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REMARKS ON DIFFERENTIABILITY OF THE NORM AND UNIFORMLY CONVEX SETS

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This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th Birthday

Abstract. In this paper, we firstly prove that the uniform Gâteaux differentiability of a Banach space is equivalent to the norm-to-weak* uniform continuity of every (normalized) duality map defined on any bounded subsets of the Banach space. Next, we investigate some convergence problem for uniformly convex sets in a reflexive strictly convex Banach space with the Kadec-Klee property.

1. Introduction

Unless other specified, throughout this paper, we assume X is a normed linear space with norm $\| \ \|$ and let X^* be it's dual space. Denote by $\langle \cdot, \cdot \rangle$ the duality product and it will be convenient to write $\langle x, x^* \rangle$ for $x^*(x)$ for every $x^* \in X^*$. Also, we denote by S_X and S_{X^*} the unit spheres of X and X^*, respectively. We use the notations $x_n \to x$ and $x_n \rightharpoonup x$ to designate the strong and respectively, weak convergence of a sequence $\{x_n\}$ to x.

Recall that X satisfies the Kadec-Klee property (or the Radon-Riesz property [6]) if, for every sequence $\{x_n\}$ in X,

$$\|x_n\| \to \|x\| \text{ and } x_n \rightharpoonup x, \quad \text{we have } \ x_n \to x. \tag{1.1}$$

The (normalized) duality map $J : X \to X^*$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{1.2}$$

for each $x \in X$. It is well known that every uniformly convex space satisfies the Kadec-Klee property.

Recall also that a set $A \subset X$ is *f-uniformly convex* if there exists a $f \in \mathcal{F}$ such that

$$B\Big(\frac{a+b}{2}, f(\|a-b\|)\Big) \subset A, \quad \forall\, a, b \in A, \tag{1.3}$$

where $\mathcal{F}$ denotes the class of all nondecreasing functions from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}_+$ that vanish only at 0 ; see Definition 5.2 of [4] (or [3]) for more details, originally due to [8]. Unless $A = X$, A is bounded. Obviously, A is convex. First, we revisit and investigate Theorem 4.3.6 of [9]. In fact, the proof was only remained as Problem 4.3.5 for the reader:

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Theorem 1.1 ([9, Takahashi])**.** *Let X be a Banach space with a uniformly Gâteaux differentiable norm. Then the duality mapping $J : X \to X^*$ is norm-to-weak* uniformly continuous on bounded subsets of X.*

Secondly, we revisit Combettes's article [3], which introduces the following interesting result due to [8]; see Proposition 2.4 (ix) in [3].

Proposition 1.2 ([3, 8])**.** *Let X be a Hilbert space and let A be a closed and uniformly convex subset of H. If $d(x_n, A) \to 0$ and $x_n \rightharpoonup x \in \partial A$, then $x_n \to x$, where ∂A is the boundary of A.*

The following questions are naturally raised. How about the converse of Theorem 1.1? Secondly, as an analogue of compactness of $A \subset X$, does the weak compactness of A ensure existence of two points $x, y \in A$ so that $\|x - y\|$ is equal to the diameter of A? How do then we extend Proposition 1.2 over general Banach spaces?

In this paper, we firstly prove that the converse of Theorem 1.1 also remains true. Next, we establish that all reflexive strictly convex Banach spaces with the Kadec-Klee property are a possible solution of our second question.

2. Preliminaries

Let X be a Banach space and consider the limit:

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}. \tag{2.1}$$

Recall we say that the norm of X is

- Gâteaux differentiable (or X is smooth) if the limit (2.1) exists for each $x, y \in S_X$;
- uniformly Gâteaux differentiable if the limit (2.1) is attained uniformly in $x \in S_X$ for each fixed $y \in S_X$;
- Fréchet differentiable if the limit (2.1) is attained uniformly in $y \in S_X$ for each fixed $x \in S_X$;
- uniformly Fréchet differentiable (or X is uniformly smooth) if the limit (2.1) is attained uniformly in $x, y \in S_X$.

Proposition 2.1 ([2, 9, 10])**.** *Let X be a real Banach space. Then its normalized duality map J satisfies the following properties:*

(i) *J is homogeneous, i.e., $J(\lambda x) = \lambda J(x)$ for $\lambda \in \mathbb{R}$ and $x \in X$.*
(ii) *J is additive if and only if X is a Hilbert space.*
(iii) *J is single-valued if and only if X is smooth.*
(iv) *J is surjective if and only if X is reflexive.*
(v) *J is injective or strictly monotone if and only if X is strictly convex.*
(vi) *J is single-valued and norm-to-norm continuous if and only if X is Fréchet differentiable.*
(vii) *if X is smooth (i.e., the norm of X is Gâteaux differentiable), then J is single-valued and norm-to-weak* continuous.*
(viii) *if the norm of X is uniformly Gâteaux differentiable, then J is single-valued and norm-to-weak* uniformly continuous on bounded sets of X.*

(vix) *J is single-valued and norm-to-norm uniformly continuous on bounded sets of X if and only if X is uniformly smooth.*

Remark 2.2. Note that the norm of X is uniformly Gâteaux differentiable if and only if

$$\lim_{\lambda\to 0}\sup_{\|x\|=1}\left|\frac{\|x+\lambda y\|-\|x\|}{\lambda}-\langle y, j_x\rangle\right|=0$$

for each $y \in S_X$, where $j_x \in S_{X^*}$ with $\langle x, j_x\rangle = 1$ for $x \in S_X$. Let $y \in S_X$ be fixed; for each $\epsilon > 0$, there exists $\delta := \delta(\epsilon, y) > 0$ such that

$$0 < |\lambda| < \delta \quad \Rightarrow \quad \big|\|x+\lambda y\| - \|x\| - \lambda\langle y, j_x\rangle\big| < \epsilon|\lambda|, \qquad x \in S_X. \tag{2.2}$$

Recall we say that X is *strictly convex* if for each $f \in X^*$ there exists at most one point in B_X at which f attains its maximum, where B_X is the unit ball of X; see [1].

Here we summarize some equivalent properties of strict convexity of X; see [1,6,7] for detailed proof.

Proposition 2.3 ([1,6,7])**.** *X is strictly convex if and only if one of the following equivalent properties holds.*

(i) *If $\|x+y\| = \|x\| + \|y\|$ and $x \neq 0$, $y = tx$ for some $t \geq 0$;*
(ii) *If $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0,1)$, namely, the unit sphere (or any sphere) contains no line segment;*
(iii) *If $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|(x+y)/2\| < 1$;*
(iv) *The function $x \to \|x\|^2$, $x \in X$, is strictly convex.*

Recall that a mapping $x \mapsto j_x$ of $X\setminus\{0\}$ to $X^*\setminus\{0\}$ is called a *support* mapping [5] whenever

(i) $\|x\| = 1$ implies $\|j_x\| = 1 = \langle x, j_x\rangle$;
(ii) $\lambda \geq 0$ implies $j_{\lambda x} = \lambda j_x$.

Proposition 2.4 ([5,6])**.** *Let $x \mapsto j_x$ be a support mapping. Then*

$$\frac{\langle y, j_x\rangle}{\|x\|} \leq \frac{\|x+\lambda y\|-\|x\|}{\lambda} \leq \frac{\langle y, j_{x+\lambda y}\rangle}{\|x+\lambda y\|}, \qquad \lambda > 0,\ x, y \in S_X. \tag{2.3}$$

The above inequalities are reversed for $\lambda < 0$.

Remark 2.5. For each $x \in X$, choose a $j_x \in J(x)$ (obviously, $j_0 = 0$). Then the mapping $x \mapsto j_x$ is called a *selection* of the (normalized) duality mapping J. Note that if $x \mapsto j_x$ of $X \setminus \{0\}$ to $X^* \setminus \{0\}$ is a support mapping, it is a selection of J. In fact, for each $x \in X$, it suffices to consider the case $\|x\| \neq 1$, $x \neq 0$; take $u := x/\|x\| \in S_X$, it follows from (i) and (ii) above that

$$\|j_u\| = 1 = \langle u, j_u\rangle \quad\Leftrightarrow\quad \|j_x\| = \|x\| \text{ and } \langle x, j_x\rangle = \|x\|^2 \quad\Leftrightarrow\quad j_x \in J(x).$$

Conversely, it is obvious that every selection of J except $0 \in X$ is a support mapping of $X \setminus \{0\}$ to $X^* \setminus \{0\}$.

Finally, we introduce the following equivalents:

Proposition 2.6 ([5,6,9])**.** *The following statements are equivalent:*

(i) *there exists a support mapping $x \mapsto j_x$ which is norm-to-norm uniformly continuous from S_X to S_{X^*};*
(ii) *the norm of X is uniformly Fréchet differentiable;*
(iii) *every support mapping $x \mapsto j_x$ is norm-to-norm uniformly continuous from S_X to S_{X^*}.*

3. Main results

First, we prove that the converse of Theorem 1.1 (or (viii) of Proposition 2.1) also holds. For this proof, the following subsequent two lemmas are crucial. At first, we give an analogue of Proposition 2.6 for uniformly Gâteaux differentiability of the norm.

Lemma 3.1. *The following statements are equivalent:*
(i) *the norm of X is uniformly Gâteaux differentiable;*
(ii) *every support mapping $x \mapsto j_x$ is norm-to-weak* uniformly continuous from S_X to S_{X^*}.*
(iii) *there exists a support mapping $x \mapsto j_x$ which is norm-to-weak* uniformly continuous from S_X to S_{X^*};*

Proof. (i)⇒(ii). We prove it by contradiction. Assume that there exists a support mapping $x \mapsto j_x$ which is not be norm-to-weak* uniformly continuous from S_X to S_{X^*}. Then there exists $\epsilon > 0$ and sequences $\{x_n\}$ and $\{y_n\}$ in S_X such that

$$\|x_n - y_n\| < \frac{1}{n} \text{ but } |\langle y, j_{x_n} - j_{y_n}\rangle| \geq 2\epsilon \text{ for some } y \in S_X. \tag{3.1}$$

Since $\|x_n - y_n\| \to 0$, we have

$$\begin{aligned} 0 &\leq 1 - \langle x_n, j_{y_n}\rangle = \langle y_n, j_{y_n}\rangle - \langle x_n, j_{y_n}\rangle \\ &= \langle y_n - x_n, j_{y_n}\rangle \leq |\langle y_n - x_n, j_{y_n}\rangle| \leq \|y_n - x_n\| \to 0 \end{aligned}$$

and so $\langle x_n, j_{y_n}\rangle \to 1$. Similarly, we can see $\langle y_n, j_{x_n}\rangle \to 1$. On the other hand, since the norm of X is uniformly Gâteaux differentiable, by (2.2) of Remark 2.2, for the above $\epsilon > 0$ and $y \in S_X$, there exists $\delta := \delta(\epsilon, y) > 0$ such that

$$0 < |\lambda| < \delta \;\Rightarrow\; \big|\|x + \lambda y\| - \|x\| - \lambda\langle y, j_x\rangle\big| < \frac{\epsilon|\lambda|}{4}, \quad x \in S_X.$$

For fixed $\lambda \in (0, \delta)$, a simple calculation yields

$$\|x + \lambda y\| - 1 \leq \frac{\lambda\epsilon}{4} + \lambda\langle y, j_x\rangle, \quad \|x - \lambda y\| - 1 \leq \frac{\lambda\epsilon}{4} - \lambda\langle y, j_x\rangle \tag{3.2}$$

for all $x \in S_X$. Since $\langle x_n, j_{y_n}\rangle \to 1$, there exists $K \in \mathbb{N}$ such that $1 - \frac{\lambda\epsilon}{2} \leq \langle x_n, j_{y_n}\rangle$ for all $n \geq K$. Then, it follows that

$$\begin{aligned} 1 - \frac{\lambda\epsilon}{2} &\leq \langle x_n, j_{y_n}\rangle = \langle x_n, j_{y_n} + j_{x_n}\rangle - 1 \\ &= \langle x_n + \lambda y, j_{y_n}\rangle + \langle x_n - \lambda y, j_{x_n}\rangle - 1 - \lambda\langle y, j_{y_n} - j_{x_n}\rangle \\ &\leq \|x_n + \lambda y\| + \|x_n - \lambda y\| - 1 - \lambda\langle y, j_{y_n} - j_{x_n}\rangle \\ &\leq 1 + \frac{\lambda\epsilon}{2} - \lambda\langle y, j_{y_n} - j_{x_n}\rangle \quad \text{by (3.2) with } x := x_n \end{aligned}$$

and we have

$$\langle y, j_{y_n} - j_{x_n} \rangle \leq \epsilon.$$

Using $\langle y_n, j_{x_n} \rangle \to 1$ and interchanging the roles of x_n and y_n above (then use (3.2) with $x := y_n$), we can similarly derive

$$\langle y, j_{x_n} - j_{y_n} \rangle \leq \epsilon.$$

Thus we have shown that $|\langle y, j_{x_n} - j_{y_n} \rangle| \leq \epsilon$, which contradicts (3.1). (ii)⇒(iii) is obvious.

(iii)⇐(i). As in the proof ("(a) ⇒ (b)") of Proposition 2.6, for any $x, y \in S_X$, set $x_\lambda := \frac{x+\lambda y}{\|x+\lambda y\|} \in S_X$ and $j_{x_\lambda} = \frac{j_{x+\lambda y}}{\|x+\lambda y\|} \in S_{X^*}$. Suppose that there exists a support mapping $x \mapsto j_x$ which is norm-to-weak* uniformly continuous from S_X to S_{X^*}, i.e., for given $\epsilon > 0$, there exists $\eta := \eta(\epsilon) > 0$ such that

$$u, v \in S_X,\ \|u - v\| < \eta \ \Rightarrow\ |\langle z, j_u - j_v \rangle| < \epsilon, \quad z \in X. \tag{3.3}$$

Since $x_\lambda \to x$ as $\lambda \to 0$, for the $\eta > 0$, there exists $\delta := \delta(\eta) > 0$ such that

$$0 < |\lambda| < \delta \ \Rightarrow\ \|x_\lambda - x\| < \eta$$

and from (3.3) it follows that $|\langle z, j_{x_\lambda} - j_x \rangle| < \epsilon$ for all $z \in X$. On the other hand, from Proposition 2.4 we observe that

$$\Big|\frac{\|x+\lambda y\| - \|x\|}{\lambda} - \langle y, j_x \rangle\Big| \leq \Big|\frac{\langle y, j_{x+\lambda y} \rangle}{\|x+\lambda y\|} - \langle y, j_x \rangle\Big|$$
$$= \Big|\langle y, \frac{j_{x+\lambda y}}{\|x+\lambda y\|} \rangle - \langle y, j_x \rangle\Big| = |\langle y, j_{x_\lambda} - j_x \rangle| < \epsilon$$

for all $\lambda \neq 0$ and $x, y \in S_X$, which directly implies that

$$x, y \in S_X, 0 < |\lambda| < \delta \ \Rightarrow\ \big|\|x+\lambda y\| - \|x\| - \lambda\langle y, j_x \rangle\big| < \epsilon|\lambda|.$$

Since (2.2) of Remark 2.2 is fulfilled, the norm of X is uniformly Gâteaux differentiable. □

Lemma 3.2. *If a support mapping $x \mapsto j_x$ is norm-to-weak* uniformly continuous from S_X to S_{X^*}, then the support mapping $x \mapsto j_x$ is norm-to-weak* uniformly continuous on bounded sets of X.*

Proof. Suppose not, (3.1) is placed with

$$x_n,\ y_n \in D,\ \|x_n - y_n\| < \frac{1}{n} \ \text{ but } \ |\langle y, j_{x_n} - j_{y_n} \rangle| \geq 2\epsilon \ \text{ for some } y \in S_X,$$

where $M := \sup_{x \in D} \|x\| > 0$. If $x_n \to 0$, then $y_n \to 0$ too. Then it yields a contradiction as

$$0 < 2\epsilon \leq |\langle y, j_{x_n} - j_{y_n} \rangle| \leq \|y\|\,\|j_{x_n} - j_{y_n}\| \leq \|y\|(\|x_n\| + \|y_n\|) \to 0.$$

So, let $x_n \not\to 0$. By passing to a subsequence if necessary, we may assume that $\|x_n\| \geq \alpha$ and also $\|y_n\| \geq \alpha$ because of $\|x_n\| - \|y_n\| \leq \|x_n - y_n\| < 1/n$. Setting $u_n := x_n/\|x_n\|$, $v_n := y_n/\|y_n\| \in S_X$; then

$$\|u_n - v_n\| \;=\; \Big\|\frac{\|y_n\|x_n - \|x_n\|y_n}{\|x_n\|\,\|y_n\|}\Big\|$$

$$\begin{aligned} &\le \frac{1}{\alpha}[|\|y_n\| - \|x_n\||\,\|x_n\| + \|x_n\|\,\|x_n - y_n\|] \\ &\le \frac{1}{\alpha}M(|\|y_n\| - \|x_n\|| + \|x_n - y_n\|) \to 0. \end{aligned}$$

On the other hand, from property (ii) of a support mapping we have $j_{u_n} = \frac{j_{x_n}}{\|x_n\|}$, $j_{v_n} = \frac{j_{y_n}}{\|y_n\|} \in S_{X^*}$ and an easy computation immediately gives

$$\begin{aligned} |\langle y, j_{u_n} - j_{v_n}\rangle| &= \Big|\langle y, \frac{j_{x_n}}{\|x_n\|} - \frac{j_{y_n}}{\|y_n\|}\rangle\Big| \\ &= \Big|\frac{1}{\|x_n\|}\langle y, j_{x_n} - j_{y_n}\rangle - \Big(\frac{1}{\|y_n\|} - \frac{1}{\|x_n\|}\Big)\langle y, j_{y_n}\rangle\Big| \\ &\ge \Big|\frac{1}{\|x_n\|}\langle y, j_{x_n} - j_{y_n}\rangle\Big| - \frac{\big|\|x_n\| - \|y_n\|\big|}{\|x_n\|\,\|y_n\|} \cdot |\langle y, j_{y_n}\rangle| \\ &\ge \frac{1}{M}|\langle y, j_{x_n} - j_{y_n}\rangle| - \frac{\|x_n - y_n\|}{\alpha}\|y\| \\ &\ge \frac{1}{M}|\langle y, j_{x_n} - j_{y_n}\rangle| - \frac{\|y\|}{n\alpha} \ge \frac{2\epsilon}{M} - \frac{\epsilon}{M} = \frac{\epsilon}{M}. \end{aligned}$$

for all $n > M\|y\|/(\alpha\epsilon)$. We have shown that there exists $\epsilon > 0$ and sequences $\{u_n\}$ and $\{v_n\}$ in S_X such that

$$\|u_n - v_n\| \to 0 \ \text{ but } \ |\langle y, j_{u_n} - j_{v_n}\rangle| \ge \frac{\epsilon}{M} \quad \text{for some } y \in S_X.$$

Therefore, $x \mapsto j_x$ is not norm-to-weak* uniformly continuous from S_X to S_{X^*}. □

Remark 3.3. Note that the converse of the above lemma is obvious.

Now, by virtue of the above two Lemmas, we also give an affirmative answer to the first question.

Theorem 3.4. *The norm of X is uniformly Gâteaux differentiable if and only if every duality mapping $J : X \to X^*$ is single-valued and norm-to-weak* uniformly continuous on bounded subsets of X.*

Proof. Since J is single-valued, it follows from Remark 2.5 that J is a support mapping of $X \setminus \{0\}$ to $X^* \setminus \{0\}$. Use Lemma 3.1 and 3.2 to induce the required conclusion. □

Next, we begin with the following interesting result for weak compactness (equivalently, weakly sequential compactness) of a subset A in reflexive Banach spaces.

Lemma 3.5. *Let X be a reflexive Banach space, let $A \subset X$ be weakly compact, and let $\delta = \operatorname{diam} A = \sup_{x,y\in A} \|x - y\|$. Then there are two diametral points $x, y \in A$ such $\|x - y\| = \delta$.*

Proof. Since A is weakly compact, it is bounded, i,e., $\delta < \infty$. Find then two sequences (x_n), $(y_n) \subset A$ so that $\|x_n - y_n\| \to \delta$. Furthermore, there are two subsequences (x_{n_k}) and (y_{n_k}) of them such that $x_{n_k} \rightharpoonup x \in A$ and $y_{n_k} \rightharpoonup y \in A$. To reach the conclusion, it suffices to show that $\|x - y\| \ge \delta$. Indeed, if $\delta = 0$, since

A is a singleton set, the conclusion is obvious. Assume $\delta > 0$, choose $\epsilon > 0$ with $0 < \epsilon < \delta$; since $\|x_{n_k} - y_{n_k}\| \to \delta$, there exists $K = K_\epsilon \in \mathbb{N}$ such that

$$0 < \delta - \epsilon < \|x_{n_k} - y_{n_k}\| < \delta + \epsilon, \quad \forall\, k \geq K.$$

For each (fixed) $k \geq K$, using the Hahn-Banach theorem, there exists $g_k \in X^*$ such that $\|x_{n_k} - y_{n_k}\| = g_k(x_{n_k} - y_{n_k})$ and $\|g_k\| = 1$. Then, since $x_{n_k} - y_{n_k} \rightharpoonup x - y$, it follows that

$$g_i(x-y) = \lim_{k\to\infty} g_i(x_{n_k} - y_{n_k}), \quad \forall\, i \geq K.$$

Taking $\sup_{i\geq K}$ on both sides, we easily compute

$$\begin{aligned}\sup_{i\geq K} g_i(x-y) &= \sup_{i\geq K} \lim_{k\to\infty} g_i(x_{n_k} - y_{n_k}) \\ &\geq \lim_{k\to\infty} g_k(x_{n_k} - y_{n_k}) = \lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = \delta.\end{aligned}$$

Clearly, $x \neq y$ because if $x = y$, it follows from the inequality above that $0 = \delta$, which contradicts to $\delta > 0$. Then it follows that

$$\|x-y\| = \sup_{\|f\|=1} f(x-y) \geq \sup_{i\geq K} g_i(x-y) \geq \delta,$$

completing the proof. □

Proposition 3.6. *Let X be a reflexive Banach space and let $A \subset X$ be closed and f-uniformly convex with $f(t) = \frac{1}{2}t$, $t \geq 0$. Then A is a closed ball with its radius $\frac{1}{2}\operatorname{diam} A$.*

Proof. If $A = X$, it is clear that the conclusion remains true. Unless $A = X$, since A is uniformly convex, it is bounded, i.e., $\delta = \operatorname{diam} A < \infty$. Since A is bounded closed convex, it is weakly compact. From Lemma 3.5 there exist two diametral points $x, y \in A$ such $\|x-y\| = \delta$. Since A is convex, $\frac{x+y}{2} \in A$ too. Finally, we claim that $A = B\left(\frac{x+y}{2}, \frac{1}{2}\delta\right)$. It is clear that $A \supset B\left(\frac{x+y}{2}, \frac{1}{2}\delta\right)$ by (1.3). The converse inclusion is clear because the length of the segment $[x, y] \subset A$ is the diameter of A. □

Now we shall establish that Proposition 1.2 still remains true on reflexive strictly convex Banach spaces which satisfy the Kadec-Klee property.

Proposition 3.7. *Let X be a reflexive strictly convex Banach space with the Kadec-Klee property, and let $A \subset X$ be closed and f-uniformly convex with $f(t) = \frac{1}{2}t$, $t \geq 0$. If $d(x_n, A) \to 0$ and $x_n \rightharpoonup x \in \partial A$, then $x_n \to x$.*

Proof. Since A is closed convex, $d(x_n, A) = \|x_n - P_A x_n\| \to 0$, where P_A denotes the metric projection of X onto A. First, assume that $x_n \notin A$ ultimately, i.e., there exists $n_0 \in \mathbb{N}$ such that $x_n \notin A$ for all $n \geq n_0$. Then, $P_A x_n \in \partial A$ for $n \geq n_0$. Since A is bounded, $\delta = \operatorname{diam} A < \infty$. From Proposition 3.6, it follows that $A = B\left(\frac{x_0+y_0}{2}, \frac{1}{2}\delta\right)$, where $x_0, y_0 \in A$ and $\|x_0 - y_0\| = \delta$. Noticing that $B\left(0, \frac{1}{2}\delta\right) = B\left(\frac{x_0+y_0}{2}, \frac{1}{2}\delta\right) - \frac{1}{2}(x_0+y_0)$, and x, $P_A x_n \in \partial A$ for all $n \geq n_0$, we observe that $\left\|P_A x_n - \frac{1}{2}(x_0+y_0)\right\| = \frac{1}{2}\delta = \left\|x - \frac{1}{2}(x_0+y_0)\right\|$ for all $n \geq n_0$. Considering the sequence $\{x_n - (x_0+y_0)/2\}$ instead of $\{x_n\}$, we readily see that $x_n - (x_0+y_0)/2 \rightharpoonup x - (x_0+y_0)/2$ and

$$\lim_{n\to\infty} \|x_n - (x_0+y_0)/2\| = \lim_{n\to\infty} \|P_A x_n - (x_0+y_0)/2\| = \|x - (x_0+y_0)/2\|,$$

which concludes that $x_n - \frac{1}{2}(x_0 + y_0) \to x - \frac{1}{2}(x_0 + y_0)$ by virtue of the Kadec-Klee property of X; hence $x_n \to x$. Next, assume that $x_n \in A$ ultimately. Then, since $x_n - (x_0 + y_0)/2 \rightharpoonup x - (x_0 + y_0)/2$ and $\|x_n - (x_0 + y_0)/2\| \leq \frac{1}{2}\delta = \|x - (x_0 + y_0)/2\|$ ultimately, we easily obtain that

$$\lim_{n\to\infty} \|x_n - (x_0 + y_0)/2\| = \|x - (x_0 + y_0)/2\|.$$

The similar argument asserts that $x_n \to x$. Finally, consider the remaining case, that is, one subsequence of $\{x_n\}$ is in A and the other is not in A. In this case we similarly concludes that $x_n \to x$ by the help of previous two cases. This completes the proof. □

Corollary 3.8. *Let X be a uniformly convex Banach space and let $A \subset X$ be closed and uniformly convex. If $d(x_n, A) \to 0$ and $x_n \rightharpoonup x \in \partial A$, then $x_n \to x$.*

4. A remark

Finally, we recall the following

Proposition 4.1 (See Problem 4.3.5 of [9])**.** *If the norm of X^* is Fréchet differentiable, then X is reflexive, strictly convex and satisfies the Kadec-Klee property.*

We found in the book [11] that the converse of Proposition 4.1 still remains true. But, we would like to suggest an easy short proof for the sake of convenience. By interchanging the roles of X and X^* in Proposition 4.1, we directly observe the following corollary in reflexive Banach spaces.

Corollary 4.2. *Let X be a reflexive Banach space. If the norm of X is Fréchet differentiable, then X^* is strictly convex and satisfies the Kadec-Klee property.*

Proposition 5.2 of [2] insists that the converse of Corollary 4.2 holds. Therefore we summarize

Proposition 4.3. *Let X be a reflexive Banach space. Then X^* is strictly convex and satisfies the Kadec-Klee property if and only if the norm of X is Fréchet differentiable.*

Since X is reflexive if and only if X^* is reflexive, and the Fréchet differentiability of the norm of X^* implies the reflexivity of X^*, by interchanging the roles of X and X^* in Proposition 4.3 again, we conclude that the converse of Proposition 4.1 holds.

Proposition 4.4. *X is reflexive, strictly convex and satisfies the Kadec-Klee property if and only if the norm of X^* is Fréchet differentiable.*

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