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FIXED POINT THEOREMS IN QUASI-METRIC SPACES AND APPLICATIONS TO MULTIDIMENSIONAL FIXED POINT THEOREMS ON G-METRIC SPACES

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Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. In this manuscript, we investigate the equivalence of the coupled fixed point theorems in quasi-metric spaces and in G-metric spaces. We also notice that coupled fixed point theorems in the setting of G-metric spaces can be derived from their corresponding versions in quasi-metric spaces. Our results generalize and unify several fixed point theorems in the context of G-metric spaces in the literature.

1. Introduction

In recent times, generalized metrics (mainly known as *G-metrics*), firstly introduced by Mustafa and Sims [33], have attracted much attention, especially in the field of Fixed Point Theory. The authors [33] associated the geometry of a G-metric space with the perimeter of triangle. The literature on this topic has exponentially raised in the last two years, in which coupled, tripled and quadrupled fixed point results have been given using different contractivity conditions. Recently, Samet *et al.* [42], and Jleli and Samet [19], reported that most of the fixed point theorems in the context of G-metric spaces can be derived from the existing ones. More precisely, the authors noticed that most of the statements of fixed point theorems in G-metric space can be written via two points. On the other hand, G-metric space supposed to tell about the geometry of three points. Later, Agarwal and Karapınar [2], and Asadi *et al.* [4], suggested new statements to which the techniques used in [19, 42] were not applicable.

One of the weakness of the notion of G-metric is that the product of G-metric spaces need not be a G-metric space unless if each factor is symmetric. Very recently, a more general notion than G-metric, namely *G^*-metric*, was firstly considered by Roldán and Karapınar [35] in order to treat this weakness of G-metric spaces. It is well-known that a G-metric is a quasi metric. Although, when we impose that two arguments must be equal, the literature on this subject using quasi-metrics has not raised to the same rate.

In this manuscript, we present some fixed point theorems in the framework of quasi-metric spaces, which can be partially ordered or not. Then, we take advantage of the previous relationship between quasi-metrics and G^*-metrics to deduce many

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coupled and tripled results on these settings. In particular, as the contractivity condition we introduce is very general, we prove that well-known results using G-metrics can be easily seen as simple consequences of our results, and they also hold using G^*-metrics. Our technique can also be employed to deduce some other results in the literature.

2. Preliminaries

Let N be a positive integer. Henceforth, let X be a non-empty set and X^N will denote the product space $X \times X \times \overset{N}{\ldots} \times X$. Throughout this manuscript, n and k will denote non-negative integers, and t and s will be non-negative real numbers. In the sequel, let $F : X^N \to X$ and $T : X \to X$ be two mappings. For brevity, $T(x)$ will be denoted by Tx.

The main aim of the present paper is to guarantee existence and uniqueness of the following class of points.

Definition 2.1. Given $T : X \to X$, we will say that a point $x \in X$ is a *fixed point of* T if $Tx = x$.

Following Gnana-Bhaskar and Lakshmikantham (see [17]), given $F : X^2 \to X$, we will say that a point $(x, y) \in X^2$ is a *coupled fixed point of* F if $F(x, y) = x$ and $F(y, x) = y$.

Following Berinde and Borcut (see [8,10]), given $F : X^3 \to X$, we will say that a point $(x, y, z) \in X^3$ is a *tripled fixed point of* F if $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$.

Following Karapınar and Luong (see [22,23]), given $F : X^4 \to X$, we will say that a point $(x, y, z, t) \in X^4$ is a *quadrupled fixed point of* F if $F(x, y, z, t) = x$, $F(y, z, t, x) = y$, $F(z, t, x, y) = z$ and $F(t, x, y, z) = t$.

A notion of *multidimensional fixed point* was given in Roldán *et al.* [36,38]. In order to guarantee existence and uniqueness of the previous kind of points, we will use the following properties and notations.

Given $N \in \{2, 3, 4\}$ and $F : X^N \to X$, let denote by $T_F^N : X^N \to X^N$ the mappings

(2.1)
$$\begin{cases} N = 2, & T_F^2(x, y) = (F(x, y), F(y, x)), \\ N = 3, & T_F^3(x, y, z) = (F(x, y, z), F(y, x, y), F(z, y, x)), \\ N = 4, & T_F^4(x, y, z, t) = (F(x, y, z, t), F(y, z, t, x), F(z, t, x, y), F(t, x, y, z)). \end{cases}$$

Definition 2.2. A *quasi-metric on* X is a function $q : X \times X \to [0, \infty)$ satisfying the following properties:

(q_1) $q(x, y) = 0$ if and only if $x = y$;
(q_2) $q(x, y) \le q(x, z) + q(z, y)$ for any points $x, y, z \in X$.

In such a case, the pair (X, q) is called a *quasi-metric space*.

Definition 2.3. Let (X, q) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. We will say that:

- $\{x_n\}$ *converges to* x (and we will denote it by $\{x_n\} \xrightarrow{q} x$) if $\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0$;

- $\{x_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.

The quasi-metric space is said to be *complete* if every Cauchy sequence is convergent.

As q is not necessarily symmetric, some authors distinguished between left/right Cauchy/convergent sequences and completeness.

Definition 2.4 (Jleli and Samet [19])**.** Let (X, q) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. We will say that:

- $\{x_n\}$ *right-converges to* x if $\lim_{n\to\infty} q(x_n, x) = 0$;
- $\{x_n\}$ *left-converges to* x if $\lim_{n\to\infty} q(x, x_n) = 0$;
- $\{x_n\}$ is a *right-Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n \geq n_0$;
- $\{x_n\}$ is a *left-Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_m, x_n) < \varepsilon$ for all $m > n \geq n_0$;
- (X, q) is *right-complete* if every right-Cauchy sequence is right-convergent;
- (X, q) is *left-complete* if every left-Cauchy sequence is left-convergent;

Remark 2.5. (1) The limit of a sequence in a quasi-metric space, if there exists, is unique. However, this is false if we consider right-limits or left-limits.
(2) If a sequence $\{x_n\}$ has a right-limit x and a left-limit y, then $x = y$, $\{x_n\}$ converges and it has an only limit (from the right and from the left). However, it is possible that a sequence has two different right-limits when it has no left-limit.

Example 2.6. Let X be a subset of $\mathbb{R}$ containing $[0, 1]$ and define, for all $x, y \in X$,

$$q(x, y) = \begin{cases} x - y, & \text{if } x \geq y, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, q) is a quasi-metric space. Notice that $\{q(1/n, 0)\} \to 0$ but $\{q(0, 1/n)\} \to 1$. Therefore, $\{1/n\}$ right-converges to 0 but it does not converge from the left. We also point out that this quasi-metric verifies the following property: if a sequence $\{x_n\}$ has a right-limit x, then it is unique.

Definition 2.7. Let (X, q) be a quasi-metric space and let $T : X \to X$ be a mapping. We will say that T is *right-continuous* if $\{q(Tx_n, Tu)\} \to 0$ for all sequence $\{x_n\} \subseteq X$ and all $u \in X$ such that $\{q(x_n, u)\} \to 0$.

Next, we introduce some preliminaries about G^*-metric spaces.

Definition 2.8 (Mustafa and Sims [33])**.** A generalized metric (or a G-metric) on X is a mapping $G : X^3 \to [0, \infty)$ verifying, for all $x, y, z \in X$:

(G_1) $G(x, x, x) = 0$.
(G_2) $G(x, x, y) > 0$ if $x \neq y$.
(G_3) $G(x, x, y) \leq G(x, y, z)$ if $y \neq z$.
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables).
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (*rectangle inequality*).

Taking into account that the product space of G-metric spaces need not be a G-metric space, Roldán *et al.* introduced the following notion.

Definition 2.9 (Roldán and Karapınar [35]). A G^*-metric on X is a mapping $G : X^3 \to [0, \infty)$ verifying (G_1), (G_2), (G_4) and (G_5).

The *open ball* $B(x, r)$ *of center* $x \in X$ *and radius* $r > 0$ in a G^*-metric space (X, G) is

$$B(x, r) = \{y \in X : G(x, x, y) < r\}.$$

A subset $A \subseteq X$ is G-open if for all $x \in A$ there exists $r > 0$ such that $B(x, r) \subseteq A$. Following classic techniques, it is possible to prove that there exists an unique topology τ_G on X such that $\beta_x = \{B(x, r) : r > 0\}$ is a neighborhood system at each $x \in X$ (see [35]). Furthermore, τ_G is a Hausdorff topology. In this topology, we characterize the notions of convergent sequence and Cauchy sequence in the following way. Let (X, G) be a G^*-metric space, let $\{x_m\} \subseteq X$ be a sequence and let $x \in X$.

- $\{x_m\}$ *G-converges to* x, and we will write $\{x_m\} \xrightarrow{G} x$, if $\lim_{m,m' \to \infty} G(x_m, x_{m'}, x) = 0$, that is, for all $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ verifying that $G(x_m, x_{m'}, x) < \varepsilon$ for all $m, m' \in \mathbb{N}$ such that $m, m' \geq m_0$.
- $\{x_m\}$ is *G-Cauchy* if $\lim_{m,m',m'' \to \infty} G(x_m, x_{m'}, x_{m''}) = 0$, that is, for all $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ verifying that $G(x_m, x_{m'}, x_{m''}) < \varepsilon$ for all $m, m', m'' \in \mathbb{N}$ such that $m, m', m'' \geq m_0$.

Remark 2.10. If (X, G) is a G^*-metric space, then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$. It follows from (G_4) and (G_5) because

$$G(x, y, y) = G(y, x, y) \leq G(y, x, x) + G(x, x, y) = 2G(y, x, x).$$

Lemma 2.11 (Roldán and Karapınar [35]). *Let (X, G) be a G^*-metric space, let $\{x_m\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.*

(a) *$\{x_m\}$ G-converges to x.*
(b) $\lim_{m \to \infty} G(x, x, x_m) = 0$.
(c) $\lim_{m \to \infty} G(x_m, x_m, x) = 0$.

Proposition 2.12 (Roldán and Karapınar [35]). *The limit of a G-convergent sequence in a G^*-metric space is unique.*

Lemma 2.13 (Roldán and Karapınar [35]). *If (X, G) is a G^*-metric space and $\{x_m\} \subseteq X$ is a sequence, then the following conditions are equivalent.*

(a) *$\{x_m\}$ is G-Cauchy.*
(b) $\lim_{m,m' \to \infty} G(x_m, x_{m'}, x_{m'}) = 0$.
(c) $\lim_{m,m' \to \infty} G(x_m, x_{m+1}, x_{m'}) = 0$.

Consider the following families of control functions.

$$\Phi = \{\ \phi : [0, \infty) \to [0, \infty) : \phi \text{ is continuous, nondecreasing, } \phi(t) = 0 \Leftrightarrow t = 0\ \},$$
$$\Psi = \{\ \psi : [0, \infty) \to [0, \infty) : \psi \text{ is lower semi-continuous, } \psi(t) = 0 \Leftrightarrow t = 0\ \},$$

Functions on Φ are called *altering distance functions* (see Khan *et al.* [25]). To conclude this section of preliminaries, we recall the following fixed point theorems in the context of quasi-metric spaces which can be found in [13] - [45].

Definition 2.14. A preorder (or a quasiorder) $\preccurlyeq$ on X is a binary relation on X that is *reflexive* (i.e., $x \preccurlyeq x$ for all $x \in X$) and *transitive* (if $x, y, z \in X$ verify $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$). In such case, we say that $(X, \preccurlyeq)$ is a preordered space (or a preordered set). If a preorder $\preccurlyeq$ is also *antisymmetric* ($x \preccurlyeq y$ and $y \preccurlyeq x$ implies $x = y$), then $\preccurlyeq$ is called a partial order, and $(X, \preccurlyeq)$ is a partially ordered space.

3. Some relationships between quasi-metrics and G^*-metrics on X^2

Before introducing the main results of the paper about sufficient conditions to ensure the existence and uniqueness of fixed points on different frameworks, we analyze the close relationships between G^*-metrics and quasi-metrics, and how to extend both notions to the product space X^2. We start showing that every G^*-metric lets us to consider two quasi-metrics.

Lemma 3.1. *Let (X, G) be a G^*-metric space and let define $q_G, q'_G : X^2 \to [0, \infty)$ by*

$$q_G(x, y) = G(x, x, y) \quad \textit{and} \quad q'_G(x, y) = G(x, y, y) \quad \textit{for all } x, y \in X.$$

Then the following properties hold.

(1) *q_G and q'_G are quasi-metrics on X. Moreover*

$$q_G(x, y) \leq 2q'_G(x, y) \leq 4q_G(x, y) \quad \textit{for all } x, y \in X. \tag{3.1}$$

(2) *In (X, q_G) and in (X, q'_G), a sequence is right-convergent (respectively, left-convergent) if, and only if, it is convergent. In such a case, its right-limit, its left-limit and its limit coincide.*

(3) *In (X, q_G) and in (X, q'_G), a sequence is right-Cauchy (respectively, left-Cauchy) if, and only if, it is Cauchy.*

(4) *In (X, q_G) and in (X, q'_G), every right-convergent (respectively, left-convergent) sequence has a unique right-limit (respectively, left-limit).*

(5) *If $\{x_n\} \subseteq X$ and $x \in X$, then $\{x_n\} \xrightarrow{G} x \iff \{x_n\} \xrightarrow{q_G} x \iff \{x_n\} \xrightarrow{q'_G} x$.*

(6) *If $\{x_n\} \subseteq X$, then $\{x_n\}$ is G-Cauchy $\iff$ $\{x_n\}$ is q_G-Cauchy $\iff$ $\{x_n\}$ is q'_G-Cauchy.*

(7) *(X, G) is complete $\iff$ (X, q_G) is complete $\iff$ (X, q'_G) is complete.*

Proof. (1) Axiom (q_1) follows from (G_1) and (G_2) and condition (q_2) holds because of properties (G_4) and (G_5) since, for all $x, y, z \in X$,

$$\begin{aligned} q_G(x, y) &= G(x, x, y) = G(y, x, x) \leq G(y, z, z) + G(z, x, x) \\ &= G(x, x, z) + G(z, z, y) = q_G(x, z) + q_G(z, y); \\ q'_G(x, y) &= G(x, y, y) \leq G(x, z, z) + G(z, y, y) = q'_G(x, z) + q'_G(z, y). \end{aligned}$$

Inequalities (3.1) follow from Remark 2.10.

(2) It follows from Lemma 2.11.

(3) It follows from Lemma 2.13.

(4) It follows from item 2 and Remark 2.5.

Other items are straightforward exercises. □

Remark 3.2. Notice that q_G and q'_G can be different quasi-metrics. For instance, q'_G is a quasi-metric even if G does not verify axiom (G_4), but q_G needs that property.

To take advantage of the previous result, we need to extend quasi-metrics and G^*-metrics to the product space X^2. The following one is an easy way to consider quasi-metrics on X^2 *via* quasi-metrics on X.

Lemma 3.3. *Let $q : X^2 \to [0,\infty)$ and $Q_s, Q_m : X^4 \to [0,\infty)$ be three mappings verifying*

$$Q_s^q((x_1,x_2),(y_1,y_2)) = q(x_1,y_1) + q(x_2,y_2) \qquad \text{and}$$
$$Q_m^q((x_1,x_2),(y_1,y_2)) = \max(q(x_1,y_1), q(x_2,y_2)) \qquad \text{for all } x_1,x_2,y_1,y_2 \in X.$$

Then the following conditions are equivalent.

(a) *q is a quasi-metric on X.*
(b) *Q_s^q is a quasi-metric on X^2.*
(c) *Q_m^q is a quasi-metric on X^2.*

In such a case, the following properties hold.

(1) *Every sequence $\{(x_n,y_n)\} \subseteq X^2$ verifies: $\{(x_n,y_n)\} \xrightarrow{Q_s^2} (x,y) \iff \{(x_n,y_n)\} \xrightarrow{Q_m^2} (x,y) \iff \big[\, \{x_n\} \xrightarrow{q} x$ and $\{y_n\} \xrightarrow{q} y \,\big]$.*
(2) *$\{(x_n,y_n)\} \subseteq X^2$ is Q_s^2-Cauchy $\iff$ $\{(x_n,y_n)\}$ is Q_m^2-Cauchy $\iff$ $\big[\{x_n\}$ and $\{y_n\}$ are q-Cauchy$\,\big]$.*
(3) *Items 1 and 2 are valid from the right and from the left.*
(4) *(X,q) is right-complete $\iff$ (X^2,Q_s^2) is right-complete $\iff$ (X^2,Q_m^2) is right-complete.*
(5) *(X,q) is left-complete $\iff$ (X^2,Q_s^2) is left-complete $\iff$ (X^2,Q_m^2) is left-complete.*
(6) *(X,q) is complete $\iff$ (X^2,Q_s^2) is complete $\iff$ (X^2,Q_m^2) is complete.*
(7) *The following conditions are equivalent.*
   (7.1) *Each right-convergent sequence in (X,q) has an unique right-limit.*
   (7.2) *Each right-convergent sequence in (X^2,Q_s^q) has an unique right-limit.*
   (7.3) *Each right-convergent sequence in (X^2,Q_m^q) has an unique right-limit.*

We can do the same construction using G^*-metrics. Notice that the following result does not hold for G-metric spaces.

Lemma 3.4. *Let $G : X^3 \to [0,\infty)$ and $G_s^2, G_m^2 : (X^2)^3 \to [0,\infty)$ be three mappings verifying*

$$G_s^2((x_1,y_1),(x_2,y_2),(x_3,y_3)) = G(x_1,x_2,x_3) + G(y_1,y_2,y_3) \qquad \text{and}$$
$$G_m^2((x_1,y_1),(x_2,y_2),(x_3,y_3)) = \max\{G(x_1,x_2,x_3), G(y_1,y_2,y_3)\}$$
$$\text{for all } x_1,x_2,x_3,y_1,y_2,y_3 \in X.$$

Then the following conditions are equivalent.

(a) *G is a G^*-metric on X.*

(b) *G_s^2 is a G^*-metric on X^2.*
(c) *G_m^2 is a G^*-metric on X^2.*

In such a case, the following properties hold.

(1) *Every sequence $\{(x_n, y_n)\} \subseteq X^2$ verifies: $\{(x_n, y_n)\} \xrightarrow{G_s^2} (x, y) \iff \{(x_n, y_n)\} \xrightarrow{G_m^2} (x, y) \iff \big[\, \{x_n\} \xrightarrow{G} x$ and $\{y_n\} \xrightarrow{G} y \big]$.*
(2) *$\{(x_n, y_n)\} \subseteq X^2$ is G_s^2-Cauchy $\iff$ $\{(x_n, y_n)\}$ is G_m^2-Cauchy $\iff$ $\big[\{x_n\}$ and $\{y_n\}$ are G-Cauchy $\big]$.*
(3) *(X, G) is G-complete $\iff$ (X^2, G_s^2) is G-complete $\iff$ (X^2, G_m^2) is G-complete.*

Following definitions in Lemmas 3.1 and 3.4, it is easy to prove the following statements.

Lemma 3.5. *If $G : X^3 \to [0, \infty)$ is a function, then $Q_s^{q_G} = q_{G_s^2}$, $Q_s^{q'_G} = q'_{G_s^2}$, $Q_m^{q_G} = q_{G_m^2}$ and $Q_m^{q'_G} = q'_{G_m^2}$.*

4. Fixed point theorems in the framework of quasi-metric spaces

In this section, we show some fixed point theorems in the framework of quasi-metric spaces, provided with a partial order or not. Firstly, we introduce the kind of control functions we will use.

Definition 4.1. We will denote by $\mathcal{F}$ the family of all pairs (ϕ, ψ), where $\phi, \psi : [0, \infty) \to [0, \infty)$ are functions, verifying the following three conditions.

($\mathcal{F}_1$) ϕ is non-decreasing.
($\mathcal{F}_2$) If there exists $t_0 \in [0, \infty)$ such that $\psi(t_0) = 0$, then $t_0 = 0$ and $\phi^{-1}(0) = \{0\}$.
($\mathcal{F}_3$) If $\{a_k\}, \{b_k\} \subset [0, \infty)$ are sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verifying $L < b_k$ and $\phi(b_k) \le (\phi - \psi)(a_k)$ for all k, then $L = 0$.

Notice that axiom ($\mathcal{F}_2$) does not imply the well-known condition $\phi(t) = 0 \Leftrightarrow t = 0 \Leftrightarrow \psi(t) = 0$. Furthermore, we do not impose any continuity condition neither on ϕ nor on ψ. The following Lemma shows some examples of pairs in $\mathcal{F}$.

Lemma 4.2. (1) *If $\phi \in \Phi$ and $\psi \in \Psi$, then $(\phi, \psi) \in \mathcal{F}$.*
(2) *If ϕ and ψ are altering distance functions, then $(\phi, \psi) \in \mathcal{F}$.*

Notice that it is not necessary the condition $\psi \le \phi$.

Proof. (1) Suppose that $\phi \in \Phi$ and $\psi \in \Psi$. Conditions ($\mathcal{F}_1$) and ($\mathcal{F}_2$) are obvious. Next, assume that $\{a_k\}, \{b_k\} \subset [0, \infty)$ are sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verifying $L < b_k$ and $\phi(b_k) \le (\phi - \psi)(a_k)$ for all k. Therefore, $\phi(b_k) \le (\phi - \psi)(a_k) = \phi(a_k) - \psi(a_k) \le \phi(a_k)$. Hence $0 \le \psi(a_k) \le \phi(a_k) - \phi(b_k)$ for all k. Letting $k \to \infty$ and taking into account that ϕ is continuous, we deduce that $\lim_{k\to\infty} \psi(a_k) = 0$. As $\{a_k\} \to L$ and ψ is lower semi-continuous, we deduce that $\psi(L) \le \liminf_{t\to L} \psi(t) \le \lim_{k\to\infty} \psi(a_k) = 0$. Hence $L = 0$.
(2) It immediately follows from item 1. □

Example 4.3. (1) If $a, b > 0$ and we define $\phi(t) = at$ and $\psi(t) = bt$ for all $t \ge 0$, then $(\phi, \psi) \in \mathcal{F}$. The case $a \ge b$ is usually included in other papers, but the case $a < b$ is new.

(2) If $\phi(t) = \psi(t) = t + 1$ for all $t \geq 0$, then $(\phi, \psi) \in \mathcal{F}$. Notice that, in this case, $(\mathcal{F}_3)$ holds because it is impossible to find such kind of sequences since $1 \leq 1 + b_k = \phi(b_k) \leq (\phi - \psi)(a_k) = 0$. In this case, the condition $\phi(t) = 0 \Leftrightarrow t = 0$ does not hold.

Some useful properties of pairs in $\mathcal{F}$ are given in the following result.

Lemma 4.4. *Let $(\phi, \psi) \in \mathcal{F}$.*

(1) *If $t, s \in [0, \infty)$ and $\phi(t) \leq (\phi - \psi)(s)$, then either $t < s$ or $t = s = 0$. In any case, $t \leq s$.*
(2) *If $t \in [0, \infty)$ and $\phi(t) \leq (\phi - \psi)(t)$ then $t = 0$.*
(3) *If $\{a_k\}, \{b_k\} \subset [0, \infty)$ are such that $\phi(a_k) \leq (\phi - \psi)(b_k)$ for all k and $\{b_k\} \to 0$, then $\{a_k\} \to 0$.*
(4) *If $\{a_k\} \subset [0, \infty)$ and $\phi(a_{k+1}) \leq (\phi - \psi)(a_k)$ for all k, then $\{a_k\} \to 0$.*

Proof. (1) Assume that $s \leq t$ and we are going to show that $t = s = 0$. Indeed, in such a case, as ϕ is non-decreasing, we have that $\phi(s) \leq \phi(t) \leq \phi(s) - \psi(s) \leq \phi(s)$. Therefore $\psi(s) = 0$. By condition $(\mathcal{F}_2)$, $s = 0$ and $\phi^{-1}(0) = \{0\}$. Then $\phi(t) \leq \phi(0) - \psi(0) = 0$, which means that $t = 0 = s$. As a consequence, both cases lead to $t \leq s$.

(2) It immediately follows from item 1.

(3) It immediately follows from item 1 taking into account that $a_k \leq b_k$ for all k.

(4) Item 1 guarantees that $\{a_k\}$ is a non-increasing sequence ($a_{k+1} \leq a_k$ for all k). Let $L = \lim_{k\to\infty} a_k \geq 0$. Hence $L \leq a_{k+1} \leq a_k$ for all k. If there exists some $k_0 \in \mathbb{N}$ such that $L = a_{k_0}$, then $a_{k_0+1} = L$. In this case, $\phi(L) = \phi(a_{k_0+1}) \leq (\phi - \psi)(a_{k_0}) = \phi(L) - \psi(L) \leq \phi(L)$, which means that $\psi(L) = 0$. Thus, $L = 0$. On the contrary, assume that $L < a_k$ for all k. Letting $b_k = a_{k+1}$ for all k, we conclude that $L = 0$ by condition $(\mathcal{F}_3)$. □

Recall that a function $\alpha : [0, \infty) \to [0, 1)$ is a *Geraghty function* if the condition $\{\alpha(t_n)\} \to 1$ implies that $\{t_n\} \to 0$

Lemma 4.5. *If α is a Geraghty function and we define $\phi(t) = t$ and $\psi(t) = (1 - \alpha(t))t$ for all $t \geq 0$, then $(\phi, \psi) \in \mathcal{F}$.*

Proof. Notice that a Geraghty function must verify $\alpha(s)s \leq s$ for all $s \geq 0$ (if $s = 0$, both members are equal, and if $s > 0$, then $\alpha(s)s < s$ since $\alpha(s) < 1$). Clearly, ϕ is non-decreasing and $\phi(t) = 0 \Leftrightarrow t = 0 \Leftrightarrow \psi(t) = 0$. Let $\{a_k\}, \{b_k\} \subset [0, \infty)$ be sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verifying $L < b_k$ and $\phi(b_k) \leq (\phi - \psi)(a_k)$ for all k. This means that $L < b_k = \phi(b_k) \leq (\phi - \psi)(a_k) = \alpha(a_k)\, a_k \leq a_k$ for all k. Letting $k \to \infty$, we deduce that $\lim_{k\to\infty} (\alpha(a_k)a_k) = L$. If $L > 0$, there is $k_0 \in \mathbb{N}$ such that $a_k \neq 0$ for all $k \geq k_0$. In such a case, we have that

$$\frac{b_k}{a_k} \leq \alpha(a_k) \leq 1 \quad \text{for all } k \geq k_0.$$

Hence $\lim_{k\to\infty} \alpha(a_k) = 1$. Since α is a Geraghty function, then $L = \lim_{k\to\infty} a_k = 0$, but this is a contradiction with $L > 0$. Therefore, $L = 0$ and $(\phi, \psi) \in \mathcal{F}$. □

In Lemmas 5.25, 5.29 and 6.2, we will show new examples of pairs in $\mathcal{F}$.

4.1. **Fixed point theorems in quasi-metric spaces.**

Definition 4.6. Let (X, q) be a quasi-metric space and let $T : X \to X$ be a mapping. We will say that T is an *$\mathcal{F}$-contractive mapping* is there exists $(\phi, \psi) \in \mathcal{F}$ such that

$$\phi(q(Tx, Ty)) \leq \phi(q(x, y)) - \psi(q(x, y)) \qquad \text{for all } x, y \in X. \tag{4.1}$$

A first property of this kind of contractive mappings is the following one.

Lemma 4.7. *Every $\mathcal{F}$-contractive mapping on a quasi-metric space into itself is a continuous mapping.*

Proof. Assume that $T : X \to X$ verifies (4.1) and let $\{y_n\} \subseteq X$ be a sequence such that $\{y_n\} \xrightarrow{q} u \in X$. Therefore, for all n,

$$\phi(q(Ty_n, Tu)) \leq (\phi - \psi)(q(y_n, u)) \quad \text{and} \quad \phi(q(Tu, Ty_n)) \leq (\phi - \psi)(q(u, y_n)).$$

By item 3 of Lemma 4.4, $\{q(Ty_n, Tu)\} \to 0$ and $\{q(Tu, Ty_n)\} \to 0$, so $\{Ty_n\} \xrightarrow{q} Tu$ and T is continuous at u. □

The following is one of the main results in this manuscript.

Theorem 4.8. *Every $\mathcal{F}$-contractive mapping on a complete quasi-metric space into itself has a unique fixed point, and it is continuous on its unique fixed point.*

In fact, if $\{x_n\}$ is any sequence such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ q-converges to the unique fixed point of T.

Proof. Part I: Existence. Let $\{x_n\}_{n\geq 0}$ be a sequence such that $x_{n+1} = Tx_n$ for all $n \geq 0$. If there exists some $n_0 \in \mathbb{N}$ such that $q(x_{n_0}, x_{n_0+1}) = 0$ or $q(x_{n_0+1}, x_{n_0}) = 0$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so x_{n_0} is a fixed point of T. In such a case, $x_n = x_{n_0}$ for all $n \geq n_0$ and $\{x_n\}$ converges to a fixed point of T. On the contrary, assume that

$$q(x_n, x_{n+1}) > 0 \quad \text{and} \quad q(x_{n+1}, x_n) > 0 \quad \text{for all } n. \tag{4.2}$$

Step 1. We claim that $\lim_{n\to\infty} q(x_n, x_{n+1}) = 0$. If we apply the contractivity condition (4.1) to $x = x_{n+1}$ and $y = x_{n+2}$, we obtain that $\phi(q(x_{n+1}, x_{n+2})) = \phi(q(Tx_n, Tx_{n+1})) \leq (\phi - \psi)(q(x_n, x_{n+1}))$ for all $n \geq 0$. By item 4 of Lemma 4.4, we have that $\{q(x_n, x_{n+1})\} \to 0$. Similarly, using $x = x_{n+2}$ and $y = x_{n+1}$, we could deduce that $\{q(x_{n+1}, x_n)\} \to 0$. Therefore, we have proved that

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = \lim_{n\to\infty} q(x_{n+1}, x_n) = 0. \tag{4.3}$$

Step 2. We claim that $\{x_n\}$ is right-Cauchy in (X, q), that is, for all $\varepsilon > 0$, there is $n_0 \in N$ such that $q(x_n, x_m) \leq \varepsilon$ for all $m > n \geq n_0$. We reasoning by contradiction. If $\{x_n\}$ is not right-Cauchy, there exist $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}_{k\in\mathbb{N}}$ and $\{x_{m(k)}\}_{k\in\mathbb{N}}$ verifying that

$$k \leq n(k) < m(k), \qquad q(x_{n(k)}, x_{m(k)}) > \varepsilon_0 \quad \text{for all } k. \tag{4.4}$$

Taking $m(k)$ as the smallest integer, greater than $n(k)$, verifying this property, we can suppose that

$$q(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 \quad \text{for all } k.$$

Therefore $\varepsilon_0 < q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)}) \leq \varepsilon_0+$

$q(x_{m(k)-1}, x_{m(k)})$, and taking limit as $k \to \infty$, it follows from (4.3) that

$$\lim_{k\to\infty} q(x_{n(k)}, x_{m(k)}) = \varepsilon_0.$$

Notice that, for all k,

$$q(x_{n(k)-1}, x_{m(k)-1}) \leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)-1}) \leq q(x_{n(k)-1}, x_{n(k)}) + \varepsilon_0,$$

and

$$\varepsilon_0 < q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)}, x_{n(k)-1}) + q(x_{n(k)-1}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)}).$$

Joining both inequalities we deduce that, for all k,

$$\varepsilon_0 - q(x_{n(k)}, x_{n(k)-1}) - q(x_{m(k)-1}, x_{m(k)}) \leq q(x_{n(k)-1}, x_{m(k)-1}) \leq q(x_{n(k)-1}, x_{n(k)}) + \varepsilon_0.$$

Letting $k \to \infty$, it follows from (4.3) that

$$\lim_{k\to\infty} q(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon_0.$$

Next, let apply the contractivity condition (4.1) to $x = x_{n(k)}$ and $y = x_{m(k)}$. We get that, for all $k \geq 0$,

$$\phi(q(x_{n(k)}, x_{m(k)})) = \phi(q(Tx_{n(k)-1}, Tx_{m(k)-1})) \leq (\phi - \psi)(q(x_{n(k)-1}, x_{m(k)-1})).$$

Using condition $(\mathcal{F}_3)$ applied to $\{a_k = q(x_{n(k)-1}, x_{m(k)-1})\}$, $\{b_k = q(x_{n(k)}, x_{m(k)})\}$ and $L = \varepsilon_0$ (notice that $b_k > L$ for all k by (4.4)), we conclude that $\varepsilon_0 = 0$, which contradicts $\varepsilon_0 > 0$. This contradiction ensures us that $\{x_n\}$ is right-Cauchy in (X, q), and Step 2 holds.

Similarly, it can be proved that $\{x_n\}$ is left-Cauchy in (X, q), so we conclude that $\{x_n\}$ is a Cauchy sequence in (X, q). As (X, q) is complete, there exists $u \in X$ such that $\{x_n\} \xrightarrow{q} u$. We show that u is a fixed point of T. Applying the contractivity condition (4.1) to $x = x_n$ and $y = u$, we have that, for all $n \geq 0$,

$$\phi(q(x_{n+1}, Tu)) = \phi(q(Tx_n, Tu)) \leq (\phi - \psi)(q(x_n, u)).$$

As $\{q(x_n, u)\} \to 0$, item 3 of Lemma 4.4 guarantees that $\{q(x_{n+1}, Tu)\} \to 0$. Similarly, it can be proved that $\{q(Tu, x_{n+1})\} \to 0$. Thus, $\{x_{n+1}\} \xrightarrow{q} Tu$ and the unicity of the limit concludes that $Tu = u$.

Part II: Unicity. Let $u, v \in X$ be any fixed points of T. Using the contractivity condition (4.1),

$$\phi(q(u, v)) = \phi(q(Tu, Tv)) \leq (\phi - \psi)(q(u, v)).$$

Item 2 of Lemma 4.4 shows that $q(u, v) = 0$, so $u = v$. Therefore, T has a unique fixed point. □

Corollary 4.9 (Jleli and Samet [19], Theorem 3.2)**.** *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be a mapping satisfying*

$$q(Tx, Ty) \leq q(x, y) - \psi(q(x, y)) \qquad \text{for all } x, y \in X,$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous with $\psi^{-1}(0) = \{0\}$. Then T has a unique fixed point.

Proof. It follows from Theorem 4.8 considering $\phi(t) = t$ for all $t \geq 0$, and taking into account that $(\phi, \psi) \in \mathcal{F}$. □

Corollary 4.10. *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be a mapping such that there exists $k \in [0, 1)$ satisfying*

$$q(Tx, Ty) \leq k\, q(x, y) \qquad \text{for all } x, y \in X,$$

Then T has a unique fixed point.

Proof. It is only necessary to take $\psi(t) = (1 - k)t$ for all $t \geq 0$ in Corollary 4.9. □

To conclude this subsection, we show how to apply the previous results to G^*-metrics spaces. For instance, the following result was proved by Aydi.

Theorem 4.11 (Aydi [3], Theorem 2.1). *Let X be a complete G-metric space. Suppose the map $T : X \to X$ satisfies for all $x, y, z \in X$*

$$\phi(G(Tx, Ty, Tz)) \leq \phi(G(x, y, z)) - \psi(G(x, y, z)), \tag{4.5}$$

where ϕ and ψ are altering distance functions. Then T has a unique fixed point (say u) and T is G-continuous at u.

We improve this theorem in the following way.

Definition 4.12. Let (X, G) be a G^*-metric space and let $T : X \to X$ be a mapping. We will say that T is an *$\mathcal{F}$-contractive mapping on the G-metric space* (X, G) if there exists $(\phi, \psi) \in \mathcal{F}$ such that

$$\phi(G(Tx, Ty, Ty)) \leq \phi(G(x, y, y)) - \psi(G(x, y, y)) \qquad \text{for all } x, y \in X. \tag{4.6}$$

Notice that the contractivity condition (4.5) obviously implies (4.6).

Corollary 4.13. *Every $\mathcal{F}$-contractive mapping on a complete G^*-metric space into itself has a unique fixed point.*

Proof. It is a consequence of Theorem 4.8 applied to the quasi-metric q'_G defined as $q'_G(x, y) = G(x, y, y)$ for all $x, y \in X$, and using item 7 of Lemma 3.1. □

Corollary 4.14. *Theorem 4.11 also holds even if G is a G^*-metric.*

Proof. It follows from item 2 of Lemma 4.2, also using Lemma 4.7. □

4.2. **Fixed point results in partially ordered quasi-metric spaces.** In this subsection we analyze the case in which the contractive condition involves a kind of functions that can be particularized to relationships more general than partial orders. We need the following notions.

Definition 4.15. We will say that a mapping $\alpha : X^2 \to [0, \infty)$ is *upper-transitive* if

$$\alpha(x, y) \geq 1,\ \alpha(y, z) \geq 1 \quad \Longrightarrow \quad \alpha(x, z) \geq 1.$$

A mapping $T : X \to X$ is said to be *α-admissible* if

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 4.16. Given a mapping $\alpha : X^2 \to [0, \infty)$, a quasi-metric space (X, q) is said to be *upper-regular with respect to α* if

$$[\ \{q(x_n, u)\} \to 0 \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1,\ \forall n\ ] \quad \Longrightarrow \quad \alpha(x_n, u) \geq 1,\ \forall n.$$

Remark 4.17. If $\alpha(x,y) \geq 1$ for all $x, y \in X$, then any mapping $T : X \to X$ is α-admissible and any quasi-metric space (X, q) is upper-regular with respect to α. In particular, this property holds when $\alpha(x,y) = 1$ for all $x, y \in X$.

Lemma 4.18. *Let $T : X \to X$ be an α-admissible mapping and let $\{x_n\}_{n\geq 0} \subseteq X$ be a sequence such that $x_{n+1} = Tx_n$ for all $n \geq 0$. If x_0 verifies $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Additionally, if α is upper-transitive, then $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$ such that $n < m$.*

Definition 4.19. Let (X, q) be a quasi-metric space and let $T : X \to X$ be a mapping. We will say that T is an $(\alpha, \mathcal{F})$-*contractive mapping* is there exist mappings $\alpha : X^2 \to \mathbb{R}$ and $\phi, \psi : [0, \infty) \to [0, \infty)$ such that:

(C_1) α is upper-transitive.
(C_2) T is α-admissible.
(C_3) $(\phi, \psi) \in \mathcal{F}$.
(C_4) for all $x, y \in X$, $\quad \alpha(x,y)\phi(q(Tx,Ty)) \leq \phi(q(x,y)) - \psi(q(x,y))$.

Remark 4.20. If $\alpha(x,y) = 1$ for all $x, y \in X$, then the notions of $(\alpha, \mathcal{F})$-contractive mapping and $\mathcal{F}$-contractive mapping are exactly the same.

The following one is the main result of this subsection.

Theorem 4.21. *Let (X, q) be a right-complete quasi-metric space in which each right-convergent sequence has an unique right-limit and let $T : X \to X$ be an $(\alpha, \mathcal{F})$-contractive mapping. Suppose that there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Also assume that, at least, one of the following conditions hold:*

(A) T is right-continuous.
(B) (X, q) is upper-regular with respect to α.

Then T has, at least, a fixed point. Additionally, assume that for all $u, v \in \operatorname{Fix} T$ there is $z \in X$ such that $\min(\alpha(z,u), \alpha(z,v)) \geq 1$. Then T has a unique fixed point.

Proof. Part I: Existence. Starting from x_0, let define $x_{n+1} = Tx_n$ for all $n \geq 0$. If there exists some $n_0 \in \mathbb{N}$ such that $q(x_{n_0}, x_{n_0+1}) = 0$ or $q(x_{n_0+1}, x_{n_0}) = 0$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so x_{n_0} is a fixed point of T. On the contrary, assume that

$$q(x_n, x_{n+1}) > 0 \quad \text{and} \quad q(x_{n+1}, x_n) > 0 \quad \text{for all } n. \tag{4.7}$$

By Lemma 4.18,

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } n, m \in \mathbb{N} \text{ such that } n < m. \tag{4.8}$$

Step 1. We claim that $\lim_{n\to\infty} q(x_n, x_{n+1}) = 0$. Let apply the contractivity condition (C_4) to $x = x_{n+1}$ and $y = x_{n+2}$, and using (4.8), we obtain that, for all $n \geq 0$,

$$\phi(q(x_{n+1}, x_{n+2})) \leq \alpha(x_n, x_{n+1})\phi(q(Tx_n, Tx_{n+1})) \leq (\phi - \psi)(q(x_n, x_{n+1})).$$

By item 4 of Lemma 4.4, we conclude that:

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0. \tag{4.9}$$

Step 2. We claim that $\{x_n\}$ is right-Cauchy in (X, q), that is, for all $\varepsilon > 0$, there is $n_0 \in N$ such that $q(x_n, x_m) \leq \varepsilon$ for all $m > n \geq n_0$. We reasoning by contradiction. If $\{x_n\}$ is not right-Cauchy, there exist $\varepsilon_0 > 0$ and two partial subsequences $\{x_{n(k)}\}_{k\in\mathbb{N}}$ and $\{x_{m(k)}\}_{k\in\mathbb{N}}$ verifying that

$$k \leq n(k) < m(k), \qquad q(x_{n(k)}, x_{m(k)}) > \varepsilon_0 \quad \text{for all } k. \tag{4.10}$$

Taking $m(k)$ as the smallest integer, greater than $n(k)$, verifying this property, we can suppose that

$$q(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 \quad \text{for all } k.$$

Therefore $\varepsilon_0 < q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)}) \leq \varepsilon_0 + q(x_{m(k)-1}, x_{m(k)})$, and taking limit as $k \to \infty$, it follows from (4.9) that

$$\lim_{k\to\infty} q(x_{n(k)}, x_{m(k)}) = \varepsilon_0.$$

Notice that, for all k,

$$\begin{aligned} q(x_{n(k)-1}, x_{m(k)-1}) &\leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)-1}) \\ &\leq q(x_{n(k)-1}, x_{n(k)}) + \varepsilon_0. \end{aligned} \tag{4.11}$$

Let apply the contractivity condition (C_4) to $x = x_{n(k)}$ and $y = x_{m(k)}$ and we obtain, using (4.8), that, for all $k \geq 0$,

$$\begin{aligned} \phi(q(x_{n(k)}, x_{m(k)})) &\leq \alpha(x_{n(k)-1}, x_{m(k)-1})\phi(q(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq (\phi - \psi)(q(x_{n(k)-1}, x_{m(k)-1})). \end{aligned} \tag{4.12}$$

By item 1 of Lemma 4.4, we have that $q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)-1}, x_{m(k)-1})$ for all k. Joining this inequality to (4.10) and (4.11), we have that, for all k,

$$\varepsilon_0 < q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)-1}, x_{m(k)-1}) \leq q(x_{n(k)-1}, x_{n(k)}) + \varepsilon_0.$$

Letting $k \to \infty$, we deduce that

$$\lim_{k\to\infty} q(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon_0.$$

Using condition $(\mathcal{F}_3)$ applied to $\{a_k = q(x_{n(k)-1}, x_{m(k)-1})\}$, $\{b_k = q(x_{n(k)}, x_{m(k)})\}$ and $L = \varepsilon_0$ (notice that $b_k > L$ for all k by (4.10)), we conclude that $\varepsilon_0 = 0$, which contradicts $\varepsilon_0 > 0$. This contradiction ensures us that $\{x_n\}$ is right-Cauchy in (X, q), and Step 2 holds.

Since (X, q) is right-complete, there exists $u \in X$ such that $\lim_{n\to\infty} q(x_n, u) = 0$. We will show that u is a fixed point of T under two different hypotheses.

Step 3. Assume that T is right-continuous. In such a case, $\lim_{n\to\infty} q(Tx_n, Tu) = 0$, and taking into account that $Tx_n = x_{n+1}$ for all n, then u and Tu are right-limits of the same sequence $\{x_n\}$. As we suppose that the right-limit in (X, q) is unique, then $Tu = u$.

Step 4. Assume that (X, q) is upper-regular with respect to α. In this case, as $\{q(x_n, u)\} \to 0$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n, we have that $\alpha(x_n, u) \geq 1$ for all n. Therefore, applying the contractivity condition (C_4) to $x = x_n$ and $y = u$, we obtain that, for all $n \geq 0$,

$$\phi(q(x_{n+1}, Tu)) \leq \alpha(x_n, u)\phi(q(Tx_n, Tu)) \leq (\phi - \psi)(q(x_n, u)).$$

As $\{q(x_n, u)\} \to 0$, item 3 of Lemma 4.4 guarantees that $\{q(x_{n+1}, Tu)\} \to 0$. Reasoning as in Step 3, we conclude that $Tu = u$.

Part II: Unicity. Let $u, v \in X$ be fixed points of T. Using the additional condition, there exists $z \in X$ such that $\min(\alpha(z,u), \alpha(z,v)) \geq 1$. Let define $z_0 = z$ and $z_{n+1} = Tz_n$ for all $n \geq 0$, and we will prove that $\{q(z_n,u)\} \to 0$ and $\{q(z_n,v)\} \to 0$. By the unicity of the right-limit, this fact will conclude that $u = v$. Using the symmetry in u and v, we will only show that $\{q(z_n,u)\} \to 0$.

Indeed, as z_0 verifies the initial condition $\alpha(z_0,u) \geq 1$ and T is α-admissible, we have that

$$\alpha(z_0,u) \geq 1 \quad \Longrightarrow \quad \alpha(Tz_0,Tu) \geq 1 \quad \Longrightarrow \quad \alpha(z_1,u) \geq 1;$$

Similarly, by induction, it can be proved that $\alpha(z_n,u) \geq 1$ for all $n \geq 0$. Hence, for all $n \geq 0$,

$$\phi(q(z_{n+1},u)) \leq \alpha(z_n,u)\phi(q(Tz_n,Tu)) \leq (\phi - \psi)(q(z_n,u)).$$

Item 4 of Lemma 4.4 guarantees that $\{q(z_n,u)\} \to 0$. This finishes the proof. □

The previous theorem can be particularized in a variety of different ways. For instance, in the following result, a transitive relation is involved. This includes the cases in which the relation is a preorder, a partial order or an equivalence relation.

Corollary 4.22. *Let (X,q) be a right-complete quasi-metric space in which each right-convergent sequence has an unique right-limit and let $T : X \to X$ be a mapping. Suppose that the following conditions are fulfilled.*

- *There exist a transitive relation $\preccurlyeq$ on X and $(\phi,\psi) \in \mathcal{F}$ satisfying*

$$\phi(q(Tx,Ty)) \leq \phi(q(x,y)) - \psi(q(x,y)) \qquad \text{for all } x,y \in X \text{ such that } x \preccurlyeq y. \tag{4.13}$$

- *T is $\preccurlyeq$-non-decreasing (that is, if $x \preccurlyeq y$, then $Tx \preccurlyeq Ty$).*
- *There exists a point $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*
- *At least, one of the following conditions hold:*
  (A) T is right-continuous.
  (B) If $\{x_n\} \subseteq X$ is a sequence in X and $u \in X$ are such that $\{q(x_n,u)\} \to 0$ and $x_n \preccurlyeq x_{n+1}$ for all n, then $x_n \preccurlyeq u$ for all n.

Then T has a fixed point.

Additionally, assume that for all $u,v \in \operatorname{Fix} T$ there is $z \in X$ such that $z \preccurlyeq u$ and $z \preccurlyeq v$. Then T has a unique fixed point.

Proof. Let define $\alpha_{\preccurlyeq} : X^2 \to \mathbb{R}$ by:

$$\alpha_{\preccurlyeq}(x,y) = \begin{cases} 1, & \text{if } x \preccurlyeq y, \\ 0, & \text{otherwise.} \end{cases}$$

As $\preccurlyeq$ is a transitive relation on X, then $\alpha_{\preccurlyeq}$ is upper-transitive. Moreover, as T is $\preccurlyeq$-non-decreasing, then T is $\alpha_{\preccurlyeq}$-admissible. Furthermore, there exists $(\phi,\psi) \in \mathcal{F}$ such that conditions (C_3) and (C_4) trivially hold. Therefore, T is an $(\alpha_{\preccurlyeq},\mathcal{F})$-contractive mapping. The condition $x_0 \preccurlyeq Tx_0$ means that $\alpha_{\preccurlyeq}(x_0,Tx_0) = 1$. Hence, Theorem 4.21 can be applied. □

5. Applications to coupled fixed point theorems in the frameworks of quasi-metric spaces and G^*-metric spaces

In this section, we show some coupled fixed point results in the frameworks of quasi-metric spaces and G^*-metric spaces, and we describe how those results can be reduced to their corresponding statements in the setting of quasi-metrics (especially, to Theorems 4.8 and 4.21). To do that, we will use the following characterization of coupled fixed point using the mapping T_F^2 defined in (2.1).

Lemma 5.1. *Given a mapping $F : X^2 \to X$, a point $(x, y) \in X^2$ is a coupled fixed point of F if, and only if, it is a fixed point of T_F^2.*

5.1. **Coupled fixed point theorems in quasi-metric spaces.** We particularize Theorem 4.8 to the case (X^2, Q_s^q) considering the mapping $T_F^2 : X^2 \times X^2 \to X^2$ by $T_F^2((x, y), (u, v)) = (F(x, y), F(u, v))$. We point out that a similar version of the following symmetric contractivity condition was firstly introduced by Berinde [7] to show the weakness of the published coupled fixed point theorems with non-symmetric contractivity condition in the framework of metric spaces, see e.g. [17]. However, Samet *et al.* understood that such coupled fixed point theorems via symmetric contractivity condition are consequence of the existing fixed point theorems.

Theorem 5.2. *Let (X, q) be a complete quasi-metric space and let $F : X^2 \to X$ be a mapping such that there exists $(\phi, \psi) \in \mathcal{F}$ satisfying, for all $x, y, u, v \in X$,*

$$\phi\Big(q(F(x,y),F(u,v)) + q(F(y,x),F(v,u))\Big) \le (\phi - \psi)\left(q(x,u) + q(y,v)\right). \tag{5.1}$$

Then F has a unique coupled fixed point, which is of the form (x, x). In particular, there exists a unique $x \in X$ such that $F(x, x) = x$.

Proof. As q is a complete quasi-metric on X, then Q_s^q is a complete quasi-metric on X^2 (see Lemma 3.3). Notice that, for all $(x, y), (u, v) \in X^2$,

$$\begin{aligned} &Q_s^q((x,y),(u,v)) = q(x,u) + q(y,v), \qquad \text{and} \\ &Q_s^q(T_F^2(x,y), T_F^2(u,v)) = Q_s^q((F(x,y),F(y,x)),(F(u,v),F(v,u))) \\ &\qquad = q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)). \end{aligned}$$

Therefore, condition (5.1) can be written as

$$\phi\left(Q_s^q(T_F^2(x,y), T_F^2(u,v))\right) \le (\phi - \psi)\left(Q_s^q((x,y),(u,v))\right) \quad \text{for all } (x,y),(u,v) \in X^2.$$

This means that T_F^2 is an $\mathcal{F}$-contractive mapping. Theorem 4.8 guarantees that T_F^2 has a unique fixed point $(x, y) \in X^2$, which is a coupled fixed point of F by Lemma 5.1. It only remains to prove that $x = y$. We have

$$\begin{aligned} \phi\left(q(x,y) + q(y,x)\right) &= \phi\left(q(F(x,y),F(y,x)) + q(F(y,x),F(x,y))\right) \\ &\le \phi(q(x,y) + q(y,x)) - \psi(q(x,y) + q(y,x)) \\ &\le \phi(q(x,y) + q(y,x)), \end{aligned}$$

which means that $\psi(q(x, y) + q(y, x)) = 0$. Therefore $q(x, y) = 0$ and $x = y$. □

We can also particularize Theorem 4.8 to the case (X^2, Q_m^q) considering the mapping T_F^2 as follows.

Theorem 5.3. *Let (X,q) be a complete quasi-metric space and let $F: X^2 \to X$ be a mapping such that there exists $(\phi,\psi) \in \mathcal{F}$ satisfying, for all $x,y,u,v \in X$,*

$$\phi(q(F(x,y),F(u,v))) \leq \max\{\phi(q(x,u)),\phi(q(y,v))\} - \psi(\max\{q(x,u),q(y,v)\}). \tag{5.2}$$

Then F has a unique coupled fixed point, which is of the form (x,x). In particular, there exists a unique $x \in X$ such that $F(x,x)=x$.

Proof. As ϕ is non-decreasing, then $\phi(\max(t,s)) = \max\{\phi(t),\phi(s)\}$ for all $t,s \in [0,\infty)$. Then

$$\begin{aligned}\phi\left(Q_m^q(T_F^2(x,y),T_F^2(u,v))\right) &= \phi\left(Q_m^q((F(x,y),F(y,x)),(F(u,v),F(v,u))\right)\\ &= \phi\Big(\max\{q(F(x,y),F(u,v)),q(F(y,x),F(v,u))\}\Big)\\ &= \max\Big(\phi(q(F(x,y),F(u,v))),\phi(q(F(y,x),F(v,u)))\Big)\\ &\leq \max\{\phi(q(x,u)),\phi(q(y,v))\} - \psi(\max\{q(x,u),q(y,v)\})\\ &= \phi\left(\max\{q(x,u),q(y,v)\}\right) - \psi(\max\{q(x,u),q(y,v)\})\\ &= (\phi-\psi)\left(Q_m^q((x,y),(u,v))\right).\end{aligned}$$

Therefore, T_F^2 is an $\mathcal{F}$-contractive mapping in the complete quasi-metric space $(X^2, q'_{G_m^2})$. The rest is similar to the proof of Theorem 5.2. □

5.2. **Coupled fixed point theorems in G-metric spaces.** We particularize Theorems 5.2 and 5.3 to the case in which $q(x,y)=G(x,y,y)$, where G is a complete G^*-metric on X. Later, we will show that this particularization lets us to prove a Shatanawi's coupled result.

Corollary 5.4. *Let (X,G) be a complete G^*-metric space and let $F: X^2 \to X$ be a mapping such that there exists $(\phi,\psi) \in \mathcal{F}$ satisfying, for all $x,y,u,v \in X$,*

$$\phi\Big(G(F(x,y),F(u,v),F(u,v)) + G(F(y,x),F(v,u),F(v,u))\Big) \leq (\phi-\psi)\left(G(x,u,u)+G(y,v,v)\right) \tag{5.3}$$

Then F has a unique coupled fixed point, which is of the form (x,x). In particular, there exists a unique $x \in X$ such that $F(x,x)=x$.

Corollary 5.5. *Let (X,G) be a complete G^*-metric (or G-metric) space and let $F: X^2 \to X$ be a mapping such that there exists $(\phi,\psi) \in \mathcal{F}$ satisfying, for all $x,y,u,v,w,z \in X$,*

$$\phi\Big(G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w))\Big) \leq (\phi-\psi)\left(G(x,u,w)+G(y,v,z)\right) \tag{5.4}$$

Then F has a unique coupled fixed point, which is of the form (x,x). In particular, there exists a unique $x \in X$ such that $F(x,x)=x$.

Proof. It follows from the fact that condition (5.4) implies condition (5.3). □

We can also particularize Theorem 5.3 to the case $q(x,y) = G(x,y,y)$ considering the mapping T_F^2 as follows.

Corollary 5.6. *Let (X,G) be a complete G^*-metric space and let $F : X^2 \to X$ be a mapping such that there exists $(\phi,\psi) \in \mathcal{F}$ satisfying, for all $x,y,u,v \in X$,*

$$(5.5)\quad \phi(G(F(x,y),F(u,v),F(u,v))) \leq \max\{\phi(G(x,u,u)),\phi(G(y,v,v))\} - \psi(\max\{G(x,u,u),G(y,v,v)\}).$$

Then F has a unique coupled fixed point, which is of the form (x,x). In particular, there exists a unique $x \in X$ such that $F(x,x) = x$.

Corollary 5.7. *Let (X,G) be a complete G^*-metric space and let $F : X^2 \to X$ be a mapping such that there exists $(\phi,\psi) \in \mathrm{F}$ satisfying, for all $x,y,u,v \in X$,*

$$(5.6)\quad \phi(G(F(x,y),F(u,v),F(w,z))) \leq \max\{\phi(G(x,u,w)),\phi(G(y,v,z))\} - \psi(\max\{G(x,u,w),G(y,v,z)\})$$

Then F has a unique coupled fixed point, which is of the form (x,x). In particular, there exists a unique $x \in X$ such that $F(x,x) = x$.

Proof. It follows from the fact that condition (5.6) implies condition (5.5). □

5.2.1. *Shatanawi's coupled fixed point results in G-metric spaces.* In [43], Shatanawi proved the following theorem.

Theorem 5.8 (Shatanawi [43])**.** *Let (X,G) be a G-complete G-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$G(F(x,y),F(u,v),F(w,z)) \leq \frac{k}{2}(G(x,u,w)+G(y,v,z)) \quad \text{for all } x,y,u,v,z,w \in X.$$

If $k \in [0,1)$, then there exists a unique $x \in X$ such that $F(x,x) = x$.

Proof. It follows from Corollary 5.5 using $\phi(t) = t$ and $\psi(t) = (1-k)t$ for all $t \geq 0$. □

We note that the previous result is also valid if G is a G^*-metric.

Corollary 5.9. *Theorem 5.8 also holds even if G is a G^*-metric.*

The following result is more general than Theorem 5.8 and it can be derived as in the previous proof using Corollary 5.4 instead of Corollary 5.5. However, we point out that it was not established in [43].

Theorem 5.10. *Let (X,G) be a G-complete G-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$G(F(x,y),F(u,v),F(u,v)) \leq \frac{k}{2}(G(x,u,u)+G(y,v,v)) \qquad \textit{for all } x,y,u,v \in X.$$

If $k \in [0,1)$, then there is a unique $x \in X$ such that $F(x,x) = x$.

In fact, we prove that the previous result admits a more general version.

Corollary 5.11. *Theorem 5.10 also holds even if G is a G^*-metric.*

5.3. Coupled fixed point theorems in partially ordered quasi-metric spaces.

In 1987, Guo and Lakshmikantham [18] introduced the notion of *coupled fixed point.* This concept was reconsidered by Gnana-Bhaskar and Lakshmikantham [17] in 2006. In this paper, they proved existence and uniqueness of a coupled fixed point of an operator $F : X \times X \to X$ on a partially ordered metric space under a condition called *mixed monotone property.*

Definition 5.12. ([17]) Let $(X, \preccurlyeq)$ be a partially ordered set and let $F : X^2 \to X$. The mapping F is said to have the *mixed monotone property with respect to* $\preccurlyeq$ if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preccurlyeq x_2 \quad \Rightarrow \quad F(x_1, y) \preccurlyeq F(x_2, y) \qquad \text{and}$$
$$y_1, y_2 \in X, \quad y_1 \preccurlyeq y_2 \quad \Rightarrow \quad F(x, y_1) \succcurlyeq F(x, y_2).$$

It is not necessary to consider a partial order $\preccurlyeq$ on X to introduce the following definition. Given a binary relation $\preccurlyeq$ on X, let define $\sqsubseteq$, for all $(x, y), (u, v) \in X^2$, by

$$(x, y) \sqsubseteq (u, v) \iff [\ x \preccurlyeq u \text{ and } y \succcurlyeq v\ ] \tag{5.7}$$

Lemma 5.13. *Let $F : X^2 \to X$ be a mapping and let $\preccurlyeq$ be a binary relation on X.*

(1) *$\preccurlyeq$ is transitive (respectively, reflexive, a preorder, a partial order) if, and only if, $\sqsubseteq$ is transitive (respectively, reflexive, a preorder, a partial order).*
(2) *If F has the mixed monotone property with respect to $\preccurlyeq$, then T_F is $\sqsubseteq$-non-decreasing.*
(3) *If $\preccurlyeq$ is reflexive, then F has the mixed monotone property with respect to $\preccurlyeq$ if, and only if, T_F is $\sqsubseteq$-non-decreasing.*

Proof. (1) We only study the transitivity. Suppose that $\preccurlyeq$ is transitive and let $(x_1, x_2) \sqsubseteq (y_1, y_2) \sqsubseteq (z_1, z_2)$. Then $x_1 \preccurlyeq y_1 \preccurlyeq z_1$ and $x_2 \succcurlyeq y_2 \succcurlyeq z_2$. Therefore $x_1 \preccurlyeq z_1$ and $x_2 \succcurlyeq z_2$, so $(x_1, x_2) \sqsubseteq (z_1, z_2)$. Conversely, assume that $\sqsubseteq$ is transitive and let $x \preccurlyeq y \preccurlyeq z$. Then $(x, z) \sqsubseteq (y, y) \sqsubseteq (z, x)$, which means that $(x, z) \sqsubseteq (z, x)$ and $x \preccurlyeq z$. Other properties are similar.

(2) Suppose that F has the mixed monotone property with respect to $\preccurlyeq$ and let $(x_1, y_1) \sqsubseteq (x_2, y_2)$. Then $x_1 \preccurlyeq x_2$ and $y_1 \succcurlyeq y_2$. Using the mixed monotone property

$$x_1 \preccurlyeq x_2 \ \Rightarrow\ F(x_1, y_1) \preccurlyeq F(x_2, y_1); \qquad y_2 \preccurlyeq y_1 \ \Rightarrow\ F(x_2, y_2) \succcurlyeq F(x_2, y_1).$$

As $\preccurlyeq$ is transitive, $F(x_1, y_1) \preccurlyeq F(x_2, y_1) \preccurlyeq F(x_2, y_2)$ implies that $F(x_1, y_1) \preccurlyeq F(x_2, y_2)$. Similarly

$$y_2 \preccurlyeq y_1 \ \Rightarrow\ F(y_2, x_2) \preccurlyeq F(y_1, x_2); \qquad x_1 \preccurlyeq x_2 \ \Rightarrow\ F(y_1, x_1) \succcurlyeq F(y_1, x_2).$$

Therefore $F(y_2, x_2) \preccurlyeq F(y_1, x_2) \preccurlyeq F(y_1, x_1)$ implies that $F(y_1, x_1) \succcurlyeq F(y_2, x_2)$. Hence

$$\begin{aligned} &F(x_1, y_1) \preccurlyeq F(x_2, y_2) \text{ and } F(y_1, x_1) \succcurlyeq F(y_2, x_2) \\ &\qquad \Leftrightarrow \quad (F(x_1, y_1), F(y_1, x_1)) \sqsubseteq (F(x_2, y_2), F(y_2, x_2)) \\ &\qquad \Leftrightarrow \quad T_F^2(x_1, y_1) \sqsubseteq T_F^2(x_2, y_2). \end{aligned}$$

Thus, T_F^2 is $\sqsubseteq$-non-decreasing.

(3) Assume that T_F is $\sqsubseteq$-non-decreasing and let $x_1, x_2, y \in X$ be such that $x_1 \preccurlyeq x_2$. As $y \preccurlyeq y$, then $(x_1, y) \sqsubseteq (x_2, y)$, so $T_F^2(x_1, y) \sqsubseteq T_F^2(x_2, y)$. This means that $(F(x_1, y), F(y, x_1)) \sqsubseteq (F(x_2, y), F(y, x_2))$ and, in particular, $F(x_1, y) \preccurlyeq F(x_2, y)$. The other condition can be proved similarly. □

Remark 5.14. From the original Ran and Reurings' theorem (see [34]), it is usual to consider partial orders to establish fixed point theorems in ordered metric spaces. However, the antisymmetric condition is not usually involved. Therefore, it could be sufficient to consider preordered spaces (where the relation is reflexive and transitive). However, as the other authors' main results have been stated in partially ordered spaces, we will also use this kind of spaces.

Taking into account the last Remark, we prefer particularize Corollary 4.22 to the ambient X^2 using T_F^2 rather than Theorem 4.21 (which could be also useful).

Theorem 5.15. *Let (X, q) be a right-complete quasi-metric space in which each right-convergent sequence has an unique right-limit and let $F : X^2 \to X$ be a mapping. Suppose that there is a preorder $\preccurlyeq$ on X such that F has the mixed monotone property and there exists $(\phi, \psi) \in \mathcal{F}$ satisfying*

$$\phi\left(q\left(F(x, y), F(u, v)\right) + q\left(F(y, x), F(v, u)\right)\right) \leq (\phi - \psi)\left(q(x, u) + q(y, v)\right) \tag{5.8}$$

for all $x, y, u, v \in X$ such that $x \preccurlyeq u$ and $y \succcurlyeq v$. Assume that there exist $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$. Also assume that, at least, one of the following conditions hold:

- *(A) F is right-continuous, or*
- *(B) (X, q) verifies the following two properties:*
  - *(B.1) If $\{x_n\} \subseteq X$ is a sequence in X and $u \in X$ are such that $\{q(x_n, u)\} \to 0$ and $x_n \preccurlyeq x_{n+1}$ for all n, then $x_n \preccurlyeq u$ for all n.*
  - *(B.2) If $\{y_n\} \subseteq X$ is a sequence in X and $v \in X$ are such that $\{q(y_n, v)\} \to 0$ and $y_n \succcurlyeq y_{n+1}$ for all n, then $y_n \succcurlyeq v$ for all n.*

Then F has, at least, a coupled fixed point. Furthermore, any coupled fixed point of F is of the form (x, x), where $x \in X$.

Additionally, assume that for all coupled fixed point (x, x) and (y, y) of F, there is $(z_1, z_2) \in X^2$ such that $z_1 \preccurlyeq x$, $z_1 \preccurlyeq y$, $z_2 \succcurlyeq x$ and $z_2 \succcurlyeq y$. Then F has a unique coupled fixed point.

Proof. As (X, q) be a right-complete, then (X^2, Q_s^q) is also right-complete, and by item 7 of Lemma 3.3, as each right-convergent sequence in (X, q) has an unique right-limit, then (X^2, Q_s^q) also verifies this property. Using the preorder $\preccurlyeq$, we could consider the preorder $\sqsubseteq$ on X^2 given by (5.7). Item 2 of Lemma 5.13 guarantees that T_F^2 is $\sqsubseteq$-non-decreasing, and the contractivity condition (5.8) can be written as

$$\begin{aligned}\phi\left(Q_s^q\left(T_F^2(x, y), T_F^2(u, v)\right)\right) &= \phi\left(q\left(F(x, y), F(u, v)\right) + q\left(F(y, x), F(v, u)\right)\right)\\ &\leq (\phi - \psi)\left(q(x, u) + q(y, v)\right)\\ &= (\phi - \psi)\left(Q_s^q\left((x, y), (u, v)\right)\right)\end{aligned}$$

for all $(x,y),(u,v)\in X^2$ such that $(x,y)\sqsubseteq(u,v)$. Moreover, $(x_0,y_0)\sqsubseteq T_F^2(x_0,y_0)$. Corollary 4.22 ensures that T_F^2 has a fixed point, which is a coupled fixed point of F.

Let (x,y) be any coupled fixed point of F. Then

$$\begin{aligned}\phi\left(q\left(x,y\right)+q\left(y,x\right)\right)&=\phi\left(q\left(F(x,y),F(y,x)\right)+q\left(F(y,x),F(x,y)\right)\right)\\&\le\left(\phi-\psi\right)\left(q(x,y)+q(y,x)\right)\\&\le\phi\left(q(x,y)+q(y,x)\right),\end{aligned}$$

which means that $\psi\left(q(x,y)+q(y,x)\right)=0$. Therefore $q(x,y)=0$ and $x=y$.

The additional condition means that $(z_1,z_2)\sqsubseteq(x,x)$ and $(z_1,z_2)\sqsubseteq(y,y)$, and we can also apply Corollary 4.22. □

The same proof is valid for the following result, in which we use Q_m^q.

Theorem 5.16. *Let (X,q) be a right-complete quasi-metric space in which each right-convergent sequence has an unique right-limit and let $F:X^2\to X$ be a mapping. Suppose that there is a preorder $\preccurlyeq$ on X such that F has the mixed monotone property and there exists $(\phi,\psi)\in\mathcal{F}$ satisfying*

$$\phi\left(q\left(F(x,y),F(u,v)\right)\right)\le\max\{\phi\left(q(x,u)\right),\phi\left(q(y,v)\right)\}-\psi\left(\max\{q(x,u),q(y,v)\}\right)\tag{5.9}$$

for all $x,y,u,v\in X$ such that $x\preccurlyeq u$ and $y\succcurlyeq v$. Assume that there exist $x_0,y_0\in X$ such that $x_0\preccurlyeq F(x_0,y_0)$ and $y_0\succcurlyeq F(y_0,x_0)$. Also assume that, at least, one of the following conditions hold:

(A) F is right-continuous, or

(B) (X,q) verifies the following two properties:

- *(B.1) If $\{x_n\}\subseteq X$ is a sequence in X and $u\in X$ are such that $\{q(x_n,u)\}\to 0$ and $x_n\preccurlyeq x_{n+1}$ for all n, then $x_n\preccurlyeq u$ for all n.*
- *(B.2) If $\{y_n\}\subseteq X$ is a sequence in X and $v\in X$ are such that $\{q(y_n,v)\}\to 0$ and $y_n\succcurlyeq y_{n+1}$ for all n, then $y_n\succcurlyeq v$ for all n.*

Then F has, at least, a coupled fixed point. Furthermore, any coupled fixed point of F is of the form (x,x), where $x\in X$.

Additionally, assume that for all coupled fixed point (x,x) and (y,y) of F, there is $(z_1,z_2)\in X^2$ such that $z_1\preccurlyeq x$, $z_1\preccurlyeq y$, $z_2\succcurlyeq x$ and $z_2\succcurlyeq y$. Then F has a unique coupled fixed point.

Proof. We only notice that the contractivity condition (5.9) is equivalent to the following one:

$$\begin{aligned}\phi\left(Q_m^q\left(T_F^2(x,y),T_F^2(u,v)\right)\right)&=\phi\left(\max\left\{q\left(F(x,y),F(u,v)\right),q\left(F(y,x),F(v,u)\right)\right\}\right)\\&=\max\left\{\phi\left(q\left(F(x,y),F(u,v)\right)\right),\phi\left(q\left(F(y,x),F(v,u)\right)\right)\right\}\\&\le\max\{\phi\left(q(x,u)\right),\phi\left(q(y,v)\right)\}-\psi\left(\max\{q(x,u),q(y,v)\}\right)\\&=\phi\left(\max\{q(x,u),q(y,v)\}\right)-\psi\left(\max\{q(x,u),q(y,v)\}\right)\\&=\left(\phi-\psi\right)\left(Q_m^q\left((x,y),(u,v)\right)\right)\end{aligned}$$

for all $(x,y),(u,v)\in X^2$ such that $(x,y)\sqsubseteq(u,v)$. □

5.4. **Coupled fixed point theorems in partially ordered G-metric spaces.** We state Theorems 5.15 and 5.16 in the case in which $q(x,y) = G(x,y,y)$ for some G^*-metric G on X, and we obtain the following results. Recall that in a G^*-metric space, right/left convergent (respectively, Cauchy) sequences are the same, and that every (right-)convergent sequence has a unique (right-)limit.

Corollary 5.17. *Let (X,G) be a complete G^*-metric space and let $F : X^2 \to X$ be a mapping. Suppose that there is a preorder $\preccurlyeq$ on X such that F has the mixed monotone property and there exists $(\phi,\psi) \in \mathcal{F}$ satisfying*

$$\phi\left(G\left(F(x,y),F(u,v),F(u,v)\right)+G\left(F(y,x),F(v,u),F(v,u)\right)\right) \leq (\phi-\psi)\left(G(x,u,u)+G(y,v,v)\right) \tag{5.10}$$

for all $x,y,u,v \in X$ such that $x \preccurlyeq u$ and $y \succcurlyeq v$. Assume that there exist $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0,y_0)$ and $y_0 \succcurlyeq F(y_0,x_0)$. Also assume that, at least, one of the following conditions hold:

- *(A) F is continuous,* or
- *(B) (X,q) verifies the following two properties:*
  - *$(B.1)$ If $\{x_n\} \subseteq X$ is a sequence in X and $u \in X$ are such that $\{x_n\} \xrightarrow{G} u$ and $x_n \preccurlyeq x_{n+1}$ for all n, then $x_n \preccurlyeq u$ for all n.*
  - *$(B.2)$ If $\{y_n\} \subseteq X$ is a sequence in X and $v \in X$ are such that $\{y_n\} \xrightarrow{G} v$ and $y_n \succcurlyeq y_{n+1}$ for all n, then $y_n \succcurlyeq v$ for all n.*

Then F has, at least, a coupled fixed point. Furthermore, any coupled fixed point of F is of the form (x,x), where $x \in X$.

Additionally, assume that for all coupled fixed point (x,x) and (y,y) of F, there is $(z_1,z_2) \in X^2$ such that $z_1 \preccurlyeq x$, $z_1 \preccurlyeq y$, $z_2 \succcurlyeq x$ and $z_2 \succcurlyeq y$. Then F has a unique coupled fixed point.

Corollary 5.18. *Let (X,G) be a complete G^*-metric space and let $F : X^2 \to X$ be a mapping. Suppose that there is a preorder $\preccurlyeq$ on X such that F has the mixed monotone property and there exists $(\phi,\psi) \in \mathcal{F}$ satisfying*

$$\phi\left(G\left(F(x,y),F(u,v),F(u,v)\right)\right) \leq \max\{\phi\left(G(x,u,u)\right),\phi\left(G(y,v,v)\right)\} - \psi\left(\max\{G(x,u,u),G(y,v,v)\}\right) \tag{5.11}$$

for all $x,y,u,v \in X$ such that $x \preccurlyeq u$ and $y \succcurlyeq v$. Assume that there exist $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0,y_0)$ and $y_0 \succcurlyeq F(y_0,x_0)$. Also assume that, at least, one of the following conditions hold:

- *(A) F is continuous,* or
- *(B) (X,q) verifies the following two properties:*
  - *$(B.1)$ If $\{x_n\} \subseteq X$ is a sequence in X and $u \in X$ are such that $\{x_n\} \xrightarrow{G} u$ and $x_n \preccurlyeq x_{n+1}$ for all n, then $x_n \preccurlyeq u$ for all n.*
  - *$(B.2)$ If $\{y_n\} \subseteq X$ is a sequence in X and $v \in X$ are such that $\{y_n\} \xrightarrow{G} v$ and $y_n \succcurlyeq y_{n+1}$ for all n, then $y_n \succcurlyeq v$ for all n.*

Then F has, at least, a coupled fixed point. Furthermore, any coupled fixed point of F is of the form (x,x), where $x \in X$.

Additionally, assume that for all coupled fixed point (x,x) and (y,y) of F, there is $(z_1,z_2)\in X^2$ such that $z_1\preccurlyeq x$, $z_1\preccurlyeq y$, $z_2\succcurlyeq x$ and $z_2\succcurlyeq y$. Then F has a unique coupled fixed point.

The following results also hold for G^*-metric spaces, but we enunciate them in G-metric spaces to respect the original versions.

5.4.1. *Choudhury and Maity's coupled fixed point results in G-metric spaces.* Choudhury and Maity [12] proved the following coupled fixed point theorems on ordered G-metric spaces.

Theorem 5.19 (Choudhury and Maity [12], Theorem 3.4)**.** *Let $(X,\preccurlyeq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Let $F: X\times X\to X$ be G-continuous mapping having the mixed monotone property on X. Suppose that there exists a $k\in[0,1)$ such that*

$$G(F(x,y),F(u,v),F(w,z))\leq\frac{k}{2}[G(x,u,w)+G(y,v,z)] \tag{5.12}$$

for all $x,y,u,v,w,z\in X$ with $x\preccurlyeq u\preccurlyeq w$ and $y\succcurlyeq v\succcurlyeq z$ where either $u\neq w$ or $v\neq z$. If there exist $x_0,y_0\in X$ such that $x_0\preccurlyeq F(x_0,y_0)$ and $F(y_0,x_0)\succcurlyeq y_0$, then F has a coupled fixed point, that is, there exists $(x,y)\in X\times X$ such that $x=F(x,y)$ and $y=F(y,x)$.

Theorem 5.20. *If in the above theorem, instead of G-continuity of F, we assume that X is ordered complete, then F has a coupled fixed point.*

We prove that the previous results can be improved as follows.

Corollary 5.21. *Theorems 5.19 and 5.20 also hold if G is a G^*-metric.*

Proof. It follows from Corollary 5.17 using $\phi(t)=t$ and $\psi(t)=(1-k)t$, for all $t\geq 0$. □

5.4.2. *Aydi* et al.'s *coupled fixed point results in G^*-metric spaces.* We consider next fixed point theorems established by Aydi *et al.* [11]. Let denote by Ω the set of functions $\varphi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(Ω_1) $\varphi^{-1}(\{0\})=0$,
(Ω_2) $\varphi(t)<t$ for all $t>0$;
(Ω_3) $\lim_{r\to t^+}\varphi(r)<t$.

As $\varphi(0)=0$, we notice that $\varphi(t)\leq t$ for all $t\geq 0$. The following property is trivial.

Lemma 5.22. *(See [11]) Let $\varphi\in\Omega$. For all $t>0$, we have $\lim_{n\to\infty}\varphi^n(t)=0$.*

Aydi *et al.* [6] proved the following fixed point theorems.

Theorem 5.23 (Aydi *et al.*, Theorem 3.1)**.** *Let $(X,\preccurlyeq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a G-complete G-metric space. Suppose that there exist $\varphi\in\Omega$ and $F: X\times X\to X$ such that*

$$G(F(x,y),F(u,v),F(w,z))\leq\varphi\left(\frac{G(x,u,w)+G(y,v,z)}{2}\right) \tag{5.13}$$

for all $x, y, u, v, w, z \in X$ with $x \preccurlyeq u \preccurlyeq w$ and $y \succcurlyeq v \succcurlyeq z$. Suppose also that F is G-continuous and has the mixed monotone property. If there exist $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, then F has a coupled fixed point, that is, there exists $(x, y) \in X^2$ such that $x = F(x, y)$ and $y = F(y, x)$.

Replacing the G-continuity of F by ordered completeness of X yields the next result.

Theorem 5.24 (Aydi *et al.*, Theorem 3.2)**.** *Let $(X, \preccurlyeq)$ be a partially ordered set and G be a G-metric on X such that $(X, G, \preccurlyeq)$ is G-complete. Suppose that there exist $\varphi \in \Omega$ and $F : X \times X \to X$ such that*

$$G(F(x,y), F(u,v), F(w,z)) \leq \varphi\left(\frac{G(x,u,w) + G(y,v,z)}{2}\right) \tag{5.14}$$

for all $x, y, u, v, w, z \in X$ with $x \preccurlyeq u \preccurlyeq w$ and $y \succcurlyeq v \succcurlyeq z$. Suppose also that F has the mixed monotone property and X is ordered complete. If there exist $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, then F has a coupled fixed point, that is, there exists $(x, y) \in X^2$ such that $x = F(x, y)$ and $y = F(y, x)$.

Next, we prove more general statements using complete G^*-metrics and a weaker contractivity condition, taking into account the following fact.

Lemma 5.25. *If $\varphi \in \Omega$ and $\phi(t) = t$ for all $t \geq 0$, then $(\phi, \phi - \varphi) \in \mathcal{F}$.*

Proof. Let $\psi = \phi - \varphi$. Clearly, ϕ is non-decreasing and $\phi(t) = 0 \Leftrightarrow t = 0 \Leftrightarrow \psi(t) = 0$. Let $\{a_k\}, \{b_k\} \subset [0, \infty)$ be sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verifying $L < b_k$ and $\phi(b_k) \leq (\phi - \psi)(a_k)$ for all k. This means that $L < b_k = \phi(b_k) \leq (\phi - \psi)(a_k) = \varphi(a_k) \leq a_k$ for all k. Letting $k \to \infty$, we deduce that $\lim_{k\to\infty} \varphi(a_k) = L$. If $L > 0$, then, by (Ω_3),

$$L = \lim_{k\to\infty} \varphi(a_k) = \lim_{r\to L^+} \varphi(r) < L,$$

which is impossible. Therefore $L = 0$ and $(\phi, \phi - \varphi) \in \mathcal{F}$. □

Corollary 5.26. *Theorems 5.23 and 5.24 also hold even if G is a G^*-metric and even replacing inequality (5.13) by*

$$G(F(x,y), F(u,v), F(u,v)) \leq \varphi\left(\frac{G(x,u,u) + G(y,v,v)}{2}\right)$$

for all $x, y, u, v \in X$ with $x \preccurlyeq u$ and $y \succcurlyeq v$.

Proof. It is only necessary to apply Corollary 5.17 and Lemma 5.25 to the case in which $\phi(t) = t$ for all $t \geq 0$ and $\psi = \phi - \varphi$. Notice that the same proof of Lemma 3.4 shows that

$$G'((x,y), (u,v), (w,z)) = \frac{G(x,u,w) + G(y,v,z)}{2} \quad \text{for all } (x,y), (u,v), (w,z) \in X^2$$

is a complete G^*-metric on X^2 verifying $G' = G_s^2/2$. □

5.4.3. *On Coupled fixed point results by Abbas* et al. *in G-metric spaces.* Let Θ be the set of functions $\theta : [0,\infty)^2 \to [0,1)$ which satisfy the condition:

$$\{\theta(t_n, s_n)\} \to 1 \quad \Rightarrow \quad [\ \{t_n\} \to 0 \text{ and } \{s_n\} \to 0\ ].$$

The following theorems have been given by Abbas *et al.* [1].

Theorem 5.27 (Abbas *et al.* [1], Theorem 3.1)**.** *Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete G-metric on X and $F : X \times X \to X$ be a continuous mapping having the mixed monotone property. Suppose that there exists $\theta \in \Theta$ such that*

$$\begin{aligned}(5.15)\quad & G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(w,z)) \\ & \qquad \leq \theta(G(x,u,w),G(y,v,z))\ [G(x,u,w) + G(y,v,z)]\end{aligned}$$

for all $x,y,z,u,v,w \in X$ for which $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$ where either $u \neq w$ or $v \neq z$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0,y_0) \quad \textit{and} \quad y_0 \succcurlyeq F(y_0,x_0),$$

then F has a coupled fixed point.

In the following result, F is not necessarily continuous.

Theorem 5.28 (Abbas *et al.* [1], Theorem 3.2)**.** *Let $(X, \preccurlyeq)$ be a partially ordered set such that there exists a complete G-metric on X and $F : X \times X \to X$ be a mapping having the mixed monotone property. Suppose that there exists $\theta \in \Theta$ such that*

$$\begin{aligned}(5.16)\quad & G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(w,z)) \\ & \qquad \leq \theta(G(x,u,w),G(y,v,z))\ [G(x,u,w) + G(y,v,z)]\end{aligned}$$

for all $x,y,z,u,v,w \in X$ for which $x \succcurlyeq u \succcurlyeq w$ and $y \preccurlyeq v \preccurlyeq z$ where either $u \neq w$ or $v \neq z$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0,y_0) \quad \textit{and} \quad y_0 \succcurlyeq F(y_0,x_0),$$

and X has the following property:

(*i*) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$,*
(*ii*) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all $n \in \mathbb{N}$,*

then F has a coupled fixed point.

We extend the previous results to the case in which G is a G^*-metric on X taking into account the following fact.

Lemma 5.29. *Let $\theta \in \Theta$ and define $\beta_\theta : [0,\infty) \to \mathbb{R}$, for all $t \geq 0$, by:*

$$\beta_\theta(t) = \begin{cases} \sup\left(\{\theta(s,r) : s + r \geq t\}\right), & \textit{if } t > 0, \\ \theta(0,0), & \textit{if } t = 0. \end{cases}$$

Then the following properties hold.

(1) *$\beta_\theta(t) \in [0,1)$ for all $t \in [0,\infty)$.*
(2) *β_θ is a Geraghty function.*
(3) *$\theta(t,s) \leq \beta_\theta(t+s)$ for all $t,s \geq 0$.*
(4) *If $\phi(t) = t$ and $\psi(t) = t - \beta_\theta(t)t$ for all $t \geq 0$, then $(\phi,\psi) \in \mathcal{F}$.*

Proof. (1) As $0 \leq \theta(t,s) < 1$ for all $t,s \in [0,\infty)$, we have that $0 \leq \beta_\theta(t) \leq 1$ for all $t \in [0,\infty)$. Clearly $\beta_\theta(0) = \theta(0,0) < 1$. Suppose that there is $t_0 > 0$ such that $\beta_\theta(t_0) = 1$. Therefore there are sequences $\{s_n\}$ and $\{r_n\}$ such that $s_n + r_n \geq t_0$ verifying $\{\theta(s_n,r_n)\} \to 1$. As $\theta \in \Theta$, then $\{s_n\} \to 0$ and $\{r_n\} \to 0$, but this is a contradiction with the fact that $s_n + r_n \geq t_0 > 0$ for all $n \in \mathbb{N}$.

(2) Let $\{t_n\} \subset [0,\infty)$ be a sequence such that $\{\beta_\theta(t_n)\} \to 1$. Then there is $n_0 \in \mathbb{N}$ such that $t_n > 0$ and $\beta(t_n) > 0$ for all $n \geq n_0$. Let define $\varepsilon_n = \min(1/n, \beta_\theta(t_n)/2) > 0$ for all $n \geq n_0$. As $\beta_\theta(t_n)$ is a supremum and $\varepsilon_n > 0$, for all $n \geq n_0$, there are $s_n, r_n \in [0,\infty)$ such that $s_n + r_n \geq t_n$ and $1 - \varepsilon_n < \theta(s_n,r_n) \leq \beta_\theta(t_n) < 1$. This process define two sequences $\{s_n\}_{n \geq n_0}$ and $\{t_n\}_{n \geq n_0}$ such that $1 - \varepsilon_n < \theta(s_n,r_n) < 1$ for all $n \geq n_0$. As $\{\varepsilon_n\} \to 0$, we have that $\{\theta(s_n,r_n)\} \to 1$. As $\theta \in \Theta$, we deduce that $\{s_n\} \to 0$ and $\{r_n\} \to 0$, so $\{t_n\} \to 0$. This proves that β_θ is a Geraghty function.

(3)If $t = s = 0$, then $\theta(0,0) = \beta_\theta(0)$. If $t + s > 0$, then $\theta(t,s) \leq \sup(\{\theta(u,v) : u + v \geq t + s\}) = \beta_\theta(t+s)$.

(4) It follows from Lemma 4.5. □

Corollary 5.30. *Theorems 5.27 and 5.28 also hold even if G is a G^*-metric.*

Proof. As $\theta \in \Theta$, item 4 of Lemma 5.29 guarantees that $(\phi,\psi) \in \mathcal{F}$, where $\phi(t) = t$ and $\psi(t) = t - \beta_\theta(t)t$ for all $t \geq 0$. Notice that $(\phi - \psi)(t) = t - (t - \beta_\theta(t)t) = \beta_\theta(t)t$ for all $t \geq 0$. Hence,

$$\begin{aligned}
&\phi\left(G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w))\right)\\
&\quad = G(F(x,y),F(u,v),F(w,z)) + G(F(y,x),F(v,u),F(z,w))\\
&\quad \leq \theta(G(x,u,w),G(y,v,z))\;[G(x,u,w) + G(y,v,z)]\\
&\quad \leq \beta_\theta(G(x,u,w) + G(y,v,z))\;[G(x,u,w) + G(y,v,z)]\\
&\quad = (\phi - \psi)\left(G(x,u,w) + G(y,v,z)\right).
\end{aligned}$$

As F is G-continuous, Corollary 5.17 implies that F has a coupled fixed point. The case in which $(X,G,\preccurlyeq)$ is regular is also included in Corollary 5.17. □

6. Applications to tripled fixed point theorems in the frameworks of quasi-metric spaces and G^*-metric spaces

Exactly the same arguments of the previous section can be applied in order to obtain tripled/quadrupled fixed point results. We only show some examples, but they can be easily generalized.

6.1. **Tripled fixed point theorem by Aydi, Karapınar and Shatanawi in partially ordered G-metric spaces.** Let Ω' be the set of all non-decreasing functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. If $\varphi \in \Omega'$, then following Matkowski [30], we have: (1) $\varphi(t) < t$ for all $t > 0$; (2) $\varphi(0) = 0$. In particular, $\varphi(t) \leq t$ for all $t \geq 0$.

Using this kind of test functions, Aydi *et al.* [5] proved the following result.

Theorem 6.1 (Aydi, Karapınar and Shatanawi [5], Theorem 2.1)**.** *Let $(X,\preccurlyeq)$ be partially ordered set and (X,G) a G-metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume there exists $\varphi \in \Omega'$*

such that, for $x, y, z, a, b, c, u, v, w \in X$, with $x \preccurlyeq a \preccurlyeq u$, $y \succcurlyeq b \succcurlyeq v$, and $z \preccurlyeq c \preccurlyeq w$, one has

(6.1)

$$G\left(F(x,y,z), F(a,b,c), F(u,v,w)\right) \leq \varphi\left(\max\left\{G(x,a,u), G(y,b,v), G(z,c,w)\right\}\right).$$

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0, z_0)$, $y_0 \succcurlyeq F(y_0, x_0, y_0)$ and $z_0 \preccurlyeq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X, that is, there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y \quad \text{and} \quad F(z,y,x) = z.$$

In order to extend this result to the framework of G^*-metric spaces, we must take into account the following considerations. Let (X, q) be a quasi-metric space and define $\tilde{Q}_s^q, \tilde{Q}_m^q : (X^3)^2 \to [0,\infty)$, for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, by:

$$\tilde{Q}_s^q((x_1,y_1,z_1),(x_2,y_2,z_2)) = q(x_1,x_2) + q(y_1,y_2) + q(z_1,z_2);$$
$$\tilde{Q}_m^q((x_1,y_1,z_1),(x_2,y_2,z_2)) = \max\left\{q(x_1,x_2), q(y_1,y_2), q(z_1,z_2)\right\}.$$

Then $(X^3, \tilde{Q}_s^q)$ and $(X^3, \tilde{Q}_m^q)$ are quasi-metric spaces. Moreover, all properties of Lemma 3.3 also hold. Given a mapping $F : X^3 \to X$, let denote by $T_F^3 : X^3 \to X^3$ the mapping

$$T_F^3(x,y,z) = (F(x,y,z), F(y,x,y), F(z,y,x)) \qquad \text{for all } (x,y,z) \in X^3.$$

A tripled fixed point of F is nothing but a fixed point of T_F^3.

Furthermore, given a binary relation $\preccurlyeq$ on X, let define

$$(x_1,y_1,z_1) \sqsubseteq (x_2,y_2,z_2) \;\Leftrightarrow\; [x_1 \preccurlyeq x_2,\; y_1 \succcurlyeq y_2 \text{ and } z_1 \preccurlyeq z_2\,].$$

As in Lemma 5.13, if F has the mixed monotone property with respect to $\preccurlyeq$, then T_F^3 is $\sqsubseteq$-non-decreasing.

The following lemma lets us to show how Theorem 6.1 is also valid if G is a G^*-metric.

Lemma 6.2. *If $\varphi \in \Omega'$ and we define $\phi(t) = t$ and $\psi(t) = t - \varphi(t)$ for all $t \geq 0$, then $(\phi, \psi) \in \mathcal{F}$.*

Proof. Clearly ϕ is non-decreasing. If there exists $t_0 \in [0,\infty)$ such that $\psi(t_0) = 0$, then $\varphi(t_0) = t_0$, which is only possible when $t_0 = 0$. In this case, $\phi^{-1}(0) = \{0\}$. Finally, to prove $(\mathcal{F}_3)$, let $\{a_k\}, \{b_k\} \subset [0,\infty)$ be sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verifying $L < b_k$ and $\phi(b_k) \leq (\phi - \psi)(a_k)$ for all k. We will prove that $L = 0$ reasoning by contradiction. Assume that $L > 0$. Therefore

$$L < b_k = \phi(b_k) \leq (\phi - \psi)(a_k) = \varphi(a_k) \leq a_k \quad \text{for all } k,$$

which means that

$$\lim_{k\to\infty} \varphi(a_k) = L.$$

As φ is non-decreasing, there exists the limit

$$L' = \lim_{s \to L^+} \varphi(s).$$

As $\{a_k\}$ is a sequence converging to L and $a_k > L$ for all k, and the previous limit exists, then

$$L' = \lim_{s \to L^+} \varphi(s) = \lim_{k \to \infty} \varphi(a_k) = L.$$

Next, we claim that $\varphi(t) > L$ for all $t > L$. Assume that there is $t_0 \in]L, \infty[$ such that $\varphi(t_0) \leq L$. As φ is non-decreasing, $L = \lim_{s \to L^+} \varphi(s) \leq \varphi(t_0) \leq L$. Hence, it follows that $\varphi(s) = L$ for all $s \in]L, t_0]$, but this is a contradiction with the fact that $L < b_k \leq \varphi(a_k)$ for all k, being $\{a_k\} \to L$. This contradiction proves that $\varphi(t) > L$ for all $t > L$. In such a case, notice that

$$a_1 > L \quad \Rightarrow \quad \varphi(a_1) > L \quad \Rightarrow \quad \varphi^2(a_1) > L \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \varphi^n(a_1) > L$$

for all $n \in \mathbb{N}$, which contradicts the fact that $\lim_{n \to \infty} \varphi^n(a_1) = 0$. This contradiction shows that $L = 0$ and $(\phi, \psi) \in \mathcal{F}$. □

Corollary 6.3. *Theorem 6.1 also holds even if G is a G^*-metric.*

Proof. It follows from Lemma 6.2 and the fact that the contractivity condition (6.1) can be seen as condition (4.13) in Corollary 4.22 using T_F^3 in the ordered G^*-metric space $(X^3, \tilde{Q}_m^{q_G}, \sqsubseteq)$, where $q_G(x, y) = G(x, y, y)$ for all $x, y \in X$. □

6.2. **Tripled fixed point theorem by Mohiuddine and Alotaibi in partially ordered G-metric spaces.** Let Θ' be the set of functions $\theta : [0, \infty)^3 \to [0, 1)$ which satisfy the condition:

$$\{\theta(t_n, s_n, r_n)\} \to 1 \quad \Rightarrow \quad [\ \{t_n\} \to 0,\ \{s_n\} \to 0 \text{ and } \{r_n\} \to 0\ ].$$

Using this kind of test functions, Mohiuddine and Alotaibi [32] presented the following result.

Theorem 6.4 (Mohiuddine and Alotaibi [32], Theorem 2.1)**.** *Let $(X, \preccurlyeq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that $F : X \times X \times X \to X$ is a continuous mapping having the mixed monotone property. Assume that there exists $\theta \in \Theta'$ such that*

$$\begin{aligned} &G(F(x,y,z), F(s,t,u), F(p,q,r)) \\ &\quad + G(F(y,x,z), F(t,s,u), F(q,p,r)) + G(F(z,y,x), F(u,t,s), F(r,q,p)) \\ &\qquad \leq \theta\,(G(x,s,p), G(y,t,q), G(z,u,r))\ [\ G(x,s,p) + G(y,t,q) + G(z,u,r)\ ] \end{aligned}$$

for all $x, y, z, s, t, u, p, q, r \in X$ with $x \succcurlyeq s \succcurlyeq p$ and $y \preccurlyeq t \preccurlyeq q$ and $z \succcurlyeq u \succcurlyeq r$, where either $s \neq p$ or $t \neq q$ or $u \neq r$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0, y_0, z_0), \quad y_0 \succcurlyeq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \preccurlyeq F(z_0, y_0, x_0)$$

then F has a tripled fixed point; that is, there exist $x, y, z \in X$ such that

$$x = F(x, y, z), \quad y = F(y, x, y) \quad \text{and} \quad z = F(z, y, x).$$

Remark 6.5. Notice that in the statement of Theorem 6.4 there is a gap. It is clear that in the expression

$$G(F(y,x,z), F(t,s,u), F(q,p,r)) \tag{6.2}$$

is not coherent with the notion of mixed monotone property. Theorem 6.4 can be corrected by replacing (6.2) by the term

$$G(F(y,x,y),F(t,s,t),F(q,p,q)).$$

The given proof in [32] is valid for our suggested version.

Extending to the tripled case the techniques showed in Subsection 5.4.3 and following the same notation as in Subsection 6.1, it is not difficult to prove the following result.

Corollary 6.6. *Theorem 6.4 also holds even if G is a G^*-metric.*

The previous technique can also be applied to prove quadrupled or even multidimensional fixed point results. For instance, in [35], Roldán and Karapınar showed how to extend unidimensional fixed point results to the multidimensional case, using a mapping $\mathbb{F}_\Upsilon : X^n \to X^n$ which is defined using $F : X^n \to X$. As the test functions we use here are more general than used in [35], we point out that all results in [35] are also consequences of Theorems 4.8 (in the non-ordered case) and 4.21 (in the partially ordered case).

7. Conclusion

In this paper, we examine several multidimensional (coupled, tripled and so on) fixed point theorem under various contraction condition in the context of G. One of the outcomes of this paper is that all discussed these multidimensional fixed point theorems are consequences of either Theorem 4.8 or 4.21. More precisely all mentioned multidimensional (coupled, tripled and so on) results in the context of G-metric spaces can be concluded from a fixed point theorem in the setting of quasi metric spaces, in particular from 4.8 or 4.21. In this case, it can be considered as a subsequent of [9, 19, 42]. Second interesting conclusion of this paper is the following: Most of the multidimensional fixed point theorems can be concluded from the uni-dimensional (one-dimensional) fixed point theorem that trend was initiated by Samet *et al.* [41], Agarwal and Karapinar [2], Karapinar *et al.* [24],Roldán *et al.* [37]. We also underline that the product of G^*-metric space is again G^*-metric space that improves the investigations of G-metric space theory.

References

[1] M. Abbas, W. Sintunavarat, P. Kumam, *Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces*, Fixed Point Theory Appl. 2012, 2012:31

[2] R. Agarwal and E. Karapinar, *Remarks on some coupled fixed point theorems in G-metric spaces*, Fixed Point Theory and Applications 2013, 2013:2

[3] H. Aydi, *A fixed point result involving a generalized weakly contractive condition in G-metric spaces*, Bulletin Math. Anal. Appl. **3** (2011), 180–188.

[4] M. Asadi, E. Karapinar and P. Salimi, A new approach to G-metric and related fixed point theorems, Journal of Inequalities and Applications (2013) 2013:454.

[5] H. Aydi, E. Karapınar and W. Shatanawi, *Tripled fixed point results in generalized metric spaces*, J. Appl. Math., **2012**, Article ID 314279, 10 pages

[6] H. Aydi, B. Damjanović, B. Samet and W. Shatanawi, *Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces*, Math. Comput. Modelling **54** (2011), 2443–2450.

[7] V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal. **74** (2011), 7347–7355.
[8] V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal. **74** (2011), 4889–4897.
[9] N. Bilgili, E. Karapinar and B. Samet, *Generalized alpha psi contractive mappings in quasi-metric spaces and related fixed point theorems*, J. Ineq. Appl., accepted.
[10] M. Borcut and V. Berinde, *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput. **218** (2012), 5929–5936.
[11] L. Ćirić and V. Lakshmikantham, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), 4341–4349.
[12] B. S. Choudhury and P. Maity, *Coupled fixed point results in generalized metric spaces*, Math. Comput. Modelling **54** (2011), 73–79.
[13] M. Deza and E. Deza, *Quasi-metrics, directed multicuts and related polyhedra*, European J. Combinatorics **21** (2000), 777–796.
[14] J. García-Falset, L. Guran and E. Llorens-Fuster, *Fixed points for multivalued contractions with respect to a w-distance*, Sci. Math. Jpn. **71** (2010), 83–91.
[15] L. Gholizadeh, *A fixed point theorem in generalized ordered metric spaces with application*, J. Nonlinear Sci. Appl. **6** (2013), 244–251.
[16] L. Gholizadeh, R. Saadati, W. Shatanawi, S. M. Vaezpour, *Contractive mapping in generalized ordered metric spaces with application in integral equations*, Math. Probl. Eng., volume 2011, Article ID 380784, 14 pages.
[17] T. Gnana-Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis **65** (2006), 1379–1393.
[18] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., Theory Methods Appl. **11** (1987), 623–632.
[19] M. Jleli and B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appl. 2012, 2012:210, 7 pages.
[20] D. Ilic, V. Rakocevic, *Common fixed points for maps on metric space with w-distance*, Applied Mathematics and Computation **199** (2008), 599–610.
[21] O. Kada, T. Suzuk and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete matrix spaces*, Math. Japonica **44** (1996), 381–391.
[22] E. Karapınar, *Quadruple fixed point theorems for weak ϕ-contractions*, ISRN Mathematical Analysis, 2011 (2011), Article ID 989423, 16 pages.
[23] E. Karapınar and N. V. Luong, *Quadruple fixed point theorems for nonlinear contractions*, Comput. Math. Appl. **64** (2012), 1839–1848.
[24] Karapinar, A. Roldan, C. I. Roldan and J. Martinez-Moreno, *A note on 'N-fixed point theorems for nonlinear contractions in partially ordered metric spaces'*, Fixed Point Theory and Appl. (2013) 2013:310
[25] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), 1–9.
[26] H. P. Künzi and M. P. Schellekens, *On the Yoneda completion of a quasi-metric space*, Theoretical Computer Science **278** (2002), 159–194.
[27] H. Lakzian, H. Aydi and B. E. Rhoades, *Fixed points for (ϕ, ψ, p)-weakly contractive mappings in metric spaces with w-distance*, Applied Mathematics and Computation **219** (2013), 6777–6782.
[28] L.-J. Lin and C.-S. Chuang, *Some new fixed point theorems of generalized nonlinear contractive multivalued maps in complete metric spaces*, Computers and Mathematics with Applications **62** (2011), 3555–3566.
[29] J. Marín, S. Romaguera and P. Tirado, *Weakly contractive multivalued maps and w-distances on complete quasi-metric spaces*, Fixed Point Theory and Applications 2011, 2011:2. Soc. **62** (1977), 344–348.
[30] J. Matkowski, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. American Math.

[31] S. G. Matthews, *Partial metric topology*, Research Report 212. Dept. of Computer Science. Univ. of Warwick, 1992.
[32] S. A. Mohiuddine and A. Alotaibi, *Some results on a tripled fixed point for nonlinear contractions in partially ordered G-metric spaces*, Fixed Point Theory Appl. 2012, 2012:179.
[33] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), 289–297.
[34] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.
[35] A. Roldán and E. Karapınar, *Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under (ψ, ϕ)-contractivity conditions*, Fixed Point Theory Appl. 2013 2013:158.
[36] A. Roldán, J. Martínez-Moreno and C. Roldán, *Multidimensional fixed point theorems in partially ordered complete metric spaces*, J. Math. Anal. Appl. **396** (2012), 536–545.
[37] A. Roldan, J. Martinez-Moreno, C. Roldán and E. Karapinar, *Some remarks on multidimensional fixed point theorems*, Fixed Point Theory (2013) (accepted).
[38] A. Roldán, J. Martínez-Moreno, C. Roldán and Y. J. Cho, *Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces*, Fuzzy Sets Syst., D.O.I. 10.1016/j.fss.2013.10.009
[39] S. Romaguera and M. Schellekens, *Quasi-metric properties of complexity spaces*, Topology and its Applications **98** (1999), 311–322.
[40] R. Saadati, S. M. Vaezpour, P. Vetro and B. E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Model. **52** (2010), 797–801.
[41] B. Samet, E. Karapinar, H. Aydi and V. C. Rajic, *Discussion on some coupled fixed point theorems*, Fixed Point Theory Appl. 2013, 2013:50
[42] B. Samet, C. Vetro and F. Vetro , *Remarks on G-metric spaces*, Int. J. Anal. **2013** (2013), Article ID 917158, 6 pages
[43] W. Shatanawi, *Coupled fixed point theorems in generalized metric spaces*, Hacettepe J. Math. Stat. **40** (2011), 441–447.
[44] H. Triebel, *A new approach to function spaces on quasi-metric spaces*, Rev. Mat. Complut. **18** (2005), 7–48.
[45] W. A. Wilson, *On quasi-metric spaces*, American J. Math. **53** (1931), 675–684

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