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ISHIKAWA ITERATIVE TECHNIQUES FOR THE SPLIT FIXED POINTS PROBLEM ASSOCIATED WITH PSEUDOCONTRACTIONS

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Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. The split common fixed points problem associated with the pseudo-contractive mappings is studied. We present an iterative using Ishikawa iterative techniques. Strong convergence analysis is shown.

1. Background and motivation

1.1. **Fixed point problem.** Many practical problems can be formulated as a fixed point problem

$$x = Tx \tag{1.1}$$

where T is a nonlinear operator (defined in a metric or normed space). The set of solutions $Fix(T)$ of this equation are called the set of fixed points of T. If T is a self-contraction defined on a complete metric space C, Banach's contraction principle establishes that T has a unique fixed point and, for any $x \in C$, the sequence of iterates, $\{T^n x\}$, called in general Picard iterates, converges strongly to the fixed point of T. However, if the mapping T is a nonexpansive self-mapping on C, then it is not true, in general, that T has a fixed point. One must assume additional conditions on T and/or the underlying space to ensure the existence of fixed points of T and, even when a fixed point of T exists, the sequence of iterates, $\{T^n x\}$, does not converge, in general. Bruck [2] is a nice survey up to the year 1983 about the asymptotic behavior of nonexpansive mappings in Hilbert and Banach spaces. It is the connection to the geometry of Banach spaces and the theory of maximal monotone and m-accretive operators (hence nonlinear evolution equations) that makes nonexpansive mappings one of the major and most active research areas of nonlinear analysis since mid-1960's. Of particular importance in recent years is the study of iterative methods for finding a solution of (1.1) when T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert or Banach space.

There are basically three types of iterative algorithms which have been investigated: Mann's algorithm, Halpern's algorithm and Ishikawa's algorithm.

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Mann's algorithm ([14])**.** *For the initial guess $x_0 \in C$, the well known Mann iteration is defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, n \in \mathbb{N} \text{ and } \alpha_n \in [0,1]. \tag{1.2}$$

Halpern's algorithm ([8])**.** *For the initial guess $x_0 \in C$ and anchor $u \in C$ arbitrary (but fixed), the Halpern iteration is defined by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, n \in \mathbb{N} \text{ and } \alpha_n \in [0,1]. \tag{1.3}$$

Ishikawa's algorithm ([11])**.** *For the initial guess $x_0 \in C$, the well known Mann iteration is defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T((1-\beta_n)x_n + \beta_n Tx_n), n \in \mathbb{N}, \tag{1.4}$$

where $\alpha_n \in [0,1]$ and $\beta_n \in [0,1]$.

In [18], Reich stated that if the underlying space is uniformly convex and has a Frechet differentiable norm, and if $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by Mann's algorithm (1.2) converges weakly to a fixed point of T (assuming that T has fixed points). However, the counterexample of Genel and Lindenstrauss ([12]) shows that Mann's algorithm can have weak convergence only (in infinite dimensional spaces). At the same time, Mann's algorithm can not in general be applicable for the iterative construction of fixed points of the pseudocontractive mappings (see [6]).

Halpern's algorithm can have strong convergence provided the underlying space is smooth enough. There are a large number references associated with the iterative approach to the fixed points of nonexpansive mappings, see, for instance, [1, 5, 15, 19–21, 24].

The importance of Ishikawa's algorithm lies in the fact that it can be be applicable for the iterative construction of fixed points of the pseudocontractive mappings. In this respect, the following result due to Ishikawa [11] is important.

Theorem 1.1. *Let C be a convex compact subset of a Hilbert space H and let $T : C \to C$ be an L-Lipschitzian pseudocontractive mapping with $Fix(T) \neq \emptyset$. Assume $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. Then the sequence $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T.*

Ishikawa's algorithm has strong convergence under the assumption that the underlying space C is a compact set. Very recently, Zegeye, Shahzad and Alghamdi [25] further studied the convergence analysis of the Ishikawa iteration (1.4). They proved ingeniously the strong convergence of the Ishikawa iteration without the compactness assumption. However, we have to assume that the interior of $Fix(T)$ is nonempty. This appears very restrictive since even in $\mathbb{R}$ with the usual norm, Lipschitz pseudocontractive maps with finite number of fixed points do not enjoy this condition that $intFix(T) \neq \emptyset$. It is therefore an interesting problem to invent iterative algorithms that can generate sequences which converge strongly to the fixed point of pseudocontractions without any additional assumptions on the underlying spaces and operators. This is one of our main motivations.

1.2. **Split fixed point problem.** Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : H_2 \to H_2$ and $T : H_1 \to H_1$ be two nonlinear mappings. Now we consider the following problem:

$$\text{Find } x^* \in Fix(T) \text{ such that } Ax^* \in Fix(S). \tag{1.5}$$

This problem referred as the split fixed points problem was first introduced by Censor and Segal [4]. The split fixed points problem is a generalization of the split feasibility problem and of the convex feasibility problem.

For solving (1.5), Censor and Segal [4] invented an algorithm which generates a sequence $\{x_n\}$ according to the iterative procedure:

$$x_{n+1} = T(x_n - \gamma A^*(I - S)Ax_n), n \in \mathbb{N}. \tag{1.6}$$

Moudafi [16] relaxed (1.6) to the following form

$$\begin{cases} y_n = x_n - \gamma A^*(I - S)Ax_n, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T(y_n), n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where S and T are demicontractive operators.

Note that (1.6) and (1.7) have weak convergence. Some related work, see, for example, [3, 7, 9, 10, 17, 22, 26]. Our another purpose of this paper is to construct iterative algorithm for solving the split fixed point problem (1.5). Our motivation is based on Ishikawa's algorithm (1.4) and Moudafi's algorithm (1.7). We devote to study a class of pseudocontractive mappings which is more general than the class of demicontractive operators. Our results extend and improve the corresponding results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H.

Recall that a mapping $T : C \to C$ is called pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2$$

for all $x, y \in C$. It is well-known that T is pseudocontractive if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \tag{2.1}$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. If $L = 1$, we call T is nonexpansive.

For all $x, y \in H$, the following conclusions hold:

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \tag{2.2}$$

where $t \in [0, 1]$.

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \tag{2.3}$$

and

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{2.4}$$

Lemma 2.1 ([27]). *Let H be a real Hilbert space, C a closed convex subset of H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then*
(i) *$Fix(T)$ is a closed convex subset of C,*
(ii) *$(I - T)$ is demiclosed at zero.*

The following lemmas can be found in [28]. For the completeness, we give the proofs.

Lemma 2.2. *Let H be a Hilbert space. Let $T : H \to H$ be an L-Lipschitzian mapping with $L > 1$. Then,*

$$Fix(T) = Fix(T((1 - \beta)I + \beta T))$$

for all $\beta \in (0, \frac{1}{L})$.

Proof. As a matter of fact, $Fix(T) \subset Fix(T((1 - \beta)I + \beta T))$ is obvious. Next, we show that $Fix(T((1 - \beta)I + \beta T)) \subset Fix(T)$.

Take any $x^* \in Fix(T((1 - \beta)I + \beta T))$. We have $T((1 - \beta)I + \beta T)x^* = x^*$. Set $S = (1 - \beta)I + \beta T$. We have $TSx^* = x^*$. Write $Sx^* = y^*$. Then, $Ty^* = x^*$. Now we show $x^* = y^*$. In fact,

$$\begin{aligned}\|x^* - y^*\| &= \|Ty^* - Sx^*\| \\ &= \|Ty^* - (1 - \beta)x^* - \beta Tx^*\| \\ &= \beta\|Ty^* - Tx^*\| \\ &\leq \beta L\|y^* - x^*\|.\end{aligned}$$

Since, $\beta < \frac{1}{L}$, we deduce $y^* = x^* \in Fix(S) = Fix(T)$. Thus, $x^* \in Fix(T)$. Hence, $Fix(T((1-\beta)I+\beta T)) \subset Fix(T)$. Therefore, $Fix(T((1-\beta)I+\beta T)) = Fix(T)$. □

Lemma 2.3. *Let H be a Hilbert space. Let $T : H \to H$ be an L-Lipschitz pseudocontractive mapping with $Fix(T) \neq \emptyset$. Then, for all $x \in H$ and all $x^* \in Fix(T)$, we have*

$$\|(1 - \alpha)x + \alpha T((1 - \beta)x + \beta Tx) - x^*\| \leq \|x - x^*\|,$$

where $0 < \alpha < \beta < \frac{1}{\sqrt{1+L^2}+1}$.

Proof. Since $x^* \in Fix(T)$, we have from (2.1) that

$$\begin{aligned}\|T((1 - \beta)I + \beta T)x - x^*\|^2 \leq\ &\|(1 - \beta)(x - x^*) + \beta(Tx - x^*)\|^2 \\ &+ \|(1 - \beta)x + \beta Tx - T((1 - \beta)x + \beta Tx)\|^2,\end{aligned} \tag{2.5}$$

and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2, \tag{2.6}$$

for all $x \in H$.

By (2.5), (2.2) and (2.6), we obtain

$$\begin{aligned}
&\|T((1-\beta)I+\beta T)x-x^*\|^2\\
&\le \|(1-\beta)(x-x^*)+\beta(Tx-x^*)\|^2\\
&\quad+\|(1-\beta)x+\beta Tx-T((1-\beta)I+\beta T)x\|^2\\
&=\|(1-\beta)(x-T((1-\beta)x+\beta Tx))+\beta(Tx-T((1-\beta)x+\beta Tx))\|^2\\
&\quad+\|(1-\beta)(x-x^*)+\beta(Tx-x^*)\|^2\\
&=(1-\beta)\|x-T((1-\beta)x+\beta Tx)\|^2+\beta\|Tx-T((1-\beta)x+\beta Tx)\|^2\\
&\quad-\beta(1-\beta)\|x-Tx\|^2+(1-\beta)\|x-x^*\|^2+\beta\|Tx-x^*\|^2\\
&\quad-\beta(1-\beta)\|x-Tx\|^2\\
&\le (1-\beta)\|x-x^*\|^2+\beta(\|x-x^*\|^2+\|x-Tx\|^2)\\
&\quad-2\beta(1-\beta)\|x-Tx\|^2+(1-\beta)\|x-T((1-\beta)x+\beta Tx)\|^2\\
&\quad+\beta\|Tx-T((1-\beta)x+\beta Tx)\|^2.
\end{aligned}$$

Since T is L-Lipschitzian and $x-((1-\beta)x+\beta Tx)=\beta(x-Tx)$, we have

$$\|Tx-T((1-\beta)x+\beta Tx)\|\le \beta L\|x-Tx\|.$$

Therefore,

$$\begin{aligned}
&\|T((1-\beta)x+\beta Tx)-x^*\|^2\\
&\le (1-\beta)\|x-x^*\|^2+\beta(\|x-x^*\|^2+\|x-Tx\|^2)\\
&\quad-2\beta(1-\beta)\|x-Tx\|^2+(1-\beta)\|x-T((1-\beta)I+\beta T)x\|^2\\
&\quad+\beta^3L^2\|x-Tx\|^2\\
&=\|x-x^*\|^2+(1-\beta)\|x-T((1-\beta)I+\beta T)x\|^2\\
&\quad-\beta(1-2\beta-\beta^2L^2)\|x-Tx\|^2.
\end{aligned}\tag{2.7}$$

Since $\beta<\frac{1}{\sqrt{1+L^2}+1}$, we deduce

$$1-2\beta-\beta^2L^2>0.$$

From (2.7), we can deduce

$$\begin{aligned}
&\|T((1-\beta)x+\beta Tx)-x^*\|^2\\
&\le \|x-x^*\|^2+(1-\beta)\|x-T((1-\beta)x+\beta Tx)\|^2,
\end{aligned}\tag{2.8}$$

for all $x\in H$ and $x^*\in Fix(T)$.

By (2.2) and (2.8), we have

$$\begin{aligned}
&\|(1-\alpha)x+\alpha T((1-\beta)x+\beta Tx)-x^*\|^2\\
&=\|(1-\alpha)(x-x^*)+\alpha(T((1-\beta)x+\beta Tx)-x^*)\|^2\\
&=(1-\alpha)\|x-x^*\|^2+\alpha\|T((1-\beta)x+\beta Tx)-x^*\|^2\\
&\quad-\alpha(1-\alpha)\|\|T((1-\beta)x+\beta Tx)-x\|^2\\
&\le\alpha[\|x-x^*\|^2+(1-\beta)\|x-T((1-\beta)x+\beta Tx)\|^2]\\
&\quad+(1-\alpha)\|x-x^*\|^2-\alpha(1-\alpha)\|\|T((1-\beta)x+\beta Tx)-x\|^2\\
&=\|x-x^*\|^2+\alpha(\alpha-\beta)\|T((1-\beta)x+\beta Tx)-x\|^2.
\end{aligned}$$

This together with $\alpha<\beta$ implies that

$$\|(1-\alpha)x+\alpha T((1-\beta)x+\beta Tx)-x^*\|\le\|x-x^*\|.$$

□

Lemma 2.4. *Let H be a Hilbert space. Let $T:H\to H$ be an L-Lipschitzian mapping with $L>1$. If $I-T$ is demiclosed at 0, then $I-T((1-\beta)I+\beta T)$ is also demiclosed at 0 when $\beta\in(0,\frac{1}{L})$.*

Proof. Let the sequence $\{x_n\}\subset H$ satisfying $x_n\rightharpoonup\tilde{x}$ and $x_n-T((1-\beta)I+\beta T)x_n\to 0$. Next, we will show that $\tilde{x}\in Fix(T((1-\beta)I+\beta T))$.

From Lemma 2.2, we only need to prove that $\tilde{x}\in Fix(T)$. As a matter of fact, since T is L-Lipschizian, we have

$$\begin{aligned}
\|x_n-Tx_n\|&\le\|x_n-T((1-\beta)I+\beta T)x_n\|+\|T((1-\beta)I+\beta T)x_n-Tx_n\|\\
&\le\|x_n-T((1-\beta)I+\beta T)x_n\|+\beta L\|x_n-Tx_n\|.
\end{aligned}$$

It follows that

$$\|x_n-Tx_n\|\le\frac{1}{1-\beta L}\|x_n-T((1-\beta)I+\beta T)x_n\|.$$

Hence,

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

Applying the demiclosedness of T (Lemma 2.1), we immediately deduce $\tilde{x}\in Fix(T)$. □

Lemma 2.5. ([23]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\le(1-\gamma_n)a_n+\delta_n, n\in\mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty}\gamma_n=\infty$;
(2) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.
Then $\lim_{n\to\infty}a_n=0$.

Lemma 2.6. ([13]) *Let $\{w_n\}$ be a sequence of real numbers. Assume $\{w_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$*

such that $w_{n_k} \leq w_{n_k+1}$ *for all* $k \geq 0$. *For every* $n \geq N_0$, *define an integer sequence* $\{\tau(n)\}$ *as*

$$\tau(n) = \max\{i \leq n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$ *and for all* $n \geq N_0$

$$\max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}.$$

3. Main results

Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $f : H_1 \to H_1$ be a ρ-contraction. Let $B : H_1 \to H_1$ be a strong positive linear bounded operator with coefficient $\xi > 2\rho$. Let $S : H_2 \to H_2$ be a nonexpansive mapping and let $T : H_1 \to H_1$ be an L-Lipschitzian pseudocontractive mapping with $L > 1$.

We use Γ to denote the set of solutions of (1.5), that is,

$$\Gamma = \{x^* | x^* \in Fix(T), Ax^* \in Fix(S)\}.$$

In the sequel, we assume $\Gamma \neq \emptyset$.

Now, we present our algorithm for finding $x^* \in \Gamma$.

Algorithm 3.1. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by:

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n - \delta A^*(I - S)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T((1 - \gamma_n)u_n + \gamma_n T u_n), n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are three real number sequences in $(0, 1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Theorem 3.2. *Assume the following conditions are satisfied:*

$(C1)$: $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$: $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C3)$: $0 < a < \beta_n < c < \gamma_n < b < \frac{1}{\sqrt{1+L^2}+1}$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.1) *converges strongly to* $x^* = P_\Gamma(f + I - B)x^*$.

Proof. Let $x^* = P_\Gamma(f + I - B)x^*$. Then we have $x^* \in Fix(T)$ and $Ax^* \in Fix(S)$. Since S is nonexpansive, we get

$$\begin{aligned} \|SAx_n - Ax^*\|^2 &= \|SAx_n - SAx^*\|^2 \\ &\leq \|Ax_n - Ax^*\|^2. \end{aligned} \tag{3.2}$$

From (2.8), we deduce

$$\begin{aligned} \|T((1 - \gamma_n)u_n + \gamma_n T u_n) - x^*\|^2 \leq{} & \|u_n - x^*\|^2 \\ & + (1 - \gamma_n)\|u_n - T((1 - \gamma_n)u_n + \gamma_n T u_n)\|^2. \end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n)u_n+\beta_n T((1-\gamma_n)u_n+\gamma_n Tu_n)-x^*\|^2\\ &= (1-\beta_n)\|u_n-x^*\|^2+\beta_n\|T((1-\gamma_n)u_n+\gamma_n Tu_n)-x^*\|^2\\ &\quad -\beta_n(1-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_n Tu_n)\|^2\\ &\le \|u_n-x^*\|^2-\beta_n(\gamma_n-\beta_n)\|T((1-\gamma_n)u_n+\gamma_n Tu_n)-x^*\|^2\\ &\le \|u_n-x^*\|^2.\end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned}\|u_n-x^*\| &= \|\alpha_n(f(x_n)-Bx^*)+(I-\alpha_n B)(x_n-x^*+\delta A^*(SAx_n-Ax_n))\|\\ &\le \alpha_n\|f(x_n)-Bx^*\|+\|I-\alpha_n B\|\|x_n-x^*+\delta A^*(SAx_n-Ax_n)\|\\ &\le \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-Bx^*\|\\ &\quad +(1-\alpha_n\xi)\|x_n-x^*+\delta A^*(SAx_n-Ax_n)\|\\ &\le \alpha_n\rho\|x_n-x^*\|+\alpha_n\|f(x^*)-Bx^*\|\\ &\quad +(1-\alpha_n\xi)\|x_n-x^*+\delta A^*(SAx_n-Ax_n)\|.\end{aligned} \tag{3.4}$$

By (2.3), we have

$$\begin{aligned}\|x_n-x^*+\delta A^*(SAx_n-Ax_n)\|^2 &= \|x_n-x^*\|^2+\delta^2\|A^*(SAx_n-Ax_n)\|^2\\ &\quad +2\delta\langle x_n-x^*, A^*(SAx_n-Ax_n)\rangle.\end{aligned} \tag{3.5}$$

Since A is a linear operator with its adjoint A^*, we have

$$\begin{aligned}&\langle x_n-x^*, A^*(SAx_n-Ax_n)\rangle\\ &= \langle A(x_n-x^*), SAx_n-Ax_n\rangle\\ &= \langle Ax_n-Ax^*+SAx_n-Ax_n-(SAx_n-Ax_n), SAx_n-Ax_n\rangle\\ &= \langle SAx_n-Ax^*, SAx_n-Ax_n\rangle-\|SAx_n-Ax_n\|^2.\end{aligned} \tag{3.6}$$

Again using (2.3), we obtain

$$\begin{aligned}&\langle SAx_n-Ax^*, SAx_n-Ax_n\rangle\\ &= \frac{1}{2}(\|SAx_n-Ax^*\|^2+\|SAx_n-Ax_n\|^2-\|Ax_n-Ax^*\|^2).\end{aligned} \tag{3.7}$$

From (3.2), (3.6) and (3.7), we get

$$\begin{aligned}&\langle x_n-x^*, A^*(SAx_n-Ax_n)\rangle\\ &= \frac{1}{2}(\|SAx_n-Ax^*\|^2+\|SAx_n-Ax_n\|^2-\|Ax_n-Ax^*\|^2)\\ &\quad -\|SAx_n-Ax_n\|^2\\ &\le \frac{1}{2}(\|Ax_n-Ax^*\|^2+\|SAx_n-Ax_n\|^2-\|Ax_n-Ax^*\|^2)\\ &\quad -\|SAx_n-Ax_n\|^2\\ &= -\frac{1}{2}\|SAx_n-Ax_n\|^2.\end{aligned} \tag{3.8}$$

So,

$$
\begin{aligned}
\|x_n - x^* + \delta A^*(SAx_n - Ax_n)\|^2 &\le \|x_n - x^*\|^2 + \delta^2\|A\|^2\|SAx_n - Ax_n\|^2 \\
&\quad + 2\delta\Big(- \frac{1}{2}\|SAx_n - Ax_n\|^2\Big) \\
&= \|x_n - x^*\|^2 + (\delta^2\|A\|^2 - \delta)\|SAx_n - Ax_n\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned} \tag{3.9}
$$

It follows that

$$
\|x_n - x^* + \delta A^*(SAx_n - Ax_n)\| \le \|x_n - x^*\|. \tag{3.10}
$$

Substituting (3.10) into (3.4) to deduce

$$
\begin{aligned}
\|u_n - x^*\| &\le \alpha_n\rho\|x_n - x^*\| + \alpha_n\|f(x^*) - Bx^*\| + (1 - \alpha_n\xi)\|x_n - x^*\| \\
&= \alpha_n\|f(x^*) - Bx^*\| + [1 - (\xi - \rho)\alpha_n]\|x_n - x^*\|.
\end{aligned} \tag{3.11}
$$

From (3.3) and (3.11), we get

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\le \|u_n - x^*\| \\
&\le \alpha_n\|f(x^*) - Bx^*\| + [1 - (\xi - \rho)\alpha_n]\|x_n - x^*\| \\
&\le \max\Big\{\|x_n - x^*\|, \frac{\|f(x^*) - Bx^*\|}{\xi - \rho}\Big\}.
\end{aligned}
$$

The boundedness of the sequence $\{x_n\}$ yields.

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m > 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \ge m$. In this case, we know that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Returning to (3.4), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 \\
&\le [\alpha_n\rho\|x_n - x^*\| + \alpha_n\|f(x^*) - Bx^*\| \\
&\quad + (1 - \alpha_n\xi)\|x_n - x^* + \delta A^*(SAx_n - Ax_n)\|]^2 \\
&= \alpha_n^2(\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|)^2 \\
&\quad + 2\alpha_n(1 - \alpha_n\xi)(\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|) \\
&\quad \times \|x_n - x^* + \delta A^*(SAx_n - Ax_n)\| \\
&\quad + (1 - \alpha_n\xi)^2\|x_n - x^* + \delta A^*(SAx_n - Ax_n)\|^2 \\
&\le \alpha_n(\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|)(3\|x_n - x^*\| + \|f(x^*) - Bx^*\|) \\
&\quad + (1 - \alpha_n\xi)\|x_n - x^* + \delta A^*(SAx_n - Ax_n)\|^2 \\
&\le M\alpha_n + (1 - \alpha_n\xi)\|x_n - x^*\|^2 \\
&\quad + (1 - \alpha_n\xi)(\delta^2\|A\|^2 - \delta)\|SAx_n - Ax_n\|^2 \\
&\le M\alpha_n + \|x_n - x^*\|^2,
\end{aligned} \tag{3.12}
$$

where $M > 0$ is a constant such that

$$
\sup_n\{(\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|)(3\|x_n - x^*\| + \|f(x^*) - Bx^*\|)\} \le M.
$$

Hence,

$$(1-\alpha_n\xi)(\delta-\delta^2\|A\|^2)\|SAx_n-Ax_n\|^2 \leq (1-\alpha_n\xi)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 +M\alpha_n.$$

Since $\lim_{n\to\infty}\|x_n-x^*\|$ exists and $\alpha_n\to 0$, we obtain

$$\lim_{n\to\infty}\|SAx_n-Ax_n\|=0. \tag{3.13}$$

Note that

$$\begin{aligned}\|u_n-x_n\| &= \|\delta A^*(S-I)Ax_n+\alpha_n(Bx_n-\delta BA^*(I-S)Ax_n-f(x_n))\| \\ &\leq \delta\|A\|\|SAx_n-Ax_n\|+\alpha_n\|Bx_n-\delta BA^*(I-S)Ax_n-f(x_n)\|.\end{aligned}$$

It follows from (3.13) that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.14}$$

From (3.3) and (3.12), we deduce

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \|u_n-x^*\|^2-\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2 \\ &\leq \|x_n-x^*\|^2+\alpha_nM-\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2.\end{aligned}$$

It follows that

$$\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\leq\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM.$$

Therefore,

$$\lim_{n\to\infty}\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|=0. \tag{3.15}$$

Observe that

$$\begin{aligned}\|u_n-Tu_n\| &\leq \|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+\|T((1-\gamma_n)u_n+\gamma_nTu_n)-Tu_n\| \\ &\leq \|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+L\gamma_n\|u_n-Tu_n\|.\end{aligned}$$

Thus,

$$\|u_n-Tu_n\|\leq\frac{1}{1-L\gamma_n}\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|.$$

This together with (3.15) implies that

$$\lim_{n\to\infty}\|u_n-Tu_n\|=0. \tag{3.16}$$

Now, we show that

$$\limsup_{n\to\infty}\langle f(x^*)-Bx^*,u_n-x^*\rangle\leq 0.$$

Choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*)-Bx^*,u_n-x^*\rangle=\lim_{i\to\infty}\langle f(x^*)-Bx^*,u_{n_i}-x^*\rangle \tag{3.17}$$

Since the sequence $\{u_{n_i}\}$ is bounded, we can choose a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}}\rightharpoonup z$. For the sake of convenience, we assume (without loss of generality) that $u_{n_i}\rightharpoonup z$. Consequently, we derive from the above conclusions that

$$x_{n_i}\rightharpoonup z \text{ and } Ax_{n_i}\rightharpoonup Az. \tag{3.18}$$

Applying Lemmas 2.1 and 2.4, we deduce

$Az\in Fix(S)$ (by (3.13) and (3.18)) and $z\in Fix(T)$ (by (3.16) and (3.18)).

That is to say, $z \in \Gamma$.

Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - Bx^*, u_n - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - Bx^*, u_{n_i} - x^*\rangle \\ &= \lim_{i\to\infty}\langle f(x^*) - Bx^*, z - x^*\rangle \\ &\le 0. \end{aligned} \tag{3.19}$$

Using (2.4), we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|(I - \alpha_n B)(x_n - \delta A^*(I - S)Ax_n - x^*) + \alpha_n(f(x_n) - Bx^*)\|^2 \\ &\le (1 - \alpha_n\xi)\|x_n - \delta A^*(I - S)Ax_n - x^*\|^2 \\ &\quad + 2\alpha_n\langle f(x_n) - Bx^*, u_n - x^*\rangle \\ &\le (1 - \alpha_n\xi)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - Bx^*, u_n - x^*\rangle \\ &= (1 - \alpha_n\xi)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), u_n - x^*\rangle \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &= (1 - \alpha_n\xi)\|x_n - x^*\|^2 + 2\alpha_n\rho\|x_n - x^*\|\|u_n - x^*\| \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\le (1 - \alpha_n\xi)\|x_n - x^*\|^2 + \alpha_n\rho\|x_n - x^*\|^2 + \alpha_n\rho\|u_n - x^*\|^2 \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle. \end{aligned} \tag{3.20}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 \\ &\le \left[1 - \frac{(\xi - 2\rho)\alpha_n}{1 - \alpha_n\rho}\right]\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n\rho}\langle f(x^*) - Bx^*, u_n - x^*\rangle. \end{aligned} \tag{3.21}$$

Applying Lemma 2.5 and (3.19) to (3.21), we deduce $x_n \to x^*$.

Case 2. Assume there exists an integer n_0 such that

$$\|x_{n_0} - x^*\| \le \|x_{n_0+1} - x^*\|.$$

Set $\omega_n = \{\|x_n - x^*\|\}$. Then, we have

$$\omega_{n_0} \le \omega_{n_0+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} | n_0 \le l \le n, \omega_l \le \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty}\tau(n) = \infty$$

and

$$\omega_{\tau(n)} \le \omega_{\tau(n)+1},$$

for all $n \ge n_0$.

By the similar argument as that of Case 1, we can obtain

$$\lim_{n\to\infty} \|SAx_{\tau(n)} - Ax_{\tau(n)}\| = 0,$$

and

$$\lim_{n\to\infty} \|u_{\tau(n)} - Tu_{\tau(n)}\| = 0.$$

This implies that

$$\omega_w(u_{\tau(n)}) \subset \Gamma.$$

Thus, we obtain

$$\limsup_{n\to\infty} \langle f(x^*) - Bx^*, u_{\tau(n)} - x^*\rangle \leq 0. \tag{3.22}$$

Since $\omega_{\tau(n)} \leq \omega_{\tau(n)+1}$, we have from (3.21) that

$$\begin{aligned} \omega_{\tau(n)}^2 \leq \omega_{\tau(n)+1}^2 \leq & \Big[1 - \frac{(\xi - 2\rho)\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}\Big]\omega_{\tau(n)}^2 \\ & + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}\langle f(x^*) - Bx^*, u_{\tau(n)} - x^*\rangle. \end{aligned} \tag{3.23}$$

It follows that

$$\omega_{\tau(n)}^2 \leq \frac{2}{\xi - 2\rho}\langle f(x^*) - Bx^*, u_{\tau(n)} - x^*\rangle. \tag{3.24}$$

Combining (3.22) and (3.24), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)} \leq 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.25}$$

By(3.23), we obtain

$$\limsup_{n\to\infty} \omega_{\tau(n)+1}^2 \leq \limsup_{n\to\infty} \omega_{\tau(n)}^2.$$

This together with (3.25) imply that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 2.6 to get

$$0 \leq \omega_n \leq \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to x^*$. This completes the proof. □

Algorithm 3.3. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by:

$$\begin{cases} u_n = \alpha_n f(x_n) + (1 - \alpha_n)(x_n - \delta A^*(I - S)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T((1 - \gamma_n)u_n + \gamma_n Tu_n), n \in \mathbb{N}, \end{cases} \tag{3.26}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.4. *Assume the following conditions are satisfied:*
$(C1)$: $\lim_{n\to\infty}\alpha_n = 0$;
$(C2)$: $\sum_{n=1}^{\infty}\alpha_n = \infty$;
$(C3)$: $0 < a < \beta_n < c < \gamma_n < b < \frac{1}{\sqrt{1+L^2}+1}$.
Then the sequence $\{x_n\}$ *generated by algorithm* (3.26) *converges strongly to* $x^* = P_\Gamma f(x^*)$.

Algorithm 3.5. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by:

$$\begin{cases} u_n = (1-\alpha_n)(x_n - \delta A^*(I-S)Ax_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n T u_n), n \in \mathbb{N}, \end{cases} \tag{3.27}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.6. *Assume the following conditions are satisfied:*
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$(C2)$: $\sum_{n=1}^{\infty}\alpha_n = \infty$;
$(C3)$: $0 < a < \beta_n < c < \gamma_n < b < \frac{1}{\sqrt{1+L^2}+1}$.
Then the sequence $\{x_n\}$ *generated by algorithm* (3.27) *converges strongly to* $x^* = P_\Gamma(0)$ *which is the minimum norm element in* Γ.

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