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WEIGHTS SHARING THE SAME EIGENVALUE

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Dedicated to Professor Wataru Takahashi, with esteem and friendship, on his seventieth birthday

ABSTRACT. Here is the simplest particular case of our main result: let $f : \mathbf{R} \to \mathbf{R}$ be a function of class C^1, with $\sup_{\mathbf{R}} f' > 0$, such that

$$\lim_{|\xi|\to+\infty} \frac{f(\xi)}{\xi} = 0 \ .$$

Then, for each $\lambda > \frac{\pi^2}{\sup_{\mathbf{R}} f'}$, the set of all $u \in H^1_0(]0,1[)$ for which the problem

$$\begin{cases} -v'' = \lambda f'(u(x))v & \text{in }]0,1[\\ \\ v(0) = v(1) = 0 \end{cases}$$

has a non-zero solution is closed and not σ-compact in $H^1_0(]0,1[)$.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain. We consider the Sobolev space $H^1_0(\Omega)$ endowed with the scalar product

$$\langle u, v\rangle = \int_\Omega \nabla u(x)\nabla v(x)dx$$

and the induced norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} \ .$$

We are interested in pairs (λ, β), where λ is a positive number and β is a measurable function, such that the linear problem

$$\begin{cases} -\Delta v = \lambda\beta(x)v & \text{in } \Omega \\ \\ v_{|\partial\Omega} = 0 \end{cases}$$

has a non-zero weak solution, that is to say a $v \in H^1_0(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla v(x)\nabla w(x)dx = \lambda \int_\Omega \beta(x)v(x)w(x)dx$$

for all $w \in H^1_0(\Omega)$.

If this happens, we say that λ is an eigenvalue related to the weight β.

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While, the structure of the set of all eigenvalues related to a fixed weight is well understood, it seems that much less is known about the structure of the set of all weights β for which a fixed positive number λ turns out to be an eigenvalue related to β.

In this very short note, we intend to give a contribution along the latter direction.

More precisely, we identify a quite general class of continuous functions $g : \mathbf{R} \to \mathbf{R}$ such that, for each λ in a suitable interval, the set of all $u \in H_0^1(\Omega)$ for which λ is an eigenvalue related to the weight $g(u(\cdot))$ is closed and not σ-compact in $H_0^1(\Omega)$.

2. Results

Let us recall that a set in a topological space is said to be σ-compact if it is the union of an at most countable family of compact sets.

For each $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$, we denote by λ_α the first eigenvalue of the problem

$$\begin{cases} -\Delta v = \lambda\alpha(x)v & \text{in } \Omega \\ \\ v_{|\partial\Omega} = 0 \ . \end{cases}$$

Let us recall that

$$\lambda_\alpha = \min_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\|v\|^2}{\int_\Omega \alpha(x)|v(x)|^2 dx} \ .$$

With the conventions $\frac{1}{+\infty} = 0$, $\frac{1}{0} = +\infty$, here is the statement of the result introduced above:

Theorem 2.1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function of class C^1 such that*

$$\max\left\{0, 2\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{|\xi|\to+\infty} \frac{f(\xi)}{\xi}\right\} < \sup_{\mathbf{R}} f' \ .$$

Moreover, if $n \geq 2$, assume that

$$\sup_{\xi \in \mathbf{R}} \frac{|f'(\xi)|}{1+|\xi|^q} < +\infty$$

for some $q > 0$, with $q < \frac{4}{n-2}$ if $n \geq 3$.

Then, for each $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$, and for every λ satisfying

$$\frac{\lambda_\alpha}{\sup_{\mathbf{R}} f'} < \lambda < \frac{\lambda_\alpha}{\max\left\{0, 2\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{|\xi|\to+\infty} \frac{f(\xi)}{\xi}\right\}}$$

the set of all $u \in H_0^1(\Omega)$ for which the problem

$$\begin{cases} -\Delta v = \lambda\alpha(x)f'(u(x))v & \text{in } \Omega \\ \\ v_{|\partial\Omega} = 0 \end{cases}$$

has a non-zero weak solution is closed and not σ-compact in $H_0^1(\Omega)$.

Remark 2.2. It is worth noticing that the linear hull of any closed and not σ-compact set in $H_0^1(\Omega)$ is infinite-dimensional. This comes from the fact that any closed set in a finite-dimensional normed space is σ-compact.

The key tool we use to prove Theorem 2.1 is Theorem 2.4 below whose proof, in turn, is entirely based on the following particular case of a result recently established in [1]:

Theorem 2.3 ([1, Theorem 10])**.** *Let $(X, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional real Hilbert space and let $I : X \to \mathbf{R}$ be a sequentially weakly lower semicontinuous, not convex functional of class C^2 such that I' is closed and* $\lim_{\|x\|\to+\infty}(I(x) + \langle z, x\rangle) = +\infty$ *for all $z \in X$.*

Then, the set

$$\{x \in X : I''(x) \text{ is not invertible}\}$$

is closed and not σ-compact.

Theorem 2.4. *Let $(X, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional real Hilbert space, and let $J : X \to \mathbf{R}$ be a functional of class C^2, with compact derivative. For each $\lambda \in \mathbf{R}$, put*

$$A_\lambda = \{x \in X : y = \lambda J''(x)(y) \ \ for \ \ some \ \ y \in X \setminus \{0\}\} \ .$$

Assume that

$$\max\left\{0, 2\limsup_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2}, \limsup_{\|x\|\to+\infty} \frac{\langle J'(x), x\rangle}{\|x\|^2}\right\} < \sup_{(x,y)\in X\times(X\setminus\{0\})} \frac{\langle J''(x)(y), y\rangle}{\|y\|^2} \ .$$

Then, for every λ satisfying

$$\begin{aligned} & \inf_{\{(x,y)\in X\times X : \langle J''(x)(y),y\rangle>0\}} \frac{\|y\|^2}{\langle J''(x)(y), y\rangle} \\ < \ & \lambda \\ < \ & \frac{1}{\max\left\{0, 2\limsup_{\|x\|\to+\infty}\frac{J(x)}{\|x\|^2}, \limsup_{\|x\|\to+\infty}\frac{\langle J'(x),x\rangle}{\|x\|^2}\right\}} \ , \end{aligned}$$

the set A_λ is closed and not σ-compact.

Proof. Fix λ satisfying

$$\begin{aligned} & \inf_{\{(x,y)\in X\times X : \langle J''(x)(y),y\rangle>0\}} \frac{\|y\|^2}{\langle J''(x)(y), y\rangle} \\ < \ & \lambda \\ < \ & \frac{1}{\max\left\{0, 2\limsup_{\|x\|\to+\infty}\frac{J(x)}{\|x\|^2}, \limsup_{\|x\|\to+\infty}\frac{\langle J'(x),x\rangle}{\|x\|^2}\right\}} \ . \end{aligned}$$

For each $x \in X$, put

$$I_\lambda(x) = \frac{1}{2}\|x\|^2 - \lambda J(x) \ .$$

Clearly, for some $(x, y) \in X \times X$, with $\langle J''(x)(y), y\rangle > 0$, we have

$$\langle y - \lambda J''(x)(y), y\rangle < 0$$

and so, since

$$I''_\lambda(x)(y) = y - \lambda J''(x)(y) \ ,$$

by a classical characterization (Theorem 2.1.11 of [2]), the functional I_λ is not convex. Now, let us show that

$$\lim_{\|x\|\to+\infty} \|x - \lambda J'(x)\| = +\infty \ . \tag{2.1}$$

Indeed, for each $x \in X \setminus \{0\}$, we have

$$\begin{aligned} \|x - \lambda J'(x)\| &= \sup_{\|y\|=1} \langle x - \lambda J'(x), y\rangle \\ &\geq \left\langle x - \lambda J'(x), \frac{x}{\|x\|}\right\rangle \\ &\geq \|x\| \left(1 - \lambda \frac{\langle J'(x), x\rangle}{\|x\|^2}\right) \ . \end{aligned} \tag{2.2}$$

On the other hand, we also have

$$\liminf_{\|x\|\to+\infty} \left(1 - \lambda \frac{\langle J'(x), x\rangle}{\|x\|^2}\right) = 1 - \lambda \limsup_{\|x\|\to+\infty} \frac{\langle J'(x), x\rangle}{\|x\|^2} > 0 \ . \tag{2.3}$$

So, (2.1) is a direct consequence of (2.2) and (2.3). Furthermore, for each $z \in X$, since

$$I_\lambda(x) + \langle z, x\rangle = \|x\|^2 \left(\frac{1}{2} - \lambda \frac{J(x)}{\|x\|^2} + \frac{\langle z, x\rangle}{\|x\|^2}\right)$$

and

$$\liminf_{\|x\|\to+\infty} \left(\frac{1}{2} - \lambda \frac{J(x)}{\|x\|^2} + \frac{\langle z, x\rangle}{\|x\|^2}\right) = \frac{1}{2} - \lambda \limsup_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} > 0 \ ,$$

we have

$$\lim_{\|x\|\to+\infty} (I_\lambda(x) + \langle z, x\rangle) = +\infty \ .$$

Since J' is compact, on the one hand, J is sequentially weakly continuous ([4, Corollary 41.9]) and, on the other hand, in view of (2.1), the operator I'_λ is closed ([3, Example 4.43]). The compactness of J' also implies that, for each $x \in X$, the operator $J''(x)$ is compact ([3, Proposition 7.33]) and so, for each $\lambda \in \mathbf{R}$, the operator $y \to y - \lambda J''(x)(y)$ is injective if and only if it is surjective ([3, Example 8.16]). At this point, the fact that A_λ is closed and not σ-compact follows directly from Theorem 2.3 which can be applied to the functional I_λ. □

Proof of Theorem 2.1. Fix $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$. For each $u \in H^1_0(\Omega)$, put

$$J_f(u) = \int_\Omega \alpha(x) F(u(x)) dx \ ,$$

where

$$F(\xi) = \int_0^\xi f(t) dt \ .$$

Our assumptions ensure that the functional J_f is of class C^2 in $H^1_0(\Omega)$, and we have

$$\langle J'_f(u), v\rangle = \int_\Omega \alpha(x) f(u(x)) v(x) dx \ ,$$

$$\langle J''_f(u)(v), w\rangle = \int_\Omega \alpha(x) f'(u(x)) v(x) w(x) dx$$

for all $u, v, w \in H^1_0(\Omega)$. Moreover, J'_f is compact. Fix $\nu > \limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2}$. Then, for a suitable constant $c > 0$, we have

$$F(\xi) \le \nu\xi^2 + c$$

for all $\xi \in \mathbf{R}$. Hence, for each $u \in H^1_0(\Omega)$, we obtain

$$J_f(u) \le \nu \int_\Omega \alpha(x)|u(x)|^2 dx + c \int_\Omega \alpha(x) dx \le \nu\lambda_\alpha^{-1}\|u\|^2 + c\int_\Omega \alpha(x)dx \ .$$

This clearly implies that

$$\limsup_{\|u\|\to+\infty} \frac{J_f(u)}{\|u\|^2} \le \lambda_\alpha^{-1} \limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^2} \ . \tag{2.4}$$

In the same way, we obtain

$$\limsup_{\|u\|\to+\infty} \frac{\langle J'_f(u), u\rangle}{\|u\|^2} \le \lambda_\alpha^{-1} \limsup_{|\xi|\to+\infty} \frac{f(\xi)}{\xi} \ . \tag{2.5}$$

Now, fix a function $\tilde{v} \in H^1_0(\Omega)$, with $\|\tilde{v}\| = 1$, such that

$$\int_\Omega \alpha(x)|\tilde{v}(x)|^2 dx = \lambda_\alpha^{-1} \ .$$

Fix also $\epsilon > 0$, $\tilde{\xi} \in \mathbf{R}$, with $f'(\tilde{\xi}) > 0$, and a closed set $C \subseteq \Omega$ so that

$$\int_C \alpha(x)|\tilde{v}(x)|^2 dx > \int_\Omega \alpha(x)|\tilde{v}(x)|^2 dx - \epsilon$$

and

$$\int_{\Omega\setminus C} \alpha(x)|\tilde{v}(x)|^2 dx < \frac{\epsilon}{\sup_{[-|\tilde{\xi}|,|\tilde{\xi}|]} |f'|} \ .$$

Finally, fix a function $\tilde{u} \in H^1_0(\Omega)$ such that

$$\tilde{u}(x) = \tilde{\xi}$$

for all $\xi \in C$ and

$$|\tilde{u}(x)| \le |\tilde{\xi}|$$

for all $\xi \in \Omega$. Then, we have

$$\begin{aligned} f'(\tilde{\xi})\left(\int_\Omega \alpha(x)|\tilde{v}(x)|^2 dx - \epsilon\right) &< f'(\tilde{\xi})\int_C \alpha(x)|\tilde{v}(x)|^2 dx \\ &= \int_\Omega \alpha(x) f'(\tilde{u}(x))|\tilde{v}(x)|^2 dx \\ &\quad - \int_{\Omega\setminus C} \alpha(x) f'(\tilde{u}(x))|\tilde{v}(x)|^2 dx \\ &\le \sup_{(u,v)\in H^1_0(\Omega)\times(H^1_0(\Omega)\setminus\{0\})} \frac{\int_\Omega \alpha(x) f'(u(x))|v(x)|^2 dx}{\|v\|^2} \\ &\quad +\epsilon \ . \end{aligned}$$

Since $\tilde{\xi}$ and ϵ are arbitrary, we then infer that

$$\lambda_\alpha^{-1}\sup_{\mathbf{R}} f' \leq \sup_{(u,v)\in H_0^1(\Omega)\times(H_0^1(\Omega)\setminus\{0\})} \frac{\int_\Omega \alpha(x)f'(u(x))|v(x)|^2dx}{\|v\|^2}. \tag{2.6}$$

Consequently, putting (2.4), (2.5) and (2.6) together, we obtain

$$\begin{aligned}
&\max\left\{0, 2\limsup_{\|u\|\to+\infty}\frac{J_f(u)}{\|u\|^2}, \frac{\langle J_f'(u), u\rangle}{\|u\|^2}\right\}\\
&\leq \lambda_\alpha^{-1}\max\left\{0, 2\limsup_{|\xi|\to+\infty}\frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{|\xi|\to+\infty}\frac{f(\xi)}{\xi}\right\}\\
&< \lambda_\alpha^{-1}\sup_{\mathbf{R}} f' \leq \sup_{(u,v)\in H_0^1(\Omega)\times(H_0^1(\Omega)\setminus\{0\})} \frac{\int_\Omega \alpha(x)f'(u(x))|v(x)|^2dx}{\|v\|^2}.
\end{aligned}$$

Therefore, we can apply Theorem 2.4 taking $X = H_0^1(\Omega)$ and $J = J_f$. Therefore, for every λ satisfying

$$\frac{\lambda_\alpha}{\sup_{\mathbf{R}} f'} < \lambda < \frac{\lambda_\alpha}{\max\left\{0, 2\limsup_{|\xi|\to+\infty}\frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{|\xi|\to+\infty}\frac{f(\xi}{\xi}\right\}},$$

the set A_λ (defined in Theorem 2.4) is closed and not σ-compact in $H_0^1(\Omega)$. But, clearly, a $u \in H_0^1(\Omega)$ belongs to A_λ if and only if the problem

$$\begin{cases} -\Delta v = \lambda\alpha(x)f'(u(x))v & \text{in } \Omega \\ \\ v_{|\partial\Omega} = 0 \end{cases}$$

has a non-zero weak solution, and the proof is complete. □

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