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APPROXIMATE FIXED POINTS OF NONEXPANSIVE SET-VALUED MAPPINGS IN UNBOUNDED SETS

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To Professor Wataru Takahashi with appreciation and respect on the occasion of his 70th birthday

Abstract. It follows from Banach's fixed point theorem that every nonexpansive self-mapping of a bounded, closed and convex set in a Banach space has approximate fixed points. This is no longer true, in general, if the set is unbounded. Nevertheless, as we have shown in a recent paper of ours, there exists an open and everywhere dense set in the space of all nonexpansive self-mappings of any closed and convex (not necessarily bounded) set in a Banach space (endowed with the natural metric of uniform convergence on bounded subsets) such that all its elements have approximate fixed points. In the present paper we prove a corresponding result for nonexpansive set-valued mappings.

1. Introduction and preliminaries

During the last fifty years or so, there has been a lot of interest in the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 3, 5, 6, 7] and the references mentioned therein. The origin of this interest lies in the classical Banach theorem [1] regarding the existence of a unique fixed point for a strict contraction. Since that seminal result, many developments have taken place in this area. We mention, for instance, existence results for fixed points of nonexpansive mappings which are not strictly contractive [5, 6]. Such results were obtained for general nonexpansive mappings in special Banach space, while for self-mappings of general complete metric spaces existence results were established for, the so-called, contractive mappings [11]. For general nonexpansive mappings in general Banach spaces the existence of a unique fixed point was established in the generic sense, using the Baire category approach [2, 3, 15, 16, 17]. More precisely, in these papers the space $\mathcal{A}$ of all nonexpansive self-mappings of a closed and convex set K in a Banach space was endowed with the natural metric of uniform convergence on bounded subsets, and it was shown that there exists a subset $\mathcal{A}' \subset \mathcal{A}$, which is a countable intersection of open and everywhere dense subsets of $\mathcal{A}$, such that every mapping in $\mathcal{A}'$ has a unique fixed point. Note that in [2, 3] the set K was assumed to be bounded, while in [15] this assumption was removed.

In [21] we considered the question of existence of *approximate* fixed points of general nonexpansive mappings in *unbounded* sets. We showed that in the above-mentioned space $\mathcal{A}$ consisting of all nonexpansive self-mappings of a closed and

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convex (not necessarily bounded) set K in a Banach space, equipped with the natural metric of uniform convergence on bounded subsets, there exists an open and everywhere dense subset $\mathcal{B} \subset \mathcal{A}$ such that any mapping in $\mathcal{B}$ has approximate fixed points. Since a translation mapping in an arbitrary Banach space has no approximate fixed points, it is clear that this result cannot be strengthened in principle.

As a matter of fact, it turns out that the result of [21] is also true for nonexpansive self-mappings of closed and convex sets in complete hyperbolic spaces, a class of metric spaces which we recall below.

In the present paper we are concerned with nonexpansive *set-valued* mappings. In several recent papers [4, 18, 19, 20] certain set-valued dynamical systems induced by such mappings have been investigated and some new iterative methods for approximating the corresponding fixed points have been obtained. Here we establish analogs of the results of [21] in the context of nonexpansive set-valued mappings.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s), c(t)) = |s-t|$ for all real s and t. The image of R^1 under a metric embedding is called a metric line and the image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping is called a metrtic segment.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \ \text{ and } \ \rho(z, y) = (1-t)\rho(x, y).$$

This point is denoted by $(1-t)x \oplus ty$. We say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\Big) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\Big) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. A set $K \subset X$ is called ρ-convex if $[x, y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in [6, 13, 14]. For information regarding those unbounded sets which have the approximate fixed point property for nonexpansive mappings in both Banach and hyperbolic spaces see, for example, [6, 8, 9, 12, 22]. Let (X, ρ, M) be a complete hyperbolic space and let K be a nonempty, closed and ρ-convex subset of X.

For each $x \in K$ and each $r > 0$, set

$$B(x, r) = \{y \in K : \ \rho(x, y) \le r\}.$$

Denote by $\mathcal{A}$ the set of all operators $A : K \to K$ such that

$$\rho(Ax, Ay) \le \rho(x, y) \text{ for all } x, y \in K.$$

Fix some $\theta \in K$.

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{U}(n) = \{(A, B) \in \mathcal{A} \times \mathcal{A} : \ \rho(Ax, Bx) \le n^{-1} \text{ for all } x \in B(\theta, n)\},$$

where n is a natural number. It is clear that the uniform space $\mathcal{A}$ is metrizable and complete.

Let $A \in \mathcal{A}$ and $\epsilon \ge 0$ be given. A point $x \in K$ is called an *ϵ-approximate fixed point* of A if $\rho(x, Ax) \le \epsilon$.

We say that a mapping A has the *bounded approximate fixed point property* (or the BAFP property, for short) if there is a nonempty bounded set $K_0 \subset K$ such that for each $\epsilon > 0$, A has an ϵ-approximate fixed point in K_0, that is, a point $x_\epsilon \in K_0$ which satisfies $\rho(x_\epsilon, Ax_\epsilon) \le \epsilon$.

The following result was established in [21].

Proposition 1.1. *Assume that $A \in \mathcal{A}$ and that $K_0 \subset K$ is a nonempty, closed, ρ-convex and bounded subset of K such that*

$$A(K_0) \subset K_0.$$

Then A has the BAFP property.

Proposition 1.1 immediately implies the following result.

Proposition 1.2. *Assume that K is bounded. Then any $A \in \mathcal{A}$ has the BAFP property.*

Proposition 1.2 does not, of course, hold if the set K is unbounded. For example, if K is a Banach space and A is a translation mapping, then A does not possess the BAFP property.

Finally, we quote the main result of [21].

Theorem 1.3. *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{A}$ such that each $A \in \mathcal{F}$ has the BAFP property.*

2. Main results

Let (X, ρ, M) be a complete hyperbolic space and let K be a nonempty, closed and ρ-convex subset of X.

For each $x \in K$ and each $r > 0$, set

$$B(x, r) = \{y \in K : \ \rho(x, y) \le r\}.$$

For each $x \in X$ and each nonempty set $D \subset X$, set

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

Denote by $S(K)$ the family of all nonempty, closed and bounded subsets of K.

For each $C, D \in S(K)$, set

$$H(C, D) := \max\{\sup\{\rho(x, D) : \ x \in C\}, \ \sup\{\rho(x, C) : \ x \in D\}\}.$$

The space $(S(K), H)$ is a metric space and its metric H is called the *Hausdorff metric*. It is known that the metric space $(S(K), H)$ is complete.

Denote by $\mathcal{M}$ the set of all mappings $A : K \to S(K)$ such that

$$H(A(x), A(y)) \le \rho(x, y) \text{ for all } x, y \in K. \tag{2.1}$$

Fix $\theta \in K$. We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\mathcal{U}(n) = \{(A, B) \in \mathcal{M} \times \mathcal{M} : \; H(A(x), B(x)) \le n^{-1} \text{ for all } x \in B(\theta, n)\}, \tag{2.2}$$

where n is a natural number. Clearly, the uniform space $\mathcal{M}$ is metrizable and complete.

Let $A \in \mathcal{M}$ and $\epsilon \ge 0$ be given. A point $x \in K$ is called an *ϵ-approximate fixed point* of A if $\rho(x, A(x)) \le \epsilon$.

We say that the mapping A has the *bounded approximate fixed point property* (or the BAFP property, for short) if there is a nonempty bounded set $K_0 \subset K$ such that for each $\epsilon > 0$, A has an ϵ-approximate fixed point in K_0, that is, a point $x_\epsilon \in K_0$ which satisfies $\rho(x_\epsilon, A(x_\epsilon)) \le \epsilon$.

For each $D \subset X$, we denote by $\mathrm{cl}(D)$ the closure of D.

For each nonempty set $D \subset X$ and each $A \in \mathcal{M}$, set

$$A(D) := \cup\{A(x) : \; x \in D\}.$$

In this paper we establish the following results.

Theorem 2.1. *Assume that $A \in \mathcal{M}$ and that $K_0 \subset K$ is a nonempty, closed, ρ-convex and bounded subset of K such that*

$$A(K_0) \subset K_0. \tag{2.3}$$

Then for each $\epsilon > 0$, there is a point $x_\epsilon \in K_0$ such that $\rho(x_\epsilon, A(x_\epsilon)) \le \epsilon$.

Thus every mapping satisfying the assumptions of Theorem 2.1 has the BAFP property.

Theorem 2.2. *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{M}$ such that each $A \in \mathcal{F}$ has the BAFP property.*

This theorem follows from Theorem 2.1 and the following result.

Theorem 2.3. *There exists an open and everywhere dense set $\mathcal{F} \subset \mathcal{M}$ such that for each $A \in \mathcal{F}$, there exists a nonempty, closed, ρ-convex and bounded set $K_A \subset K$ such that*

$$A(K_A) \subset K_A.$$

The rest of our paper is organized as follows. The next section contains several auxiliary results. Theorem 2.1 is proved in Section 4 and Theorem 2.3 is proved in Section 5.

3. Auxiliary results

Lemma 3.1. *Let $A \in \mathcal{M}$ and let $C \subset K$ be a nonempty bounded set. Then $\cup\{A(z) : \; z \in C\}$ is also a bounded set.*

Proof. There is $M > 0$ such that

$$C \subset B(\theta, M). \tag{3.1}$$

Let

$$x \in \cup\{A(z) : \; z \in C\}. \tag{3.2}$$

Then there is a point

$$z \in C \tag{3.3}$$

such that

$$x \in A(z). \tag{3.4}$$

By (2.1), (3.1), (3.3) and (3.4),

$$\rho(x, A(\theta)) \leq H(A(z), A(\theta)) \leq \rho(z, \theta) \leq M.$$

Since the above relation holds for any x satisfying (3.2), we conclude that

$$\cup\{A(z) : \; z \in C\} \subset \cup\{B(z, M+1) : \; z \in A(\theta)\}.$$

Lemma 3.1 is proved. □

Lemma 3.2. *Let $A \in \mathcal{M}$, $\gamma \in (0,1)$, $\eta \in K$ and for each $x \in K$, set*

$$\tilde{A}(x) := cl(\{(1-\gamma)z \oplus \gamma\eta : \; z \in A(x)\}). \tag{3.5}$$

Then $\tilde{A} \in \mathcal{M}$ and for all $x, y \in K$,

$$H(\tilde{A}(x), \tilde{A}(y)) \leq (1-\gamma)\rho(x, y). \tag{3.6}$$

Proof. Clearly, $\tilde{A}(x) \in S(K)$ for all $x \in K$. Let $x, y \in K$. We claim that (3.6) holds. To verify this, it is sufficient to show that for any $\xi \in \tilde{A}(x)$,

$$\rho(\xi, \tilde{A}(y)) \leq (1-\gamma)\rho(x, y). \tag{3.7}$$

In view of (3.5), it suffices to show that (3.7) holds for any

$$\xi \in \{(1-\gamma)z \oplus \gamma\eta : \; z \in A(x)\}. \tag{3.8}$$

Let $\xi \in K$ satisfy (3.8). Then there is

$$z \in A(x) \tag{3.9}$$

such that

$$\xi = (1-\gamma)z \oplus \gamma\eta. \tag{3.10}$$

By (2.1), (3.5), (3.9) and (3.10),

$$\begin{aligned}
\rho(\xi, \tilde{A}y) &= \rho((1-\gamma)z \oplus \gamma\eta, \tilde{A}(y)) \\
&\leq \inf\{\rho((1-\gamma)z \oplus \gamma\eta, (1-\gamma)u \oplus \gamma\eta) : \; u \in A(x)\} \\
&\leq \inf\{(1-\gamma)\rho(z, u) : \; u \in A(x)\} = (1-\gamma)\rho(z, A(x)) \\
&\leq (1-\gamma)H(A(x), A(y)) \leq (1-\gamma)\rho(x, y),
\end{aligned}$$

as asserted. □

Lemma 3.3. *Assume that $A \in \mathcal{M}$, $\gamma \in (0,1)$ and that for all $x, y \in K$,*

$$H(A(x), A(y)) \leq \gamma\rho(x, y). \tag{3.11}$$

Then there exists a number $M_1 > 1$ such that

$$A(B(\theta, M_1)) \subset B(\theta, M_1 - 1).$$

Proof. For each nonempty, closed and bounded set $C \subset K$, set

$$\widehat{A}(C) := \mathrm{cl}(A(C)). \tag{3.12}$$

By Lemma 3.1, the set $\widehat{A}(C)$ is bounded for all $C \in S(K)$ and $\widehat{A} : S(K) \to S(K)$. We now show that for each $C_1, C_2 \in S(K)$,

$$H(\widehat{A}(C_1), \widehat{A}(C_2)) \le \gamma H(C_1, C_2). \tag{3.13}$$

To this end, let

$$C_1, C_2 \in S(K) \ \text{ and } \ z \in \widehat{A}(C_1).$$

In order to prove (3.13), it is sufficient to show that

$$\rho(z, \widehat{A}(C_2)) \le \gamma H(C_1, C_2). \tag{3.14}$$

In view of (3.12), we may assume without loss of generality that

$$z \in A(C_1).$$

Thus there is a point

$$\xi \in C_1 \tag{3.15}$$

such that

$$z \in A(\xi). \tag{3.16}$$

Let ϵ be an arbitrary positive number. There is a point

$$v \in C_2 \tag{3.17}$$

such that

$$\rho(\xi, v) \le \rho(\xi, C_2) + \epsilon \le H(C_1, C_2) + \epsilon. \tag{3.18}$$

(The last inequality follows from (3.15).) By (3.11) and (3.18),

$$H(A(\xi), A(v)) \le \gamma\rho(\xi, v) \le \gamma H(C_1, C_2) + \gamma\epsilon. \tag{3.19}$$

By (3.16) and (3.19), there is a point

$$u \in A(v) \tag{3.20}$$

such that

$$\rho(z, u) \le \gamma H(C_1, C_2) + 2\gamma\epsilon. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\rho(z, A(v)) \le \gamma H(C_1, C_2) + 2\gamma\epsilon.$$

When combined with (3.12) and (3.17), this inequality implies that

$$\rho(z, \widehat{A}(C_2)) \le \rho(z, A(v)) \le \gamma H(C_1, C_2) + 2\gamma\epsilon.$$

Since ϵ is an arbitrary positive number, we conclude that (3.14) indeed holds. Thus (3.13) is true for all $C_1, C_2 \in S(K)$. By (3.13) and the Banach fixed point theorem [1], there is

$$C_* \in S(K)$$

such that

$$\widehat{A}(C_*) = C_*. \tag{3.22}$$

Choose $M_0 > 0$ such that

$$C_* \subset B(\theta, M_0) \tag{3.23}$$

and

$$M_1 > (2M_0 + 1)(1-\gamma)^{-1}. \tag{3.24}$$

Assume that

$$x \in B(\theta, M_1). \tag{3.25}$$

Fix

$$y \in C_*. \tag{3.26}$$

By (3.11), (3.23), (3.25) and (3.26),

$$\begin{aligned} H(A(x), A(y)) \leq & \quad \gamma\rho(x,y) \leq \gamma(\rho(x,\theta) + \rho(\theta, y)) \\ \leq & \quad \gamma(M_1 + M_0) \leq \gamma M_1 + M_0. \end{aligned} \tag{3.27}$$

In view of (3.27), for each $\xi \in A(x)$,

$$\rho(\xi, A(y)) \leq \gamma M_1 + M_0$$

and by (3.22) and (3.26),

$$\rho(\xi, C_*) \leq \gamma M_1 + M_0. \tag{3.28}$$

It follows from (3.23), (3.24) and (3.28) that for each $\xi \in A(x)$,

$$\rho(\xi, \theta) \leq \rho(\xi, C_*) + \sup\{\rho(h,\theta) :\ \theta \in C_*\} \leq \gamma M_1 + 2M_0 < M_1 - 1.$$

This implies that $A(x) \subset B(\theta, M_1 - 1)$ for all $x \in B(\theta, M_1)$. Lemma 3.3 is proved. □

4. Proof of Theorem 2.1

Let $\epsilon > 0$ be given. Set

$$d_0 = \sup\{\rho(y,z) :\ y, z \in K_0\}. \tag{4.1}$$

Choose $\gamma \in (0,1)$ such that

$$\gamma(d_0 + 1) < \epsilon. \tag{4.2}$$

Fix a point

$$\tilde{x} \in K_0. \tag{4.3}$$

For each $x \in K$, set

$$\tilde{A}(x) := \mathrm{cl}(\{(1-\gamma)z \oplus \gamma\tilde{x} :\ z \in A(x)\}). \tag{4.4}$$

By (2.3), (4.3) and (4.4), for all $x \in K$, $\tilde{A}(x) \in S(K)$ and

$$\tilde{A}(x) \subset K_0,\ x \in K_0. \tag{4.5}$$

By Lemma 3.2, for all $x, y \in K$,

$$H(\tilde{A}(x), \tilde{A}(y)) \leq (1-\gamma)\rho(x,y). \tag{4.6}$$

By (2.3), (4.1), (4.2), (4.3) and (4.4), we have for all $x \in K_0$,

$$\begin{aligned} H(A(x), \tilde{A}(x)) &= H(A(x), \mathrm{cl}(\{(1-\gamma)z \oplus \gamma\tilde{x} : z \in A(x)\}) \\ &\le \sup\{\rho(z, (1-\gamma)z \oplus \gamma\tilde{x}) : z \in A(x)\} \\ &\le \gamma \sup\{\rho(z, \tilde{x}) : z \in A(x)\} \le d_0\gamma < \epsilon. \end{aligned} \tag{4.7}$$

By (4.5), (4.6) and Nadler's fixed point theorem [10] (see also [23, page 38]), there exists a point x_ϵ such that

$$x_\epsilon \in K_0, \text{ and } x_\epsilon \in \tilde{A}(x_\epsilon).$$

When combined with (4.7), this implies that

$$\rho(x_\epsilon, A(x_\epsilon)) \le \rho(x_\epsilon, \tilde{A}(x_\epsilon)) + H(\tilde{A}(x_\epsilon), A(x_\epsilon)) < \epsilon.$$

This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.3

Let $A \in \mathcal{M}$ and let n be a natural number. It is not difficult to see that in order to prove the theorem it is sufficient to show that there exist $\tilde{A} \in \mathcal{M}$ and a natural number k such that the following properties hold:

$$(A, \tilde{A}) \in \mathcal{U}(n);$$

there is a nonempty, bounded, closed and ρ-convex set $M \subset K$ such that

$$B(M) \subset M \text{ for each } B \in \mathcal{M} \text{ satisfying } (B, \tilde{A}) \in \mathcal{U}(k).$$

Choose a number $\gamma \in (0, 1)$ such that

$$\gamma(n + \sup\{\rho(\xi, \theta) : \xi \in A(\theta)\}) < (2n)^{-1}. \tag{5.1}$$

For each point $x \in K$, set

$$\tilde{A}(x) := \mathrm{cl}(\{(1-\gamma)z \oplus \gamma\theta : z \in A(x)\}). \tag{5.2}$$

By Lemma 3.2, $\tilde{A} \in \mathcal{M}$ and

$$H(\tilde{A}(x), \tilde{A}(y)) \le (1-\gamma)\rho(x, y) \text{ for all } x, y \in K. \tag{5.3}$$

By Lemma 3.3, there exists $M_1 > 1$ such that

$$\tilde{A}(B(\theta, M_1)) \subset B(\theta, M_1 - 1). \tag{5.4}$$

In view of (5.1) and (5.2), for each $x \in B(\theta, n)$,

$$\begin{aligned} H(A(x), \tilde{A}(x)) &= H(A(x), \mathrm{cl}(\{(1-\gamma)z \oplus \gamma\theta : z \in A(x)\})) \\ &\le \sup\{\rho(z, (1-\gamma)z \oplus \gamma\theta) : z \in A(x)\} \\ &\le \gamma \sup\{\rho(z, \theta) : z \in A(x)\} \\ &\le \gamma \sup\{\rho(z, A(\theta)) + \sup\{\rho(\xi, \theta) : \xi \in A(\theta)\} : z \in A(x)\} \\ &\le \gamma \sup\{\rho(\xi, \theta) : \xi \in A(\theta)\} + \gamma \sup\{\rho(z, A(\theta)) : z \in A(x)\} \\ &\le \gamma \sup\{\rho(\xi, \theta) : \xi \in A(\theta)\} + \gamma H(A(x), A(\theta)) \\ &\le \gamma \sup\{\rho(\xi, \theta) : \xi \in A(\theta)\} + \gamma\rho(x, \theta) \le (2n)^{-1} \end{aligned}$$

and

$$(A, \tilde{A}) \in \mathcal{U}(n).$$

Choose a natural number

$$k > M_1 + 1 \tag{5.5}$$

and assume that

$$x \in B(\theta, M_1), \tag{5.6}$$

$$B \in \mathcal{M} \text{ and } (B, \tilde{A}) \in \mathcal{U}(k). \tag{5.7}$$

Let

$$y \in B(x). \tag{5.8}$$

By (5.5), (5.6), (5.7) and (5.8),

$$\rho(y, \tilde{A}(x)) \le H(B(x), \tilde{A}(x)) \le k^{-1}. \tag{5.9}$$

In view of (5.4) and (5.6),

$$\tilde{A}(x) \subset B(\theta, M_1 - 1).$$

When combined with (5.9), this inclusion implies that $y \in B(\theta, M_1)$ and therefore

$$B(x) \subset B(\theta, M_1)$$

for all x satisfying (5.6) and all B satisfying (5.7). This completes the proof of Theorem 2.3.

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