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UNIFORMLY LIPSCHITZIAN MAPPINGS IN $\mathbb{R}$-TREES

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Dedicated to Wataru Takahashi on the occasion of his 70th birthday

Abstract. Let (X,ρ) be a complete $\mathbb{R}$-tree, and suppose $T : X \to X$ has bounded orbits and satisfies for all $n \in \mathbb{N}$ sufficiently large,

$$\rho(T^n x, T^n y) \le k_n \rho(x,y),$$

for all $x, y \in X$. A. Aksoy and M. A. Khamsi [Sci. Math. Jpn. 65(2007), 31-41, e:2006, 1143-1153] have shown that if $\limsup_{n\to\infty} k_n < 2$ then T has a fixed point. The main result of this paper shows that if, in addition, T is assumed to be continuous, then it suffices merely to assume that $\limsup_{n\to\infty} k_n < \infty$.

1. Introduction

Let (X,ρ) be a metric space. The balls in X are said to be *c-regular for* $c \ge 1$ if the following holds: For any $k < c$ there exist numbers $\mu, \alpha \in (0,1)$ such that for any $x, y \in X$ and $r > 0$ with $\rho(x,y) \ge (1-\mu) r$, there exists $z \in X$ such that

$$B(x; (1+\mu) r) \cap B(y; k(1+\mu) r) \subseteq B(z; \alpha r).$$

The Lifšic constant $\kappa(X)$ of X is the number

$$\kappa(X) = \sup\{c \ge 1 : \text{ the balls in } X \text{ are } c\text{-regular}\}.$$

Lifšic proved in [11] that if (X,ρ) is a bounded complete metric space, and if for some $k < \kappa(X)$, $T : X \to X$ satisfies

$$\rho(T^n x, T^n y) \le k\rho(x,y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$, then T has a fixed point.

A related result for an $\mathbb{R}$-tree (defined below) is found in [1], where it is shown that if (X,ρ) is a complete $\mathbb{R}$-tree, and if $T : X \to X$ has bounded orbits and satisfies for all $n \in \mathbb{N}$ sufficiently large,

$$\rho(T^n x, T^n y) \le k_n \rho(x,y),$$

for all $x, y \in X$, where $\limsup_{n\to\infty} k_n < 2$, then T has a fixed point.

Our objective in this note is to extend both the Lifšic and Aksoy-Khamsi results. In particular we show that Lifšic's assumption $\rho(T^n x, T^n y) \le k\rho(x,y)$ can be replaced with the weaker assumption $\rho(T^n x, T^n y) \le k_n \rho(x,y)$ where

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$\limsup_{n\to\infty} k_n < \kappa(X)$. Regarding the Aksoy-Khamsi result, we show that if T is continuous, $\limsup_{n\to\infty} k_n < 2$ can be replace with the much weaker assumption $\limsup_{n\to\infty} k_n < \infty$.

2. The Lifšic extension

Theorem 2.1 (Lifšic)**.** *Suppose (X,ρ) is a complete metric space, and suppose $T : X \to X$ has bounded orbits and satisfies for all $n \in \mathbb{N}$ sufficiently large,*

$$\rho(T^n x, T^n y) \le k_n \rho(x,y) \tag{2.1}$$

for all $x, y \in X$, with $\limsup_{n\to\infty} k_n < \kappa(X)$. Then T has a fixed point.

Proof. (Except for the final paragraph, this is identical to the proof given in [7, p. 172].) If $\kappa(X) = 1$ then, for sufficiently large n, T^n is a contraction mapping and there is nothing to prove. So, suppose $\kappa(X) > 1$. For each $x \in X$, set

$$r(x) = \inf\{r > 0 : B(x;r) \text{ contains an orbit of } T\}.$$

Now let $\limsup_{n\to\infty} k_n < k < \kappa(X)$, and let $\mu, \alpha \in (0,1)$ be the numbers associated with k in the definition of k-regular balls. Then given any $x \in X$ there is an integer $m \in \mathbb{N}$ such that

$$\rho(x, T^m x) \ge (1-\mu) r(x)$$

and there is also a point $y \in X$ such that

$$\rho(x, T^n y) \le (1+\mu) r(x), \qquad n = 1, 2, \ldots.$$

Since the balls are k-regular there exists $z \in X$ and $\alpha < 1$ such that

$$D := B(x; (1+\mu) r(x)) \cap B(T^m x; k(1+\mu) r(x)) \subseteq B(z; \alpha r(x)).$$

Next observe that for m sufficiently large,

$$\rho(T^m x, T^n y) \le k\rho\left(x, T^{n-m} y\right) \le k(1+\mu) r(x).$$

for all $n > m$. This shows that $\{T^n y\}_{n>m}$ is contained in D, and hence in $B(z; \alpha r(x))$. This in turn implies that

$$r(z) \le \alpha r(x).$$

Also, for any $u \in D$,

$$\begin{aligned}
\rho(z,x) &\le \rho(z,u) + \rho(u,x) \\
&\le \alpha r(x) + (1+\mu) r(x) \\
&= A r(x)
\end{aligned}$$

where $A = \alpha + 1 + \mu$.

By setting $x = x_0$ and $z = z(x_0)$, it is possible to define a sequence $\{x_n\}$ with $x_{n+1} = z(x_n)$, where $z(x_n)$ is defined via the above procedure. Thus $r(x_n) \le \alpha^n r(x_0)$ and $\rho(x_n, x_{n+1}) \le A r(x_n) \le \alpha^n r(x_0)$. This proves that $\{x_n\}$ is a Cauchy sequence which has limit, say x^*. Now choose $N \in \mathbb{N}$ so that both T^N and T^{N+1} are lipschitzian. Since $B(x^*; \varepsilon)$ contains an orbit of T for any $\varepsilon > 0$ there exists a sequence $\{y_n\}$ also converging to x^* for which $\lim_{n\to\infty} \rho\left(T^N y_n, T^{N+1} y_n\right) = 0$. It follows that $T^N x^* = T^{N+1} x^*$; hence $T^N x^*$ is a fixed point of T. □

For most metric spaces M, $\kappa(M) = 1$ and for such spaces Lifšic's theorem is equivalent to the Banach Contraction Principle. However as we observe below there are spaces for which $\kappa(M) > 1$.

Now let X be a Banach space. We define the *uniform Lifšic constant,* $\kappa_0(X)$, of X as follows

$$\kappa_0(X) = \sup\left\{c \geq 1 : \begin{array}{l} \exists\alpha < 1 \text{ such that } \forall x,\ \|x\| \leq 1,\ \exists\lambda \in [0,1] \\ \text{such that } B(0;1) \cap B(x;c) \subset B(\lambda x;\alpha) \end{array}\right\}.$$

Lifšic proved that $\kappa_0(H) \geq \sqrt{2}$ if H is a Hilbert space, and this estimate is sharp.

The Lifšic constant is also known to be larger than one in certain geodesic spaces, specifically the class of geodesic spaces called the CAT(κ) spaces for $\kappa \leq 0$. A geodesic space (X, d) is said to be a *$CAT(\kappa)$ space* (the term is due to M. Gromov– see, e.g., [2], p. 159) if it is geodesically connected and has constant curvature bounded above by κ. More precisely, every geodesic triangle in X is at least as 'thin' as its comparison triangle in M_κ^2, where for $\kappa < 0$ M_κ^2 is the real hyperbolic space $\mathbb{H}^2$ with the distance function scaled by a factor of $1/\sqrt{-\kappa}$, and if $\kappa = 0$, M_κ^2 is the Euclidean plane. For precise definitions and a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [2] or Burago, et al. [3]. We note in particular that the complex Hilbert ball with a hyperbolic metric (see [8]; also inequality (4.3) of [12] and subsequent comments) is a CAT(0) space.

There are interesting spaces which are CAT(κ) for all $\kappa \leq 0$.

Definition 2.2. An $\mathbb{R}$-tree (or metric tree) is a metric space M such that

(i) there is a unique geodesic (metric) segment denoted by $[x, y]$ joining each pair of points x *and* y *in* M*; and*

(ii) $[y, x] \cap [x, z] = \{x\} \Rightarrow [y, x] \cup [x, z] = [y, z]$.

In [5] it was proved that the Lifšic constant $\kappa(X)$ for any CAT(κ) space X with $\kappa \leq 0$ satisfies $\kappa(X) \geq \sqrt{2}$, and $\kappa(X) = 2$ if X is an $\mathbb{R}$-tree.

In view of this observation and Theorem 2.1 we have the following:

Theorem 2.3. *Let (X, ρ) be a complete $CAT(0)$, and let $T : X \to X$ have bounded orbits and satisfies for all $n \in \mathbb{N}$ sufficiently large,*

$$\rho(T^n x, T^n y) \leq k_n \rho(x, y)$$

for all $x, y \in X$, where $\limsup_{n\to\infty} k_n < \sqrt{2}$. *Then T has a fixed point.*

In view of the fact that $\kappa(X) = 2$ if X is an $\mathbb{R}$-tree, Theorem 2.1 also yields the following result of Aksoy and Khamsi.

Theorem 2.4 ([1])**.** *Let (X, ρ) be a complete $\mathbb{R}$-tree, and suppose $T : X \to X$ has bounded orbits and satisfies for all $n \in \mathbb{N}$ sufficiently large,*

$$\rho(T^n x, T^n y) \leq k_n \rho(x, y),$$

for all $x, y \in X$, where $\limsup_{n\to\infty} k_n < 2$. *Then T has a fixed point.*

Question 1. It is natural to ask whether 2 the optimal constant for Theorem 2.4.

In the next section we show that the answer to Question 1 in some spaces is 'no'. We also show that if T is *continuous* the assumption $\limsup_{n\to\infty} k_n < 2$ may be remarkably relaxed. In this case it is enough to assume $\lim_{n\to\infty} k_n < \infty$. This is the main result of the paper.

3. An $\mathbb{R}$-tree extension

Throughout this section we use $O(x)$ to denote the orbit of a mapping $T : X \to X$ at a point $x \in X$; thus $O(x) = \{x, Tx, T^2x, \dots\}$.

Our extension of Theorem 2.4 is an application of the following fundamental fact. For a proof see [9].

Theorem 3.1. *Every continuous mapping T of a complete geodesically bounded $\mathbb{R}$-tree X into itself has a fixed point.*

We should remark that Theorem 3.1 is actually a special case of a theorem of G. S. Young [13]. For further discussion see [10].

Theorem 3.2. *Let (X, ρ) be a complete $\mathbb{R}$-tree. Suppose $T : X \to X$ is continuous and has bounded orbits, and suppose for all $n \in \mathbb{N}$ sufficiently large,*

$$\rho(T^n x, T^n y) \le k_n \rho(x, y) \tag{3.1}$$

for all $x, y \in X$, with $\limsup_{n\to\infty} k_n < \infty$. Then some bounded convex subset of X is T-invariant; hence T has a fixed point.

This will be an immediate consequence of Theorem 3.1 and the following result.

Theorem 3.3. *Let (X, ρ) be an $\mathbb{R}$-tree. Suppose $T : X \to X$ is continuous and has bounded orbits, and suppose for all $n \in \mathbb{N}$ sufficiently large,*

$$\rho(T^n x, T^n y) \le k_n \rho(x, y) \tag{3.2}$$

for all $x, y \in X$, with $\limsup_{n\to\infty} k_n < \infty$. Then some bounded subtree of X is T-invariant.

Proof. Fix $x \in X$ and choose $m \in \mathbb{N}$ and $k > 0$ with $\limsup_{n\to\infty} k_n < k$ so that $\rho(T^n u, T^n v) \le k\rho(u, v)$ for all $u, v \in [x, Tx]$ and $n \ge m$. Let $Y = \bigcup_{i=1}^{\infty} T^i([x, Tx])$. Since each $T^i([x, Tx])$ is an arcwise connected subset of X, Y is an arcwise connected subset of X; hence Y itself is an $\mathbb{R}$-tree which is clearly T-invariant. We show that Y is bounded.

Let $\xi(z) = \sup\{\rho(z, T^n z) : n \ge m\}$ for each $z \in [x, Tx]$. By assumption $\xi(z) < \infty$ for each $z \in [x, Tx]$. If $z, w \in [x, Tx]$ then

$$\begin{aligned} \rho(w, T^n w) &\le \rho(w, z) + \rho(z, T^n z) + \rho(T^n z, T^n w) \\ &\le \rho(w, z) + \xi(z) + k\rho(z, w) \end{aligned}$$

for each $n \ge m$. Thus $\xi(w) \le \xi(z) + (1 + k)\rho(z, w)$. Reversing the roles of z and w, we conclude

$$|\xi(z) - \xi(w)| \le (1 + k)\rho(z, w)$$

for all $z, w \in [x, Tx]$. Thus ξ is continuous, and since $[x, Tx]$ is compact,

$$\xi := \sup\{\xi(z) : z \in [x, Tx]\} < \infty.$$

Now for $1 \leq i < m$, let $m_i = \sup \left\{ \rho \left(z, T^i z \right) : z \in [x, Tx] \right\}$ and let

$$\beta = \max \{ m_i : i = 1, \cdots, m-1 \} .$$

Since T is continuous, $\beta < \infty$. Also, by construction, given $y \in Y$ there is at least one point $z \in [x, Tx]$ such that $y \in O(z)$. It follows that $\rho(z, y) \leq \beta + \xi$. Therefore Y is bounded. Specifically, $Y \subset B(x; d)$ where $d = \rho(x, Tx) + \beta + \xi$. □

Since a nonexpansive mapping satisfies (3.1) for $k_n \equiv 1$ we have the following corollary.

Corollary 3.4 (Theorem 4.5 (i) of [6])**.** *A nonexpansive mapping of a complete $\mathbb{R}$-tree into itself with bounded orbits always has a fixed point.*

Remark 3.5. Under the assumptions of Theorem 3.2 it is enough to assume that one orbit of T is bounded. Indeed, the following is true.

Proposition 3.6. *Let (X, ρ) be a metric space and suppose $T : X \to X$ has a bounded orbit. Suppose that for all n sufficiently large,*

$$\rho(T^n x, T^n y) \leq k_n \rho(x, y)$$

for all $x, y \in X$. Suppose also that $\limsup_{n\to\infty} k_n < \infty$. Then all orbits of T are bounded.

Proof. Assume there exist $x \in X$ and $r > 0$ such that $O(x) \subset B(x; r)$. Choose $k > 0$ so that $\limsup_{n\to\infty} k_n < k$. Then if $y \in X$ it is possible to choose $m \in \mathbb{N}$ so that for all $n \geq m$,

$$\rho(T^n x, T^n y) \leq k \rho(x, y) .$$

Then for $n \geq m$,

$$\rho(x, T^n y) \leq \rho(x, T^n x) + \rho(T^n x, T^n y) \leq r + k\rho(x, y) .$$

This proves that $(T^n y)_{n \geq m} \subset B(x; d)$ where $d = r + k\rho(x, y)$. Let

$$d' = \max \left\{ \rho \left(x, T^i y \right) : i = 1, \cdots, m-1 \right\} .$$

Then $O(y) \subset B(x; d^*)$ where $d^* = \max \{d, d'\}$. Since y is arbitrary, all orbits of T are bounded. □

We now show that the answer to Question 1 is negative. The simplest complete $\mathbb{R}$-tree is a closed real line interval. For this case we recall a classical theorem due to Ralph DeMarr.

Theorem 3.7 (DeMarr [4])**.** *Let I be a closed real line interval, and let f and g be commuting continuous mappings of I into itself which have respective Lipschitz constants α and β satisfying the condition $\beta(\alpha - 1) < (\alpha + 1)$. Then f and g have at least one common fixed point.*

If we take $\alpha = \beta$ in DeMarr's condition we find that the condition $\beta(\alpha - 1) < (\alpha + 1)$ reduces to $\alpha < 1 + \sqrt{2}$. This leads to the following result which shows that in Theorem 2.4 the constant 2 is not always optimal. Notice that here we are not assuming T is continuous.

Theorem 3.8. *Let I be a closed real line interval and suppose $T : I \to I$ satisfies all $n \in \mathbb{N}$ sufficiently large,*

$$|T^n x - T^n y| \leq k_n |x - y|$$

for each $x, y \in I$. If $\limsup_{n\to\infty} k_n < 1 + \sqrt{2}$, *then T has a fixed point.*

Proof. Choose N so large that both T^N and T^{N+1} have Lipschitz constant less than $1+\sqrt{2}$. Since T^N and T^{N+1} are continuous and commute, by DeMarr's Theorem there exists $x_0 \in I$ such that $T^N x_0 = T^{N+1} x_0 = x_0$. This clearly implies $Tx_0 = x_0$. □

Theorem 3.9. *Let (X, ρ) be a complete geodesically bounded $\mathbb{R}$-tree, and suppose $T : X \to X$ satisfies for all $n \in \mathbb{N}$ sufficiently large,*

$$\rho(T^n x, T^n y) \leq k_n \rho(x, y)$$

for each $x, y \in X$, where $\limsup_{n\to\infty} k_n < 2$. *Then T has a fixed point.*

Proof. For n sufficiently large, T^n is lipschitzian and hence has a fixed point by Theorem 3.1. Therefore T has a bounded orbit, so the conclusion follows from Theorem 2.4 and Proposition 3.6 □

Question 2. Can the assumption that X is geodesically bounded in Theorem 3.1 be replaced with the assumption that T has bounded orbits?

Question 3. Can the assumption that $\limsup_{n\to\infty} k_n < 2$ in Theorem 3.9 be replaced with the assumption that $\limsup_{n\to\infty} k_n < \infty$?

It is easy to see that if a continuous mapping $f : \mathbb{R} \to \mathbb{R}$ has a bounded orbit, then it has a fixed point. Suppose $\{f^n x\}$ is bounded. If this sequence is monotone, then clearly $\lim_{n\to\infty} f^n x$ is a fixed point of f. Otherwise there exists $n \in \mathbb{N}$ such that $f^n x \leq f^{n+1} x$ and $f^{n+2} x \leq f^{n+1} x$, in which case f has a fixed point in the interval $\left[f^n x, f^{n+1} x\right]$, or such that $f^{n+1} x \leq f^n x$ and $f^{n+1} x \leq f^{n+2} x$, in which case f has a fixed point in $\left[f^{n+1} x, f^n x\right]$.

Question 4. If T is a continuous mapping of a complete $\mathbb{R}$-tree X into itself, and if T has a bounded orbits, then does T have a fixed point?

Remark 3.10. The answer to Question 4 is 'no' if just a single orbit is assumed to be bounded even if that orbit is a periodic point. Let $X = [0, \infty) \cup [x, y]$ where $x = (0, -1)$ and $y = (0, 1)$. Let $u = \left(0, -\frac{1}{2}\right)$ and $v = \left(0, \frac{1}{2}\right)$, and let 0 denote the origin. Reflect x and y in the origin, stretch the intervals $[x, u]$ and $[y, v]$ so that u and v touch the origin, and shift the intervals $[u, 0]$ and $[v, 0]$ onto the x-axis pushing $[0, \infty)$ to the right. By moving u and v nearer to each other an example can be constructed for which the mapping is lipschitzian with Lipschitz constant arbitrarily near 1.

Theorem 2.4 applied to the case when X is the real line $\mathbb{R}$ asserts that if $T : \mathbb{R} \to \mathbb{R}$ has bounded orbits and satisfies

$$|T^n x - T^n y| \leq k_n |x - y|$$

for each $x, y \in \mathbb{R}$, where $\limsup_{n\to\infty} k_n < 2$, then T has a fixed point. This raises the obvious question of whether 2 can be replaced with the estimate $1 + \sqrt{2}$ of Theorem 3.8. (Of course this result is of interest only for discontinuous mappings T, since it is easy to see that a continuous mapping of $\mathbb{R}$ into $\mathbb{R}$ with a bounded orbit always has a fixed point.)

Remark 3.11. Continuity of T was crucial to the *proof* of Theorem 3.2. However we do not know whether this assumption is essential.

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