valued mapping $f : K \to H$

$$x_0 \in K, \ x_{n+1} = P_K(x_n - f x_n), \tag{2.4}$$

where P_K is the projection onto K.

We define an order on the class of subsets of ordered vector space H.

Definition 2.3. Let A and B be two nonempty subsets of a Hilbert space H. Then, $A \leq B$ if and only if $a \leq b$ for every $a \in A$ and for every $b \in B$.

Definition 2.4. Let $T : K \rightsquigarrow H$ be a set-valued mapping, $\{x_n\} \subseteq K$ and $\{z_n\}$ be such that $z_n \in T x_n$. The set-valued mapping T is said to be upper hemicontinuous if a sequence $\{x_n\}$ converges to x and $\{z_n\}$ converges to z implies that $z \in Tx$.

Definition 2.5. Let $(H, \langle ., . \rangle)$ be a Hilbert space $K \subseteq H$ a closed convex cone and $\leq$ be the preorder. The mapping $T : K \rightsquigarrow H$ is called (a) strongly monotone decreasing if for all $x, y \in K$, $x \leq y$ implies that $Ty \leq Tx$; (b) strongly monotone increasing if for all $x, y \in K$, $x \leq y$ implies that $Tx \leq Ty$.

Example 2.6. Let $H = \mathbb{R}$ and $K = \mathbb{R}^+$ be an isotone projection cone. Then, the set-valued mapping defined by

$$Tx = [1 - x, 2 - x],$$

is strongly monotone decreasing.

Definition 2.7. Let $(H, \langle ., . \rangle)$ be a Hilbert space $K \subseteq H$ a closed convex cone and $\leq$ a preorder. The set-valued mapping $T : K \rightsquigarrow H$ is called (a) weakly monotone decreasing if for all $x, y \in K$ with $x \leq y$ and for every $y' \in Ty$, there exists a $x' \in Tx$ such that $y' \leq x'$; (b) weakly monotone increasing if for all $x, y \in K$ with $x \leq y$ for every $x' \in Tx$ there exists a $y' \in Ty$ such that $x' \leq y'$.

Note that every strongly monotone decreasing (increasing) set-valued mapping is weakly monotone decreasing (increasing) mapping. Converse does not hold as shown in the following example.

Example 2.8. Let $H = \mathbb{R}$ and $K = \mathbb{R}^+$. A set-valued mapping $T : K \rightsquigarrow H$ defined by

$$Tx = \begin{cases} [0, 2-x], & \text{if } 0 \le x \le 2, \\ [x-2, 2], & \text{if } 2 < x \le 4, \\ [6-x, 2], & \text{if } 4 < x, \end{cases}$$

is not strongly monotone decreasing as for $3 < 4$, $2 \in T4 = \{2\}$ and $1 \in T3 = [1, 2]$, we have $2 \nleq 1$. On the other hand, A mapping T is weakly monotone decreasing as for all x, y with $x < y$, there is an element 2 in Tx such that $y^{'} \le 2$ for all $y^{'} \in Ty$.

Németh [17] introduced the notion of pseudomonotone decreasing mapping $f : K \to H$. Following is set-valued version of the concept of pseudomonotone decreasing mapping.

Definition 2.9. Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone and $\le$ a preorder. A mapping $T : K \rightsquigarrow H$ is called strongly pseudomonotone decreasing if for all $x, y \in K$ with $x \le y$, $\{0\} \le Ty$ implies $\{0\} \le Tx$.

Definition 2.10. Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone and $\le$ a preorder. The set-valued mapping $T : K \rightsquigarrow H$ is called weakly pseudomonotone decreasing if for all $x, y \in K$ with $x \le y$ and for all $y^{'} \in Ty$ such that $0 \le y^{'}$, there exists $x^{'} \in Tx$ such that $0 \le x^{'}$.

Remark 2.11. (1)
(a) Every strongly monotone decreasing mapping is weakly monotone decreasing and strongly pseudomonotone decreasing, and hence weakly pseudomonotone decreasing.
(b) Every strongly pseudomonotone decreasing set-valued mapping is weakly pseudomonotone decreasing.
(c) If $T(k) \subseteq K$ for each $k \in K$, then T is strongly pseudomonotone decreasing and hence weakly pseudomonotone decreasing. Indeed, for any $x \in K$, we have $\{0\} \le Tx$.

Define a set $T^{-1}(K) = \{x \in K : T(x) \subseteq K\}$.

Example 2.12. Let $H = \mathbb{R}$ and $K = \mathbb{R}^+$. The set-valued mapping $T : K \rightsquigarrow H$ defined by

$$Tx = [0, \ln(x+1)]$$

is not weakly monotone decreasing because for all $x \le y$ we have $\ln(x+1) \le \ln(y+1)$. Since $Tx \subseteq K$ for all x in K, T is strongly pseudomonotone decreasing.

Example 2.13. Let $H = \mathbb{R}$ and $K = \mathbb{R}^+$. A mapping $T : K \rightsquigarrow H$ defined by

$$Tx = \{0, -x, x\}$$

is weakly pseudomonotone decreasing but is not strongly pseudomonotone decreasing.

3. Main results

We propose the following iterative scheme to generate a sequence of points in K.

Algorithm 3.1. Let $(H, \langle .,. \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone and $T : K \rightsquigarrow H$ a set-valued mapping.

(3.1)
$$\begin{cases} \text{For } x_0 \in K, \text{ pick } z_0 \in Tx_0, \text{ and } x_1 = P_K(x_0 - z_0); \\ \text{Pick } z_1 \in Tx_1 \text{ such that } z_0 \geq z_1, \text{ and } x_2 = P_K(x_1 - z_1); \\ \quad\vdots \\ \text{In general, pick } z_n \in Tx_n \text{ such that } z_{n-1} \geq z_n, \text{ define } x_{n+1} = P_K(x_n - z_n), \end{cases}$$

where $n \in \mathbb{N}$.

In this section we study some conditions under which the sequences $\{x_n\}$ and $\{z_n\}$ are convergent. If the sequences $\{x_n\}$ and $\{z_n\}$ converge to x^* and z^*, respectively, and $T : K \rightsquigarrow H$ is upper hemicontinuous then taking limit as n approaches infinity in Algorithm 3.1, we obtain that x^* is a fixed point of a mapping $P_K \circ (I - T)$. In view of Lemma 2.2, x^* is a solution of GSNCP$-(K, T)$.

Throughout this paper, we assume that T is upper hemicontinuous. Now we give two lemmas essential to prove our main results.

Lemma 3.2. *Let $(H, \langle .,. \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone and $T : K \rightsquigarrow H$ a set-valued mapping. If sequences $\{x_n\}$ and $\{z_n\}$ converges to x^* and z^*, respectively, then x^* is a solution of GSNCP$-(K, T)$.*

Proof. Since P_K is nonexpansive, hence continuous, therefore by taking limit as n tends to infinity in Algorithm 3.1, we obtain $z^* \in Tx^*$ and

$$x^* = P_K(x^* - z^*) \in (P_K \circ (I - T))x^*.$$

Hence, by Lemma 2.2, x^* is a solution of GSNCP$-(K, T)$. □

Lemma 3.3. *Let $(H, \langle .,. \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex regular cone and $T : K \rightsquigarrow H$ a set-valued mapping. If the sequences $\{x_n\}$ and $\{z_n\}$ are monotone decreasing, then x^* is a solution of GSNCP$-(K, T)$.*

Proof. Since K is regular, and $\{x_n\}$ and $\{z_n\}$ are monotone decreasing sequence, $\{x_n\}$ and $\{z_n\}$ converge to some x^* and z^*, respectively. Hence, by Lemma 3.2, x^* is a solution of GSNCP$-(K, T)$. □

Now we give the main result of this section.

Theorem 3.4. *Let $(H, \langle .,. \rangle)$ be a Hilbert space, $K \subseteq H$ an isotone projection cone and $T : K \rightsquigarrow H$ be a set-valued mapping. Consider Algorithm 3.1 starting from $x_0 \in K \cap T^{-1}(K)$. Then, a sequence $\{x_n\}$ is convergent and its limit x^* is a solution of GSNCP$-(K, T)$ provided that T is strongly pseudomonotone decreasing.*

Proof. Since K is isotone projection cone, so is regular. The sequence $\{z_n\}$ defined by recursion formula (3.1) is monotone decreasing, so it is enough to show that that sequence $\{x_n\}$ is monotone decreasing. We will prove that $z_n \in K$ for all $n \in \mathbb{N}$.

Indeed, if $z_n \in K$ for all $n \in \mathbb{N}$, then $x_n-(x_n-z_n) \in K$ implies that $x_n-z_n \leq x_n$. Using the fact that K is isotone projection cone, we obtain

$$x_{n+1} = P_K(x_n - z_n) \leq P_K(x_n) = x_n.$$

Hence, the sequence $\{x_n\}$ is monotone decreasing.

Now we will prove the following proposition

$$z_n \in K \text{ holds for all } n \in \mathbb{N}. \tag{3.2}$$

Since $x_0 \in K \cap T^{-1}(K)$, $Tx_0 \subseteq K$. As $z_0 \in Tx_0$, so (3.2) holds true for $n = 0$. Let $z_n \in K$ and we will show that $z_{n+1} \in K$. Since $z_n \in Tx_n$, therefore, by the construction of $\{z_n\}$, we obtain $z_{n+1} \in Tx_{n+1}$ such that $z_n \geq z_{n+1}$. Also, $x_{n+1} \leq x_n$. Since T is strongly pseudomonotone decreasing, therefore, $z_{n+1} \geq 0$, that is, $z_{n+1} \in K$. Hence, the result follows. □

Example 3.5. Let $H = \mathbb{R}$ and $K = \mathbb{R}^+$. Define $T : K \rightsquigarrow H$ by

$$Tx = [0, |\ln(x+1) - 1|].$$

Since $Tx \subseteq K$ for all x in K, T is strongly pseudomonotone decreasing. By recursion formula (3.1), we obtain the sequence $\{x_n\}$ and $\{z_n\}$ as follows:

$$x_n \in K,\ z_n = |\ln(x_n+1) - 1|\,,\ \text{define } x_{n+1} = \max\{0, x_n - |\ln(x_n+1) - 1|\}.$$

It can be easily checked that $x^* \in \{0, e-1\}$ is a solution of GSNCP$-(K,T)$, where $e - 1 = 1.71828182845904523536 0....$ Following is the stopping criteria we used:

$$|x_{n+1} - x_n| \leq 10^{-8}.$$

- **-:** If we start the algorithm from $x_0 = 1$, then it stops at the third step with the solution $x^* = 0$.
- **-:** If we start the algorithm from $x_0 = 1.5$, then it stops at the sixth step with the solution $x^* = 0$.
- **-:** If we start the algorithm from $x_0 = 2$, then it stops at the thirty sixth step with the solution $x^* = 1.718281849044202$.
- **-:** If we start the algorithm from $x_0 = 2.536968$, then it stops at the thirty ninth step with the solution $x^* = 1.718281845949004$.
- **-:** If we start the algorithm from $x_0 = 3$, then it stops at the fortieth step with the solution $x^* = 1.718281847980126$.
- **-:** If we start the algorithm from $x_0 = 69$, then it stops at the seventieth step with the solution $x^* = 1.718281845602664$.

We observe that if we start from any number less than $e - 1$, the algorithm converges to 0. If we start from any number greater than or equal to $e - 1$, the algorithm converges to $e - 1$.

Remark 3.6. Analyzing Theorem 3.4, it is noted that $x^* = 0$ is a solution of GSNCP$-(K,T)$ for a mapping T satisfying the conditions of Theorem 3.4. Note that $0 \leq x_0$ and $0 \leq z'$ for all $z' \in Tx_0$. Since T is strongly pseudomonotone decreasing, $0 \leq y'$ for all $y' \in T(0)$. This implies that $y' \in K$. Since K is isotone projection cone, so $K \subseteq K^*$. This further gives that $y' \in K^*$. Hence, there exists $x^* = 0 \in K$ such that $\langle y', x^*\rangle = 0$ and $y' \in Tx^* \cap K^*$, that is, $x^* = 0$ is a solution of GSNCP$-(K,T)$.

The next theorem gives sufficient conditions for the Algorithm 3.1 to be convergent to a nonzero solution.

Theorem 3.7. *Let H be a Hilbert space, $K \subseteq H$ an isotone projection cone and $T : K \rightsquigarrow H$ a strongly pseudomonotone decreasing such that $T^{-1}(K) \cap K \neq \emptyset$. Let $J : K \to H$ be the inclusion mapping defined by $J(x) = x$. If there are $x^{\triangleleft} \in T^{-1}(K) \cap K$ and $u \in x^{\triangleleft} + K$ such that*

$$(P_K \circ (J - T))w \subseteq x^{\triangleleft} + K,$$

for all $w \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$. Then, $x^{\triangleleft}$ is a solution of GSNCP$-(K, T)$ for any $x_0 \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$. The recursion formula (3.1) *starting from x_0 is convergent and its limit x^* is a solution of GSNCP$-(K, T)$ such that $x^{\triangleleft} \leq x^* \leq u$. In particular, if $x^{\triangleleft} \neq 0$, then recursion formula* (3.1) *is convergent to a nonzero solution.*

Proof. As $x^{\triangleleft} \in T^{-1}(K) \cap K$, there exists $y^{\triangleleft} \in Tx^{\triangleleft}$ such that $y^{\triangleleft} \geq 0$. So, we obtain

$$x^{\triangleleft} - y^{\triangleleft} \leq x^{\triangleleft}. \tag{3.3}$$

Since $x^{\triangleleft} \in T^{-1}(K) \cap K$ and $u \in x^{\triangleleft} + K$, we have

$$x^{\triangleleft} \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K). \tag{3.4}$$

By hypothesis, we have

$$P_K \circ (J - T)w \subseteq x^{\triangleleft} + K, \tag{3.5}$$

for all $w \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$. Relations (3.3), (3.4) and (3.5) imply the following:

$$x^{\triangleleft} \leq P_K(x^{\triangleleft} - y^{\triangleleft}) \leq P_K(x^{\triangleleft}) = x^{\triangleleft}.$$

This implies

$$x^{\triangleleft} = P_K(x^{\triangleleft} - y^{\triangleleft}) \in P_K \circ (I - T)x^{\triangleleft},$$

that is, $x^{\triangleleft}$ is a solution of GSNCP$-(K, T)$. From the proof of Theorem 3.4, we know that

$$x_n \in K \cap T^{-1}(K), \tag{3.6}$$

$$z_n \in K, \tag{3.7}$$

for all $n \in \mathbb{N}$. Now we prove that

$$x^{\triangleleft} \leq x_n \leq u, \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Since $x_0 \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$, (3.8) holds true for $n = 0$. Suppose that it holds true for n. Then, (3.6) and (3.8) imply that

$$x_n \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K). \tag{3.9}$$

Thus, from (3.9) and by given assumption, we get

$$x_{n+1} = P_K(x_n - z_n) = P_K(Jx_n - z_n) \in P_K(J - T)x_n \subseteq x^{\triangleleft} + K.$$

On the other hand, from (3.6) and (3.8), we obtain $x_n - z_n \leq x_n \leq u$. Consequently, we get $x_{n+1} = P_K(x_n - z_n) \leq P_K(u) = u$. Hence, (3.8) holds for all $n \in \mathbb{N}$. On taking limit as n tends to ∞, we get $x^{\triangleleft} \leq x^* \leq u$. □

Definition 3.8. Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, $T : K \rightsquigarrow H$ a set-valued mapping and $L > 0$. The mapping T is called generalized order weakly $L-$Lipschitz of type-1 if

$$y \leq x \text{ implies } x' - y' \leq L(x - y), \text{ for all } x' \in Tx \text{ and } y' \in Ty.$$

If $L = 1$, then T is called generalized order weakly nonexpansive of type-1.

Proposition 3.9. *Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, $T : K \rightsquigarrow H$ a set-valued mapping and $L > 0$. A mapping T is generalized order weakly $L-$Lipschitz of type-1 if and only if the mapping $S : K \rightsquigarrow K$ defined by $Sx = (LI - T)x$ is strongly monotone increasing.*

Proof. Let $y \leq x$. Suppose that S is strongly monotone increasing. So, $Ly - T(y) \leq Lx - T(x)$ implies $Ly - y' \leq Lx - x'$. Hence,

$$x' - y' \leq L(x - y).$$

This implies that T is generalized order weakly $L-$Lipschitz of type-1.

Conversely, suppose that T is generalized order weakly $L-$Lipschitz of type-1, then $y \leq x$ implies

$$x' - y' \leq L(x - y), \quad \text{for all } x' \in Tx \text{ and } y' \in Ty.$$

Thus, $Ly - y' \leq Lx - x'$, consequently, $Sy \leq Sx$. □

Definition 3.10. Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, $T : K \rightsquigarrow H$ a set-valued mapping and $L > 0$. Then, T is called generalized projection order weakly $L-$Lipschitz of type-1 if and only if the mapping $S_P : K \rightsquigarrow K$ defined by $S_P x = P_K(Lx - Tx)$ is strongly monotone increasing.

If $L = 1$, then T is called generalized projection order weakly nonexpansive of type-1.

Remark 3.11. In Definition 3.10 if K is isotone projection cone then every generalized order weakly $L-$Lipschitz mapping of type-1 is generalized projection order weakly $L-$Lipschitz of type-1, and every generalized order weakly nonexpansive mapping of type-1 is generalized projection order weakly nonexpansive of type-1.

Theorem 3.12. *Let H be a Hilbert space, $K \subseteq H$ an isotone projection cone, and $T : K \rightsquigarrow H$ a strongly pseudomonotone decreasing and generalized projection order weakly $L-$Lipschitz mapping of type-1 such that $T^{-1}(K) \cap K \neq \emptyset$. Let $x^{\triangleleft}$ is a solution of GSNCP$-(K, T)$ and $\lambda = \frac{1}{L}$. Then, for any $x_0 \in (x^{\triangleleft} + K) \cap T^{-1}(K)$, the following recursion*

$$\begin{cases} x_0 \in K, \ \lambda z_0 \in \lambda T x_0, \ x_1 = P_K(x_0 - \lambda z_0); \\ \textit{pick } \lambda z_1 \in \lambda T x_1 \textit{ such that } z_0 \geq z_1, \textit{ define } x_2 = P_K(x_1 - \lambda z_1); \\ \vdots \\ \textit{in general, pick } \lambda z_n \in \lambda T x_n \textit{ such that } z_{n-1} \geq z_n, \\ \textit{and define } x_{n+1} = P_K(x_n - \lambda z_n), \end{cases} \tag{3.10}$$

starting from x_0 is convergent and its limit x^* is a solution of GSNCP$-(K, T)$ such that $x^{\triangleleft} \leq x^*$. In particular, if $x^{\triangleleft} \neq 0$, then recursion formula (3.10) *is convergent to a nonzero solution.*

Proof. Since P_K is a projection onto K and K is closed convex cone, so for any $\alpha > 0$, we have

$$P_K(\alpha x) = \alpha P_K(x), \quad \text{for all } x \in H.$$

Also, GSNCP$-(K,T)$ is equivalent to GSNCP$-(K,\lambda T)$. Denote $S = \lambda T = \frac{1}{L}T$, then the recursion formula (3.10) becomes

$$\left\{\begin{array}{l} \text{Let } x_0 \in K,\ z_0 \in Sx_0,\ \text{define } x_1 = P_K(x_0 - z_0); \\ \text{Pick } z_1 \in Sx_1 \text{ such that } z_0 \geq z_1, \text{ define } x_2 = P_K(x_1 - z_1); \\ \qquad \vdots \\ \text{Pick } z_n \in Sx_n \text{ such that } z_{n-1} \geq z_n,\ \text{define } x_{n+1} = P_K(x_n - z_n). \end{array}\right. \tag{3.11}$$

Let $J : K \to H$ be the inclusion mapping defined by $J(x) = x$ and $u \in x^{\triangleleft} + K$ be arbitrary. Using Theorem 3.7 with mapping S, we obtain that any solution of GSNCP$-(K,S)$ is a solution of GSNCP$-(K,T)$. The only condition which must hold for the mapping S is the relation

$$(P_K \circ (J - S))w \subseteq x^{\triangleleft} + K, \tag{3.12}$$

for all $w \in (x^{\triangleleft} + K) \cap (u - K) \cap S^{-1}(K)$. For any $x \in K$, we have

$$P_K(x - Sx) = P_K\left(x - \frac{1}{L}Tx\right) = P_K\left(\frac{1}{L}(Lx - Tx)\right) = \frac{1}{L}P_K(Lx - Tx). \tag{3.13}$$

Since T is generalized projection order weakly $L-$Lipschitz, so by (3.13) and the scale invariance of the ordering induced by K, it follows that the mapping S is generalized projection order weakly $L-$Lipschitz. Also, $x^{\triangleleft}$ is a solution of GSNCP$-(K,S)$. Hence, for each $x \in (x^{\triangleleft} + K) \cap (u - K) \cap S^{-1}(K)$, we have $x^{\triangleleft} \in P_K(x^{\triangleleft} - Sx^{\triangleleft}) \subseteq P_K(x - Sx)$, that is,

$$(P_K \circ (J - S))(x) \subseteq x^{\triangleleft} + K.$$

Consequently, (3.12) holds. □

The following result is the corollary of Theorem 3.7.

Corollary 3.13. *Let H be a Hilbert space, $K \subseteq H$ an isotone projection cone and $T : K \rightsquigarrow H$ a strongly pseudomonotone decreasing mapping such that $T^{-1}(K) \cap K \neq \emptyset$. Let $J : K \to H$ be the inclusion mapping defined by $J(x) = x$. If there are $x^{\triangleleft} \in T^{-1}(K) \cap K$ and $u \in x^{\triangleleft} + K$ such that*

$$(P_K \circ (J - T))w \subseteq x^{\triangleleft} + K,$$

for all $w \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$. Then, $x^{\triangleleft}$ is a solution of GSNCP$-(K,T)$ for any $x_0 \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$. The recursion formula (3.1) *starting from x_0 is convergent and its limit x^* is a solution of GSNCP$-(K,T)$ such that $x^{\triangleleft} \leq x^* \leq u$. In particular, if $x^{\triangleleft} \neq 0$, then recursion formula* (3.1) *is convergent to a nonzero solution.*

Note that in Algorithm 3.1, we extracted a monotone decreasing sequence $\{z_n\}$ and proved the convergence result for strongly pseudomonotone decreasing set-valued mapping T.

Now we propose the following algorithm and prove the convergence result for the weakly pseudomonotone decreasing set-valued mapping. In this case, we will assume that $I - T$ is weakly monotone decreasing.

Algorithm 3.14. Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, and $T : K \rightsquigarrow H$ a set-valued mapping. Let

$$\begin{cases} x_0 \in K, \ z_0 \in Tx_0, \ x_1 = P_K(x_0 - z_0), \\ \text{in general, } x_{n+1} = P_K(x_n - z_n), \ \text{where } z_n \in Tx_n, \end{cases} \tag{3.14}$$

where $n \in \mathbb{N}$.

Now we prove the existence of a solution of GSNCP$-(K,T)$ for weakly pseudomonotone set-valued mapping T.

Theorem 3.15. *Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ an isotone projection cone and $T : K \rightsquigarrow H$ a set-valued mapping. Consider the recursion formula* (3.14) *starting from $x_0 \in K \cap T^{-1}(K)$. Then, the sequence $\{x_n\}$ is convergent and its limit x^* is a solution of GSNCP$-(K,T)$ provided that T is weakly pseudomonotone decreasing and one of the following conditions holds:*

(D1) *The sequence $\{z_n\}$ in recursion formula* (3.14) *is decreasing.*
(D2) *$(I - T)$ is weakly monotone decreasing.*

Proof. Since K is an isotone projection cone, it is regular. It is enough to show that the sequences $\{x_n\}$ and $\{z_n\}$ defined by the recursion formula (3.14) are monotone decreasing. First we prove that $z_n \in K$ for all $n \in \mathbb{N}$.

Indeed, if $z_n \in K$ for all $n \in \mathbb{N}$, then $x_n - (x_n - z_n) \in K$ implies that $x_n - z_n \leq x_n$. Since K is isotone projection cone, we have

$$x_{n+1} = P_K(x_n - z_n) \leq P_K(x_n) = x_n.$$

Hence, the sequence $\{x_n\}$ is monotone decreasing.

Now we will prove that that the proposition

$$z_n \in K \text{ holds for all } n \in \mathbb{N}. \tag{3.15}$$

Since $x_0 \in K \cap T^{-1}(K)$ gives $Tx_0 \subseteq K$, which further implies $z_0 \in Tx_0 \subseteq K$. Hence, (3.15) holds true for $n = 0$.

Let $z_n \in K$ and we will show that $z_{n+1} \in K$. Since $x_{n+1} \leq x_n$ and T is weakly pseudomonotone decreasing, we have $z_{n+1} \in K$. Hence, for all $n \in \mathbb{N}$, $z_n \in K$.

Let (D1) holds, then the sequence $\{z_n\}$ is monotone decreasing and the result follows.

Let (D2) holds. Again, since $x_{n+1} \leq x_n$ and $S = (I - T)$ is weakly monotone decreasing, therefore for all $w_n \in Sx_n$, there exists $w_{n+1} \in Sx_{n+1}$ such that

$$w_n \leq w_{n+1}. \tag{3.16}$$

Since $w_n \in Sx_n$ and $w_{n+1} \in Sx_{n+1}$, there exist $z_n \in Tx_n$ and $z_n \in Tx_n$ such that $w_n = x_n - z_n$ and $w_{n+1} = x_{n+1} - z_{n+1}$. Now, (3.16) implies

$$x_n - z_n \leq x_{n+1} - z_{n+1},$$

which further implies that $x_n - x_{n+1} \leq z_n - z_{n+1}$. Now, $x_n - x_{n+1} \geq 0$ gives $z_n - z_{n+1} \geq 0$. Consequently, $z_n \geq z_{n+1}$ and the result follows. □

Example 3.16. Let $H = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. For $x, y \in K$ such that $x = (x_1, x_2), y = (y_1, y_2)$, we define $x \leq y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. A set-valued mapping $T : \mathbb{R}^2_+ \rightsquigarrow \mathbb{R}^2$ is given by

$$Tx = T(x_1, x_2) = \left\{(0,0), ((9 - x_1 x_2)e^{x_1 + x_2}, (6 - x_2)(1 + (x_1 + x_2)e^{x_1}))\right\}.$$

Since $(0,0) \in Tx$ for all $x \in K$, T is weakly pseudomonotone decreasing. By recursion formula (3.14), we can write

$$\begin{aligned}
&x^0 = \left(x_1^0, x_2^0\right) \in K \\
&z^0 = \left(z_1^0, z_2^0\right) = \left(\left(9 - x_1^0 x_2^0\right) e^{x_1^0 + x_2^0}, \left(6 - x_2^0\right)\left(1 + (x_1^0 + x_2^0)e^{x_1^0}\right)\right) \\
&x_1^1 = \max\left\{0, x_1^0 - \left(9 - x_1^0\right) e^{x_1^0 + x_2^0}\right\} \\
&x_2^1 = \max\left\{0, x_2^0 - \left(6 - x_2^0\right)\left(1 + \left(x_1^0 + x_2^0\right) e^{x_1^0}\right)\right\}.
\end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned}
&x^n = (x_1^n, x_2^n) \in K \\
&z^n = (z_1^n, z_2^n) = \left((9 - x_1^n x_2^n)\, e^{x_1^n + x_2^n}, (6 - x_2^n)\left(1 + (x_1^n + x_2^n)e^{x_1^n}\right)\right) \\
&x_1^{n+1} = \max\left\{0, x_1^n - (9 - x_1^n x_2^n)\, e^{x_1^n + x_2^n}\right\} \\
&x_2^{n+1} = \max\left\{0, x_2^n - (6 - x_2^n)\left(1 + (x_1^n + x_2^n)\, e^{x_1^n}\right)\right\}.
\end{aligned}$$

It can be easily checked that $x^* \in \{(0,0), (0,6), (1.5,6)\}$ is a solution of GSNCP$-(K,T)$. For initial guess, we take $x^0 = (x_1^0, x_2^0) \in K \cap T^{-1}(K)$, that is, $Tx^0 \in K$ and $x^0 \in K$. This further implies

$$\begin{aligned}
\left(9 - x_1^0 x_2^0\right) e^{x_1^0 + x_2^0} &\geq 0, \\
\left(6 - x_2^0\right)\left(1 + (x_1^0 + x_2^0)e^{x_1^0}\right) &\geq 0.
\end{aligned}$$

This means $x_1^0 \leq 1.5$ and $x_2^0 \leq 6$. The stopping criteria we used is

$$\left|x_i^{n+1} - x_i^{n+1}\right| \leq 10^{-4} \text{ for all } i = 1, 2.$$

- **-:** If we start the Algorithm 3.14 from $x_0 = (1.4, 5.999)$, then it stops at the sixth step with the solution $x^* = (0,0)$.
- **-:** If we start the Algorithm 3.14 from $x_0 = (1.499999, 6)$, then it stops at the third step with the solution $x^* = (0,6)$.
- **-:** If we start the Algorithm 3.14 from $x_0 = (1.5, 6)$, then it stops at the second step with the solution $x^* = (1.5, 6)$.

Note that if we start from different starting points we get the convergence of Algorithm 3.14 to a different solution of GSNCP$-(K,T)$.

Remark 3.17. Analyzing Theorem 3.15, it is noted that $x^* = 0$ is a solution of GSNCP$-(K,T)$ for a mapping T given in Theorem 3.15. Note that $0 \leq x_0$ and $0 \leq z'$ for all $z' \in Tx_0$. Since T is weakly pseudomonotone decreasing, there exists $y' \in T0$ such that $0 \leq y'$. This implies that $y' \in K$. Since K is isotone projection cone, $K \subseteq K^*$. This further gives that $y' \in K^*$. Hence, there exists $x^* = 0 \in K$ such that $\langle y', x^* \rangle = 0$ and $y' \in Tx^* \cap K^*$, that is, $x^* = 0$ is a solution of GSNCP$-(K,T)$.

The next theorem gives a sufficient condition for the recursion formula (3.14) to be convergent to a nonzero solution.

Theorem 3.18. *Let H be a Hilbert space, $K \subseteq H$ an isotone projection cone and $T : K \rightsquigarrow H$ a set-valued mapping such that $T^{-1}(K) \cap K \neq \emptyset$. Let $J : K \to H$ be the inclusion mapping defined by $J(x) = x$. If there are $x^{\triangleleft} \in T^{-1}(K) \cap K$ and $u \in x^{\triangleleft} + K$ such that*

$$(P_K \circ (J - T))w \subseteq x^{\triangleleft} + K, \tag{3.17}$$

for all $w \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$. Then, $x^{\triangleleft}$ is a solution of GSNCP$-(K, T)$ for any $x_0 \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$ provided that T is weakly pseudomonotone decreasing and one of the following conditions holds:

(D1) *The sequence $\{z_n\}$ in recursion formula* (3.14) *is decreasing.*
(D2) *$(I - T)$ is weakly monotone decreasing.*

The recursion formula (3.14) *starting from x_0 is convergent and its limit x^* is a solution of GSNCP$-(K, T)$ such that $x^{\triangleleft} \le x^* \le u$. In particular, if $x^{\triangleleft} \neq 0$, then recursion formula* (3.14) *is convergent to a nonzero solution.*

Proof. Since $x^{\triangleleft} \in T^{-1}(K) \cap K$, there exists $y^{\triangleleft} \in Tx^{\triangleleft}$ such that

$$x^{\triangleleft} - y^{\triangleleft} \le x^{\triangleleft}. \tag{3.18}$$

Since $x^{\triangleleft} \in T^{-1}(K) \cap K$ and $u \in x^{\triangleleft} + K$, we have

$$x^{\triangleleft} \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K). \tag{3.19}$$

By condition (3.17) and relations (3.18), (3.19) and (**??**), we have

$$x^{\triangleleft} \le P_K(x^{\triangleleft} - y^{\triangleleft}) \le P_K(x^{\triangleleft}) = x^{\triangleleft}.$$

This implies that

$$x^{\triangleleft} = P_K(x^{\triangleleft} - y^{\triangleleft}) \in (P_K \circ (I - T))x^{\triangleleft},$$

that is, $x^{\triangleleft}$ is a solution of GSNCP$-(K, T)$. From the proof of Theorem 3.15, we know that

$$x_n \in K \cap T^{-1}(K), \tag{3.20}$$

$$z_n \in K, \tag{3.21}$$

for all $n \in \mathbb{N}$. Now we prove that

$$x^{\triangleleft} \le x_n \le u, \quad \text{for all } n \in \mathbb{N}. \tag{3.22}$$

Note that

$$x_0 \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K).$$

This implies that (3.22) holds true for $n = 0$. Now suppose that it holds true for n. Then, (3.20) and (3.22) imply that

$$x_n \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K). \tag{3.23}$$

Thus,

$$x_{n+1} = P_K(x_n - z_n) = P_K(Jx_n - z_n) \in P_K(J - T)x_n \subseteq x^{\triangleleft} + K.$$

On the other hand, from (3.20) and (3.22), we obtain $x_n - z_n \le x_n \le u$. Consequently, $x_{n+1} = P_K(x_n - z_n) \le P_K(u) = u$. Hence, (3.22) holds for all $n \in \mathbb{N}$. On taking limit as n tends to ∞, we get $x^{\triangleleft} \le x^* \le u$. $\square$

Definition 3.19. Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, $T : K \rightsquigarrow H$ a set-valued mapping and $L > 0$. A mapping T is called generalized order weakly $L-$Lipschitz of type-2 if $y \leq x$ and for every $y' \in Ty$, there exists $x' \in Tx$ such that

$$x' - y' \leq L(x - y).$$

If $L = 1$, then T is called generalized order weakly nonexpansive of type-2.

Proposition 3.20. *Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, and $L > 0$. A mapping $T : K \rightsquigarrow H$ is generalized order weakly $L-$Lipschitz of type-2 if and only if mapping $S : K \rightsquigarrow K$ defined by $Sx = Lx - Tx$ is weakly monotone increasing.*

Proof. Suppose that $y \leq x$ and S is weakly monotone increasing. Then, for every $Ly - y' \in Ly - Ty = Sy$, there exists $Lx - x' \in Lx - Tx = Sx$ such that $Ly - y' \leq Lx - x'$. This implies that $Ly - y' \leq Lx - x'$. Hence, for every $y' \in Ty$, there exists $x' \in Tx$ such that

$$x' - y' \leq L(x - y).$$

Thus, T is generalized order weakly $L-$Lipschitz of type-2.

Conversely, let T be a generalized order weakly $L-$Lipschitz of type-2. Then, $y \leq x$ implies that for every $y' \in Ty$, there exists $x' \in Tx$ such that

$$x' - y' \leq L(x - y).$$

Therefore, $Ly - y' \leq Lx - x'$. Consequently, S is weakly monotone decreasing. □

Definition 3.21. Let $(H, \langle ., . \rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex cone, and $L > 0$. A set-valued mapping $T : K \rightsquigarrow H$ is called generalized projection order weakly $L-$Lipschitz of type-2 if and only if the mapping $S_P : K \rightsquigarrow K$ defined as $S_P x = P_K(Lx - Tx)$ is weakly monotone increasing.

If $L = 1$ then T is called generalized projection order weakly nonexpansive of type-2.

Remark 3.22. In above definition if K is isotone projection cone then every generalized order weakly $L-$Lipschitz mapping of type-2 is generalized projection order weakly $L-$Lipschitz of type-2 and every generalized order weakly nonexpansive mapping of type-2 is generalized projection order weakly nonexpansive of type-2.

Theorem 3.23. *Let H be a Hilbert space, $K \subseteq H$ an isotone projection cone and $T : K \rightsquigarrow H$ a generalized projection order weakly $L-$Lipschitz mapping of type-2 with $T^{-1}(K) \cap K \neq \emptyset$. If $x^{\triangleleft}$ is a solution of GSNCP$-(K, T)$ and $\lambda = \frac{1}{L}$, then for any $x_0 \in (x^{\triangleleft} + K) \cap T^{-1}(K)$, the recursion*

$$x_0 \in K, \ x_{n+1} = P_K(x_n - \lambda z_n), \ \lambda z_n \in \lambda T x_n, \tag{3.24}$$

starting from x_0 is convergent and its limit x^* is a solution of GSNCP$-(K, T)$ such that $x^{\triangleleft} \leq x^*$. In particular, if $x^{\triangleleft} \neq 0$, then the recursion formula (3.24) is convergent to a nonzero solution provided that T is weakly pseudomonotone decreasing and one of the following conditions hold:

(D1) *The sequence $\{z_n\}$ in recursion formula (3.14) is decreasing.*
(D2) *$(I - T)$ is weakly monotone decreasing.*

Proof. Since P_K is the projection onto K and K is closed convex cone, for any $\alpha > 0$, we have

$$P_K(\alpha x) = \alpha P_K(x), \quad \text{for all } x \in H.$$

Note that GSNCP$-(K,T)$ is equivalent to GSNCP$-(K,\lambda T)$. If $S = \lambda T = \frac{1}{L}T$, then the recursion formula (3.24) becomes

$$x_0 \in K, \; x_{n+1} = P_K(x_n - z_n), \; z_n \in Sx_n. \tag{3.25}$$

Let $J : K \to H$ be the inclusion mapping and $u \in x^{\triangleleft} + K$ be arbitrary. Using Theorem 3.15 with mapping S, we obtain that any solution of GSNCP$-(K,S)$ is a solution of GSNCP$-(K,T)$. The only condition which must hold for the mapping S is the relation

$$P_K \circ (J - S)w \subseteq x^{\triangleleft} + K, \tag{3.26}$$

for all $w \in (x^{\triangleleft} + K) \cap (u - K) \cap S^{-1}(K)$. For any $x \in K$, we have

$$P_K(x - Sx) = P_K\left(x - \frac{1}{L}Tx\right) = P_K\left(\frac{1}{L}(Lx - Tx)\right) = \frac{1}{L}P_K(Lx - Tx). \tag{3.27}$$

Since T is generalized projection order weakly $L-$Lipschitz of type-2, so by (3.27) and the scale invariance of the ordering induced by K, it follows that the mapping S is generalized projection order weakly $L-$Lipschitz of type-2. As $x^{\triangleleft}$ is a solution of GSNCP$-(K,S)$, so for each $x \in (x^{\triangleleft} + K) \cap (u - K) \cap S^{-1}(K)$, we have $x^{\triangleleft} \in P_K(x^{\triangleleft} - Sx^{\triangleleft}) \subseteq P_K(x - Sx)$, that is,

$$(P_K \circ (J - S))(x) \subseteq x^{\triangleleft} + K.$$

Hence, (3.26) holds. □

The following corollary is derived from Theorem 3.23.

Corollary 3.24. *Let H be a Hilbert space, $K \subseteq H$ an isotone projection cone and $T : K \rightsquigarrow H$ be a mapping such that $T^{-1}(K) \cap K \neq \emptyset$. Let $J : K \to H$ be the inclusion mapping. If there are $x^{\triangleleft} \in T^{-1}(K) \cap K$ and $u \in x^{\triangleleft} + K$ such that*

$$(P_K \circ (J - T))w \subseteq x^{\triangleleft} + K,$$

for all $w \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$. Then, $x^{\triangleleft}$ is a solution of GSNCP$-(K,T)$ for any $x_0 \in (x^{\triangleleft} + K) \cap (u - K) \cap T^{-1}(K)$ provided that T is weakly pseudomonotone decreasing and one of the following conditions hold:

(D1) *The sequence $\{z_n\}$ in recursion formula* (3.14) *is decreasing.*
(D2) $(I - T)$ *is weakly monotone decreasing.*

The recursion formula (3.14) *starting from x_0 is convergent and its limit x^* is a solution of GSNCP$-(K,T)$ such that $x^{\triangleleft} \leq x^* \leq u$. In particular, if $x^{\triangleleft} \neq 0$, then recursion formula* (3.14) *is convergent to a nonzero solution.*

Remark 3.25. The proposed algorithms to obtain the solution of a generalized nonlinear complementarity problem for a class of multivalued mappings defined on an isotone projection cones, depend on the order induced by a cone as well as on the computation of projections on them at each step of iteration. The results obtained in this paper can be viewed as an extension of a scope of the study initiated in [17] Nemeth and Nemth [16] remarked that computation of projections on to the cone is a difficult problem and they observed that the projection of a given point onto an isotone projection cones in $\mathbb{R}^n$ can be reduced to a finite number of projections onto subspaces of decreasing dimension. The proposed technique dealing with implementable numerical methods of computation of projections is applicable to the algorithms defined herein.

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