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HALPERN-TYPE STRONG CONVERGENCE THEOREM FOR WIDELY MORE GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES AND APPLICATIONS

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Dedicated to Prof. Wataru Takahashi on the occasion of his 70th birthday

Abstract. In this paper, using the concept of attractive points of a nonlinear mapping, we obtain a strong convergence theorem of Halpern's type [6] for a wide class of nonlinear mappings which contains nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. Using this result, we obtain new strong convergence theorems of Halpern's type in a Hilbert space. In particular, we obtain an extension of Suzuki's theorem [16] and also solve a problem posed by Kurokawa and Takahashi [13].

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. A mapping $T : C \to H$ is called generalized hybrid [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. We call such a mapping (α, β)-generalized hybrid. The class of generalized hybrid mappings covers many well-known mappings. For example, a (1, 0)-generalized hybrid mapping T is nonexpansive [4], i.e.,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

It is nonspreading [11, 12] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is hybrid [20] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

We know that a nonspreading mapping and a hybrid mapping are not continuous in general. In fact, we can give the following example [8] of nonspreading mappings in a Hilbert space. Let H be a real Hilbert space. Set $E = \{x \in H : \|x\| \le 1\}$,

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$D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $S : C \to C$ as follows:

$$Sx = \begin{cases} 0, & x \in D, \\ P_E x, & x \notin D. \end{cases}$$

Then S is a nonspreading mapping which is not continuous. Motivated by generalized hybrid mappings, Kawasaki and Takahashi [9] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [10] if $\alpha+\beta = -\gamma-\delta = 1$ and $\varepsilon = \zeta = \eta = 0$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. For a mapping $T : C \to H$, we denote by $F(T)$ the set of fixed points of T. In 1992, Wittmann [24] proved the following strong convergence theorem of Halpern's type [6] in a Hilbert space; see also [18] and [22].

Theorem 1.1. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0, 1]$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Kurokawa and Takahashi [13] also proved the following strong convergence theorem for nonspreading mappings in a Hilbert space; see also Hojo and Takahashi [7] for generalized hybrid mappings.

Theorem 1.2. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let T be a nonspreading mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)}u$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Recently, Takahashi and Takeuchi [21] introduced the concept of *attractive points* of a mapping in a Hilbert space. Let C be a nonempty subset of H. For a mapping T of C into H, we denote by $A(T)$ the set of all *attractive points* of T, i.e.,

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|,\ \forall x \in C\}.$$

They proved a mean convergence theorem of Baillon's type [3] without convexity for generalized hybrid mappings. Guu and Takahashi [5] proved a weak convergence theorem of Mann's type [15] and a strong convergence theorem of Kurokawa and Takahashi's type [13] for widely more generalized hybrid mappings in a Hilbert space which generalizes Hojo and Takahashi's result [7] for generalized hybrid mappings. Akashi and Takahash [1] also proved a strong convergence theorem of Halpern's type for nonexpansive mappings on star-shaped sets in a Hilbert space. However, they used essentially the properties of nonexpansiveness in the proof.

In this paper, motivated by [1, 7, 13, 24], we obtain a strong convergence theorem of Halpern's type for finding attractive points of widely more generalized hybrid mappings in a Hilbert space. Using this result, we obtain new strong convergence theorems in a Hilbert space. In particular, we obtain an extension of Suzuki's theorem [16] and also solve a problem posed by Kurokawa and Takahashi [13].

2. Preliminaries and lemmas

Let H be a real Hilbert space. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We know that for $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we know that for all $x, y, z, w \in H$,

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2. \tag{2.3}$$

Let D be a closed and convex subset of H. For every $x \in H$, there exists a unique nearest point in D denoted by $P_D x$, that is, $\|x - P_D x\| \leq \|x - y\|$ for every $y \in D$. This mapping P_D is called the *metric projection* of H onto D. It is known that P_D is firmly nonexpansive, that is, the following hold:

$$0 \leq \langle x - P_D x, P_D x - y\rangle \quad \text{and} \quad \|x - P_D x\|^2 + \|P_D x - y\|^2 \leq \|x - y\|^2$$

for any $x \in H$ and $y \in D$; see [17–19]. Takahashi and Takeuchi [21] proved the following useful lemma.

Lemma 2.1 ([21])**.** *Let C be a nonempty subset of H and let T be a mapping from C into H. Then, $A(T)$ is a closed and convex subset of H.*

We note that a mapping $T : C \to H$ in Lemma 2.1 may be not necessarily nonexpansive. The following theorem was proved by Guu and Takahashi [5].

Theorem 2.2 ([5])**.** *Let H be a Hilbert space, let C be a nonempty subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *or* (2)*:*

(1) *$\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \geq 0$ and $\zeta + \eta \geq 0$;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$, $\zeta+\eta \geq 0$ *and* $\varepsilon+\eta \geq 0$.

Then $A(T) \neq \emptyset$ *if and only if there exists* $z \in C$ *such that* $\{T^n z : n \in \mathbb{N}\}$ *is bounded.*

To prove our main result, we need two lemmas.

Lemma 2.3 ([14])**.** *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $i \in \mathbb{N}$*. Define the sequence* $\{\tau(n)\}_{n \geq n_0}$ *of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ *such that* $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$*. Then, the following hold:*

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \dots$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Lemma 2.4 ([2]; see also [23])**.** *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers, let* $\{\alpha_n\}$ *be a sequence of* $[0,1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$*, let* $\{\beta_n\}$ *be a sequence of nonnegative real numbers with* $\sum_{n=1}^{\infty} \beta_n < \infty$*, and let* $\{\gamma_n\}$ *be a sequence of real numbers with* $\limsup_{n\to\infty} \gamma_n \leq 0$*. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n \in \mathbb{N}$*. Then* $\lim_{n\to\infty} s_n = 0$.

3. Strong convergence theorem of Halpern's type

In this section, using the technique of [22], we prove a strong convergence theorem of Halpern's type [6] for finding attractive points of widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following lemma which was proved by Guu and Takahashi [5].

Lemma 3.1 ([5])**.** *Let* H *be a Hilbert space and let* C *be a nonempty subset of* H*. Let* $T : C \to H$ *be an* $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$*-widely more generalized hybrid mapping. Suppose that it satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma>0$ *and* $\varepsilon+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$ *and* $\zeta+\eta \geq 0$.

If $x_n \rightharpoonup z$ *and* $x_n - Tx_n \to 0$*, then* $z \in A(T)$.

Theorem 3.2. *Let* H *be a Hilbert space and let* C *be a convex subset of* H*. Let* T *be an* $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$*-widely more generalized hybrid mapping from* C *into itself with* $A(T) \neq \emptyset$ *and let* $P_{A(T)}$ *be the metric projection of* H *onto* $A(T)$*. Suppose that it satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma>0$ *and* $\varepsilon+\eta \geq 0$;

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$ *and* $\zeta+\eta \geq 0$.

Let $\{z_n\}$ *be a sequence in* C *such that* $z_n \to z \in H$ *and let* $\{x_n\}$ *be a sequence in* C *defined by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0,1)$ *such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in A(T)$, *where* $\bar{x} = P_{A(T)}z$.

Proof. We first remark from Lemma 2.1 that $A(T)$ is closed and convex. Thus $P_{A(T)}$ is well defined. Let $x_1 \in C$ and $u \in A(T)$. We have that $\|Tx_n - u\| \leq \|x_n - u\|$ from $u \in A(T)$. Define $y_n = \beta_n x_n + (1-\beta_n)Tx_n$. Then we have from (2.2) that

$$\begin{aligned} \|y_n - u\|^2 &= \|\beta_n x_n + (1-\beta_n)Tx_n - u\|^2 \\ &= \beta_n\|x_n - u\|^2 + (1-\beta_n)\|Tx_n - u\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\leq \beta_n\|x_n - u\|^2 + (1-\beta_n)\|x_n - u\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - u\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\leq \|x_n - u\|^2. \end{aligned} \tag{3.1}$$

Let $K = \sup\{\|z_n - u\| : n \in \mathbb{N}\}$ and put $M = \max\{\|x_1 - u\|, K\}$. It is obvious that $\|x_1 - u\| \leq M$. Suppose that $\|x_k - u\| \leq M$ for some $k \in \mathbb{N}$. Then we have from (3.1) that

$$\begin{aligned} \|x_{k+1} - u\| &= \|\alpha_k z_k + (1-\alpha_k)y_k - u\| \\ &\leq \alpha_k\|z_k - u\| + (1-\alpha_k)\|y_k - u\| \\ &\leq \alpha_k\|z_k - u\| + (1-\alpha_k)\|x_k - u\| \\ &\leq \alpha_k M + (1-\alpha_k)M \\ &= M. \end{aligned}$$

By mathematical induction, we have that $\|x_n - u\| \leq M$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. Hence $\{Tx_n\}$ is bounded. Take $\bar{x} = P_{A(T)}z$. We have from (3.1) that

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &\leq \alpha_n\|z_n - \bar{x}\|^2 + (1-\alpha_n)\|y_n - \bar{x}\|^2 \\ &\leq \alpha_n\|z_n - \bar{x}\|^2 + (1-\alpha_n)(\|x_n - \bar{x}\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2) \\ &\leq \alpha_n\|z_n - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2. \end{aligned} \tag{3.2}$$

We have from (3.2) that

$$\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \leq \alpha_n\|z_n - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|x_{n+1} - \bar{x}\|^2. \tag{3.3}$$

We also have that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n z_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n - x_n\| \\ &\leq \alpha_n\|z_n - x_n\| + (1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|. \end{aligned} \tag{3.4}$$

Case A: Put $\Gamma_n = \|x_n - \bar{x}\|^2$ for all $n \in \mathbb{N}$. Suppose that there exists a positive integer N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. From $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ and (3.3), we have that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.5}$$

We have from $\lim_{n\to\infty}\alpha_n = 0$, (3.4) and (3.5) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle z-\bar{x}, x_n-\bar{x}\rangle = \lim_{i\to\infty}\langle z-\bar{x}, x_{n_i}-\bar{x}\rangle. \tag{3.7}$$

Without loss of generality, we may assume that $x_{n_i} \rightharpoonup v$. By (3.5) and Lemma 3.1, we have that $v \in A(T)$. We have from (3.7) that

$$\limsup_{n\to\infty}\langle z-\bar{x}, x_n-\bar{x}\rangle = \langle z-\bar{x}, v-\bar{x}\rangle \le 0. \tag{3.8}$$

On the other hand, since $x_{n+1}-\bar{x} = \alpha_n(z_n-\bar{x}) + (1-\alpha_n)(y_n-\bar{x})$, we have from (2.1) and (3.1) that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\le (1-\alpha_n)\|y_n-\bar{x}\|^2 + 2\alpha_n\langle z_n-\bar{x}, x_{n+1}-\bar{x}\rangle\\ &\le (1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle z_n-\bar{x}, x_{n+1}-\bar{x}\rangle\\ &= (1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle z_n-\bar{x}, x_{n+1}-x_n\rangle\\ &\quad + 2\alpha_n\langle z_n-\bar{x}, x_n-\bar{x}\rangle\\ &= (1-\alpha_n)\|x_n-\bar{x}\|^2 + 2\alpha_n\langle z_n-\bar{x}, x_{n+1}-x_n\rangle\\ &\quad + 2\alpha_n\langle z_n-z, x_n-\bar{x}\rangle + 2\alpha_n\langle z-\bar{x}, x_n-\bar{x}\rangle.\end{aligned} \tag{3.9}$$

By $\sum_{n=1}^{\infty}\alpha_n = \infty$, (3.6), $z_n \to z$, (3.8), (3.9) and Lemma 2.4, we have that $\lim_{n\to\infty} x_n = \bar{x}$.

Case B: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then it follows from Lemma 2.3 that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$. We have from (3.3) that

$$\begin{aligned}&\beta_{\tau(n)}(1-\beta_{\tau(n)})\|x_{\tau(n)}-Tx_{\tau(n)}\|^2\\ &\qquad \le \alpha_{\tau(n)}\|z_{\tau(n)}-\bar{x}\|^2 + \|x_{\tau(n)}-\bar{x}\|^2 - \|x_{\tau(n)+1}-\bar{x}\|^2\\ &\qquad \le \alpha_{\tau(n)}\|z_{\tau(n)}-\bar{x}\|^2.\end{aligned} \tag{3.10}$$

By $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ and (3.10), we have that

$$\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\| = 0. \tag{3.11}$$

We have from (3.9) that

$$\|x_{\tau(n)+1}-\bar{x}\|^2 \le (1-\alpha_{\tau(n)})\|x_{\tau(n)}-\bar{x}\|^2 + 2\alpha_{\tau(n)}\langle z_{\tau(n)}-\bar{x}, x_{\tau(n)+1}-\bar{x}\rangle.$$

From this inequality, we have that

$$\begin{aligned}\|x_{\tau(n)+1}-\bar{x}\|^2 - \|x_{\tau(n)}-\bar{x}\|^2 + \alpha_{\tau(n)})\|x_{\tau(n)}-\bar{x}\|^2&\\ \le 2\alpha_{\tau(n)}\langle z_{\tau(n)}-\bar{x}, x_{\tau(n)+1}-\bar{x}\rangle&.\end{aligned} \tag{3.12}$$

From $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$, (3.12) and $\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}\|x_{\tau(n)}-\bar{x}\|^2 &\le 2\langle z_{\tau(n)}-\bar{x}, x_{\tau(n)+1}-\bar{x}\rangle\\ &= 2\langle z_{\tau(n)}-\bar{x}, x_{\tau(n)+1}-x_{\tau(n)}\rangle + 2\langle z_{\tau(n)}-\bar{x}, x_{\tau(n)}-\bar{x}\rangle\\ &= 2\langle z_{\tau(n)}-\bar{x}, x_{\tau(n)+1}-x_{\tau(n)}\rangle + 2\langle z_{\tau(n)}-z, x_{\tau(n)}-\bar{x}\rangle\\ &\quad + 2\langle z-\bar{x}, x_{\tau(n)}-\bar{x}\rangle.\end{aligned} \tag{3.13}$$

On the other hand, by $\lim_{n\to\infty} \alpha_n = 0$, (3.4) and (3.11), we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{3.14}$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence $\{x_{\tau(n_i)}\}$ such that

$$\limsup_{n\to\infty} \langle z - \bar{x}, x_{\tau(n)} - \bar{x}\rangle = \lim_{i\to\infty} \langle z - \bar{x}, x_{\tau(n_i)} - \bar{x}\rangle. \tag{3.15}$$

Following the same argument as the proof of Case A for $\{x_{\tau(n_i)}\}$, we have that

$$\limsup_{n\to\infty} \langle z - \bar{x}, x_{\tau(n)} - \bar{x}\rangle \leq 0. \tag{3.16}$$

Using (3.13), (3.14), $z_{\tau(n)} \to z$ and (3.16), we have that

$$\lim_{n\to\infty} \|x_{\tau(n)} - \bar{x}\| = 0. \tag{3.17}$$

By (3.14) we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - \bar{x}\| = 0. \tag{3.18}$$

Using Lemma 2.3 for (3.18) again, we have that

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = 0.$$

This completes the proof. □

4. Applications

In this section, using Theorem 3.2, we establish new strong convergence theorems of Halpern's type in a Hilbert space. We first prove a strong convergence theorem for finding fixed points of widely more generalized hybrid mappings which generalizes Suzuki's theorem [16].

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$. Suppose that it satisfies the following conditions (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$.

Let $\{z_n\}$ be a sequence in C such that $z_n \to z$ and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z_n + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)T x_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0.$$

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in F(T)$, where $\bar{x} = P_{F(T)} z$.

Proof. Since T is $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalised hybrid, we have that

$$\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\ +\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2\le 0$$

for all $x,y\in C$. Replacing x by a fixed point u of T, we have that for any $y\in C$,

$$\alpha\|u-Ty\|^2+\beta\|u-Ty\|^2+\gamma\|u-y\|^2+\delta\|u-y\|^2 \\ +\zeta\|y-Ty\|^2+\eta\|y-Ty\|^2\le 0$$

and hence

$$(\alpha+\beta)\|u-Ty\|^2+(\gamma+\delta)\|u-y\|^2+(\zeta+\eta)\|y-Ty\|^2\le 0.$$

Suppose that it satisfies the following condition (2):

$$\alpha+\beta+\gamma+\delta\ge 0,\ \alpha+\beta>0 \quad \textit{and} \quad \zeta+\eta\ge 0.$$

From $\zeta+\eta\ge 0$, we have that $(\alpha+\beta)\|u-Ty\|^2+(\gamma+\delta)\|u-y\|^2\le 0$. Furthermore, since $\alpha+\beta+\gamma+\delta\ge 0$ and $\alpha+\beta>0$, we have that for any $y\in C$,

$$\|u-Ty\|^2\le\frac{-(\gamma+\delta)}{\alpha+\beta}\|u-y\|^2\le\|u-y\|^2.$$

This implies that $F(T)\subset A(T)$. Since $F(T)\neq\emptyset$, we have that $A(T)$ is nonempty. From Theorem 3.2, it follows that $\{x_n\}$ converges strongly to $\bar{x}\in A(T)$. Since C is closed and $x_n\to\bar{x}$, we have $\bar{x}\in C$. From $\bar{x}\in A(T)\cap C$, we have that

$$\|T\bar{x}-\bar{x}\|\le\|\bar{x}-\bar{x}\|=0$$

and hence $\bar{x}\in F(T)$. Furthermore, we have from Theorem 3.2 that $\bar{x}=P_{A(T)}z$. Thus we have that

$$\|z-\bar{x}\|=\min\{\|z-u\|:u\in A(T)\}\le\min\{\|z-u\|:u\in F(T)\}$$

and hence $\bar{x}=P_{F(T)}z$.

Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta\ge 0$, $\alpha+\gamma>0$ and $\varepsilon+\eta\ge 0$. This completes the proof. □

As direct consequences of Theorems 3.2 and 4.1, we have the following results.

Theorem 4.2. *Let H be a Hilbert space and let C be a convex subset of H. Let T be a nonexpansive mapping from C into itself, i.e.,*

$$\|Tx-Ty\|\le\|x-y\|,\quad \forall x,y\in C.$$

Assume $A(T)\neq\emptyset$ and let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $\{z_n\}$ be a sequence in C such that $z_n\to z$ and let $\{x_n\}$ be a sequence in C defined by $x_1\in C$ and

$$x_{n+1}=\alpha_n z_n+(1-\alpha_n)(\beta_n x_n+(1-\beta_n)Tx_n),\quad \forall n\in\mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that

$$\lim_{n\to\infty}\alpha_n=0,\quad \sum_{n=1}^{\infty}\alpha_n=\infty,\quad \textit{and}\quad \liminf_{n\to\infty}\beta_n(1-\beta_n)>0.$$

Then $\{x_n\}$ converges strongly to $\bar{x}=P_{A(T)}z$. Additionally, if C is closed and convex, then $\{x_n\}$ converges strongly to $\bar{x}=P_{F(T)}z$.

Proof. Since a (1, 0, 0, -1, 0, 0, 0)-widely more generalized hybrid mapping T is nonexpansive and it satisfies $\alpha+\beta+\gamma+\delta = 1-1 \geq 0$, $\alpha+\gamma = 1 > 0$ and $\varepsilon+\eta \geq 0$, we have the desired result from Theorems 3.2 and 4.1. □

Theorem 4.3. *Let H be a Hilbert space and let C be a convex subset of H. Let T be a nonspreading mapping from C into itself, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Assume $A(T) \neq \emptyset$ and let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $\{z_n\}$ be a sequence in C such that $z_n \to z$ and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{A(T)}z$. Additionally, if C is closed and convex, then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(T)}z$.

Proof. Since a (2, -1, -1, 0, 0, 0, 0)-widely more generalized hybrid mapping T is nonspreading and it satisfies $\alpha+\beta+\gamma+\delta = 2-1-1 \geq 0$, $\alpha+\gamma = 2-1 > 0$ and $\varepsilon+\eta \geq 0$, we have the desired result from Theorems 3.2 and 4.1. □

Theorem 4.4. *Let H be a Hilbert space and let C be a convex subset of H. Let T be a hybrid mapping from C into itself, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Assume $A(T) \neq \emptyset$ and let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $\{z_n\}$ be a sequence in C such that $z_n \to z$ and let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{A(T)}z$. Additionally, if C is closed and convex, then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(T)}z$.

Proof. Since a $(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0)$-widely more generalized hybrid mapping T is hybrid and it satisfies $\alpha+\beta+\gamma+\delta = \frac{3}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2} \geq 0$, $\alpha+\gamma = \frac{3}{2}-\frac{1}{2} > 0$ and $\varepsilon+\eta \geq 0$, we have the desired result from Theorems 3.2 and 4.1. □

If C is closed and convex and $F(T)$ is nonempty in Theorem 4.2, then the result is Suzuki's theorem in the setting of Hilbert space. Theorem 4.3 solves a problem posed by Kurokawa and Takahashi [13].

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