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APPROXIMATING COMMON FIXED POINTS OF (a, b)-MONOTONE AND NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Dedicated to Prof. W. Takahashi on the occasion of his 70th birthday

Abstract. In this paper, we prove weak and strong convergence theorems for Moudafi's iterative scheme including (a, b)-monotone and nonexpansive mappings in Hilbert spaces and give some examples to illustrate our main results. The results in this paper improve and extend the recent results of Iemoto and Takahashi, Lin and Wang and some others.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the set of fixed points of a mapping $T : C \to C$ by $F(T)$. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if the set of fixed points of T is nonempty and

$$\|Tx - y\| \leq \|x - y\|$$

for all $x \in C$ and $y \in F(T)$. If $T : C \to C$ is nonexpansive and the set of fixed points of T is nonempty, then T is quasi-nonexpansive. Furthermore, F is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$. Every firmly nonexpansive mapping is nonexpansive. For some important examples of nonexpansive mappings and mappings of the from $F = \frac{1}{2}(I + T)$ with a nonexpansive mapping T, see [2, 3].

In 2008, Kohsaka and Takahashi [6] studied the existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces. They [7] also introduced the class of mappings called the class of nonspreading mappings.

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Let E be a real smooth, strictly convex and reflexive Banach space and j denote the duality mapping of E. Let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$ for all $x, y \in E$. If E is a Hilbert space, then we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. Thus a nonspreading mapping $S : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \le \|Sx - y\|^2 + \|x - Sy\|^2 \tag{1.1}$$

for all $x, y \in C$. It is well known ([4]) that (1.1) is equivalent to

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + 2\langle x - Sx, y - Sy\rangle \tag{1.2}$$

for all $x, y \in C$.

We know that, in a Hilbert space H, every firmly nonexpansive mapping is nonspreading and, if the set of fixed points of a nonspreading mapping is nonempty, then every nonspreading mapping is quasi-nonexpansive ([7]). In 2010, Takahashi [16] introduced the class of hybrid mappings in Hilbert spaces, that is, a mapping $T : C \to C$ in a Hilbert space H is said to be *hybrid* if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|x - Ty\|^2$$

for all $x, y \in C$. The class of hybrid mappings contains the class of firmly nonexpansive mappings in Hilbert spaces. Recently, Takahashi and Yao [17] introduced the new class of mappings $T : C \to C$ in a Hilbert space H, that is, a mappings $T : C \to C$ is called a TY-*mapping* if

$$2\|Sx - Sy\|^2 \le \|x - y\|^2 + \|Sx - y\|^2 \tag{1.3}$$

for all $x, y \in C$.

Very recently, Lin and Wang [8] introduced the new class of mappings $T : C \to C$ in a Hilbert space H, that is, a mappings $T : C \to C$ is said to be (a, b)-*monotone* if

$$\langle x - y, Tx - Ty\rangle \ge a\|Tx - Ty\|^2 + (1 - a)\|x - y\|^2 - b\|x - Tx\|^2 - b\|y - Ty\|^2 \tag{1.4}$$

for all $x, y \in C$, where $a \in (\frac{1}{2}, \infty)$ and $b \in (-\infty, a)$. This class contains the classes of nonspeading mappings, hybrid mappings and TY-mappings (for more details, see [8]).

Remark 1.1. From [8], it follows that every (a, b)-monotone mapping is not necessary quasi-nonexpansive, nonspreadings, TY, hybrid and λ-hybrid mapping.

On the other hand, weak convergence theorems for two nonexpansive mappings T_1, T_2 of C into itself were discussed by Takahashi and Tamura in [15]. They considered the following iterative procedure:

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary,} \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1(\beta_n T_2 x_n + (1 - \beta_n)x_n) \end{cases} \tag{1.5}$$

for all $n \geq 1$, where $F(T_1) \cap F(T_2)$ is nonempty. In 2007, Moudafi [9] considered another iterative procedure for two nonexpansive mappings T_1, T_2 of C into itself:

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n T_1 x_n + (1-\beta_n)T_2 x_n) \end{cases} \tag{1.6}$$

for all $n \geq 1$, where $F(T_1)$ and $F(T_2)$ are nonempty.

In 2009, Iemoto and Takahashi [4] extended the result of [9] for the approximation of common fixed points of nonexpansive mappings and nonspreading mappings in Hilbert spaces by using Moudafi's iterative scheme as follows:

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n S x_n + (1-\beta_n)T x_n) \end{cases} \tag{1.7}$$

for all $n \geq 1$, where S is a nonspreading mapping, T is a nonexpansive mapping and $F(S) \cap F(S)$ is nonempty.

The aim of this paper is to study the approximation of common fixed points of (a,b)-monotone mappings and nonexpansive mappings in Hilbert spaces by using the Moudafi's iterative scheme. The main result of this paper extend and generalize the corresponding results given by Iemoto and Takahashi [4], Lin and Wang [8] and some others in the literature.

2. Preliminaries

Throughout this paper, we denote $\mathbb{R}$ by the set of real numbers. Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$, respectively. First, we start with a brief recollection of some basic concepts and results in Hilbert spaces for our main results in this paper.

In a Hilbert space H, it is well known that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle. \tag{2.1}$$

and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$ (see, for instance, [13]). Further, in a Hilbert space H, we have

$$2\langle x-y, z-w \rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$ (see [4]). We know that a Hilbert space H satisfies *Opial's property* ([11]), that is, for any sequence $\{x_n\}$ in H with $x_n \rightharpoonup x$ implies

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.4}$$

for all $y \in H$ with $y \neq x$. We say that a mapping $T : C \to C$ have the *condition* (A) ([12]) if there exists a nondecreasing $f : [0, \infty] \to [0, \infty]$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, F(T)) \leq \|x - Tx\|$$

for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

In 2005, Khan and Fukhar-ud-din [5] modified the condition (A) for two mapping as follows:

Two mappings $S, T : C \to C$ have the *condition* (A') if there exists a nondecreasing $f : [0, \infty] \to [0, \infty]$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, \mathcal{F})) \le \frac{1}{2}(\|x - Tx\| + (\|x - Sx\|)$$

for all $x \in C$, where $d(x, \mathcal{F}) = \inf\{\|x - x^*\| : x^* \in \mathcal{F})\}$ and $\mathcal{F} := F(T) \cap F(S)$.

Lemma 2.1 ([18])**.** *Suppose that $\{s_n\}$ and $\{e_n\}$ are the sequences of nonnegative real numbers such that $s_{n+1} \le s_n + e_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} e_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.*

Proposition 2.2 ([8])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and T be a mapping from C to itself.*

(1) *If T is a nonspreading mapping, then T is a $(1, \frac{1}{2})$-monotone mapping.*
(2) *If T is a hybrid mapping, then T is a $(\frac{3}{2}, \frac{1}{2})$-monotone mapping.*
(3) *If T is a TY- mapping, then T is a $(2, \frac{1}{2})$-monotone mapping.*

Proposition 2.3 ([8])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and T be a (a, b)-monotone mapping defined on C. Then we have*

$$\|x - p\|^2 \ge \|Tx - p\|^2 + \frac{1 - 2b}{2a - 1}\|x - Tx\|^2 \tag{2.5}$$

for all $x \in C$ and $p \in F(T)$.

Theorem 2.4 ([8])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and T be a (a,b)-monotone mapping defined on C. If a sequence $\{x_n\} \subseteq C$ with $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, then $x^* = Tx^*$.*

Theorem 2.5. ([14]) *Let H be a Hilbert space and $\{x_n\}$ be a bounded sequence in H. Then $\{x_n\}$ is weakly convergent if and only if each weakly convergent subsequence of $\{x_n\}$ has the same weak limit, that is, for any $x \in H$,*

$$x_n \rightharpoonup x \iff [x_{n_i} \rightharpoonup y \implies x = y].$$

3. Weak convergence theorems

In this section, we prove the approximation of common fixed points of (a, b)-monotone mappings and nonexpansive mappings in a Hilbert space by the using Moudafi's iterative scheme.

Theorem 3.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a (a,b)-monotone mapping of C into itself and T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \ne \emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n S x_n + (1 - \beta_n) T x_n) \end{cases} \tag{3.1}$$

for all $n \geq 1$, *where* $\{\alpha_n\} \subseteq (0, 1)$ *with* $\alpha_n > \frac{2b-1}{2a-1}$ *and* $\{\beta_n\} \subset [0, 1]$. *If*

$$\liminf_{n\to\infty} (1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big) > 0, \quad \sum_{n=1}^{\infty}(1-\beta_n) < \infty,$$

then the sequence $\{x_n\}$ *converges weakly to a point* $v \in F(S)$.

Proof. Notice first that there exists a sequence $\{\alpha_n\}$ satisfies our assumptions. Indeed, $b < a$ and $\frac{2b-1}{2a-1} < 1$ and so there exists a constant $\alpha \in \mathbb{R}$ such that $\frac{2b-1}{2a-1} < \alpha < 1$. If we take $\alpha_n = \alpha$ for all $n \geq 1$, then $\{\alpha_n\} \subseteq (0, 1)$ such that

$$\alpha_n > \frac{2b-1}{2a-1}, \quad \liminf_{n\to\infty} (1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big) > 0.$$

Moreover, for each $n \geq 1$, we have

$$\alpha_n + \frac{1-2b}{2a-1} > 0. \tag{3.2}$$

Next, we show that $\{x_n\}$ is a bounded sequence in C. Since S is a (a, b)-monotone mapping, by Proposition 2.3, for each $p \in F(S) \cap F(T)$ and $x \in C$, we get

$$\|x-p\|^2 \geq \|Sx-p\|^2 + \frac{1-2b}{2a-1}\|x-Sx\|^2. \tag{3.3}$$

Let $U_n = \beta_n S + (1-\beta_n)T$ for each $n \geq 1$. Then, for all $x, y \in C$, we have

$$\begin{aligned} \|U_nx-U_ny\|^2 &= \|\beta_n(Sx-Sy) + (1-\beta_n)(Tx-Ty)\|^2 \\ &= \beta_n\|Sx-Sy\|^2 + (1-\beta_n)\|Tx-Ty\|^2 \\ &\quad -\beta_n(1-\beta_n)\|(Sx-Sy)-(Tx-Ty)\|^2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|x-U_nx\|^2 &= \|x-Sx+Sx-U_nx\|^2 \\ &= \|x-Sx\|^2 + \|Sx-U_nx\|^2 + 2\langle x-Sx, Sx-U_nx\rangle \\ &= \|x-Sx\|^2 + \|Sx-\beta_nSx-(1-\beta_n)Tx\|^2 \\ &\quad +2\langle x-Sx, Sx-\beta_nSx-(1-\beta_n)Tx\rangle \\ &= \|x-Sx\|^2 + (1-\beta_n)^2\|Sx-Tx\|^2 \\ &\quad +2(1-\beta_n)\langle x-Sx, Sx-Tx\rangle \end{aligned} \tag{3.5}$$

Using (2.2), (3.3), (3.4) and (3.5), we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|U_nx_n-p\|^2 - \alpha_n(1-\alpha_n)\|x_n-U_nx_n\|^2 \\ &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)[\beta_n\|Sx_n-p\|^2 + (1-\beta_n)\|Tx_n-p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|Sx_n-Tx_n\|^2] - \alpha_n(1-\alpha_n)\|x_n-U_nx_n\|^2 \\ &\leq \alpha_n\|x_n-p\|^2 + (1-\alpha_n)[\beta_n\|Sx_n-p\|^2 + (1-\beta_n)\|x_n-p\|^2] \\ &\quad -(1-\alpha_n)\beta_n(1-\beta_n)\|(Sx_n-x_n)+(x_n-Tx_n)\|^2 \\ &\quad -\alpha_n(1-\alpha_n)[\|x_n-Sx_n\|^2 + (1-\beta_n)^2\|Sx_n-Tx_n\|^2 \\ &\quad +2(1-\beta_n)\langle x_n-Sx_n, Sx_n-Tx_n\rangle] \\ &\leq \alpha_n\|x_n-p\|^2 + (1-\alpha_n)[\beta_n\|x_n-p\|^2 \end{aligned}$$

$$
\begin{aligned}
& -\beta_n \frac{1-2b}{2a-1}\|x_n - Sx_n\|^2 + (1-\beta_n)\|x_n - p\|^2] \\
& -(1-\alpha_n)\beta_n(1-\beta_n)[\|Sx_n - x_n\|^2 + \|x_n - Tx_n\|^2 \\
& +2\langle Sx_n - x_n, x_n - Tx_n\rangle] \\
& -\alpha_n(1-\alpha_n)[\|x_n - Sx_n\|^2 + (1-\beta_n)^2\|Sx_n - Tx_n\|^2 \\
& +2(1-\beta_n)\langle x_n - Sx_n, Sx_n - Tx_n\rangle] \\
= {} & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\|^2 - \beta_n\frac{1-2b}{2a-1}\|x_n - Sx_n\|^2\Big] \\
& -(1-\alpha_n)\beta_n(1-\beta_n)[\|Sx_n - x_n\|^2 + \|x_n - Tx_n\|^2 \\
& -2\langle x_n - Sx_n, x_n - Tx_n\rangle] \\
& -\alpha_n(1-\alpha_n)[\|x_n - Sx_n\|^2 + (1-\beta_n)^2\|Sx_n - Tx_n\|^2 \\
& +(1-\beta_n)[\|x_n - Tx_n\|^2 - \|x_n - Sx_n\|^2 - \|Sx_n - Tx_n\|^2]] \\
= {} & \|x_n - p\|^2 - (1-\alpha_n)\beta_n\frac{1-2b}{2a-1}\|x_n - Sx_n\|^2 \\
& -(1-\alpha_n)\beta_n(1-\beta_n)[\|x_n - Sx_n\|^2 + \|x_n - Tx_n\|^2 \\
& -[\|x_n - Tx_n\|^2 + \|x_n - Sx_n\|^2 - \|Sx_n - Tx_n\|^2]] \\
& -\alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2 - \alpha_n(1-\alpha_n)(1-\beta_n)^2\|Sx_n - Tx_n\|^2 \\
& -\alpha_n(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 + \alpha_n(1-\alpha_n)(1-\beta_n)\|x_n - Sx_n\|^2 \\
& +\alpha_n(1-\alpha_n)(1-\beta_n)\|Sx_n - Tx_n\|^2 \\
= {} & \|x_n - p\|^2 - (1-\alpha_n)\beta_n\frac{1-2b}{2a-1}\|x_n - Sx_n\|^2 \\
& -(1-\alpha_n)\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2 \\
& -\alpha_n(1-\alpha_n)(\beta_n)\|x_n - Sx_n\|^2 + \alpha_n(1-\alpha_n)(1-\beta_n)\beta_n\|Sx_n - Tx_n\|^2 \\
& -\alpha_n(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 \\
= {} & \|x_n - p\|^2 - (1-\alpha_n)\beta_n\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 \\
& -(1-\alpha_n)^2\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2 \\
& -\alpha_n(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 \\
\leq {} & \|x_n - p\|^2
\end{aligned} \tag{3.6}
$$

for all $n \geq 1$. Thus $\lim_{n\to\infty}\|x_n - p\|$ exists, say $c = \lim_{n\to\infty}\|x_n - p\|$, and hence $\{x_n\}$ is bounded. Moreover, $\{\|Tx_n - Sx_n\|\}$ is bounded. In deed, by (3.3), we have

$$
\begin{aligned}
& (1-\alpha_n)^2\beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2 \\
\leq {} & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - (1-\alpha_n)\beta_n\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 \\
& -\alpha_n(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 \\
\leq {} & \|x_n - p\|^2 - \|x_{n+1} - p\|^2
\end{aligned}
$$

Without loss of generality, we may assume that $\beta_n \neq 0$ and $\beta_n \neq 1$ for all $n \geq 1$ and hence, by using the boundedness of $\{x_n\}$, then $\|Tx_n - Sx_n\|$ is also bounded.

Let $z_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n$ for all $n \geq 1$. Then we get

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &= \|\alpha_n x_n + (1-\alpha_n)Ux_n - \alpha_n x_n - (1-\alpha_n)Sx_n\| \\ &= (1-\alpha_n)\|\beta_n Sx_n + (1-\beta_n)Tx_n - Sx_n\| \\ &= (1-\alpha_n)(1-\beta_n)\|Tx_n - Sx_n\| \\ &\leq (1-\beta_n)\|Tx_n - Sx_n\|. \end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$ and $\{\|Tx_n - Sx_n\|\}$ is bounded, we have

$$\sum_{n=1}^{\infty} \|x_n - z_n\| < \infty. \tag{3.7}$$

Therefore, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0 \tag{3.8}$$

and hence

$$\lim_{n\to\infty} \|z_n - p\| = \lim_{n\to\infty} \|x_n - p\| = c. \tag{3.9}$$

Since

$$\begin{aligned} \|z_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sx_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Sx_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\|^2 - \frac{1-2b}{2a-1}\|x_n - Sx_n\|^2\Big] \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2 \\ &= \|x_n - p\|^2 - (1-\alpha_n)\frac{1-2b}{2a-1}\|x_n - Sx_n\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2 \\ &= \|x_n - p\|^2 - (1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2, \end{aligned} \tag{3.10}$$

we have

$$(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 \leq \|x_n - p\|^2 - \|z_{n+1} - p\|^2$$

and hence

$$\lim_{n\to\infty}(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 = 0. \tag{3.11}$$

Thus, from (3.11) and $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n + \frac{1-2b}{2a-1}) > 0$, we conclude that

$$\lim_{n\to\infty}\|x_n - Sx_n\|^2 = 0.$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $v \in C$ such that $x_{n_i} \rightharpoonup v$. Since S is a (a,b)-monotone mapping, by Theorem 2.4, we obtain $v \in F(S)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ which converges weakly to $v^* \in C$. By the same argument as above, we can see that $v^* \in F(S)$.

Finally, we show that $v = v^*$. Before proving this, we prove that, for any $z \in F(S)$, $\lim_{n\to\infty} \|x_n - z\|$ exists. In the same way of the inequality (3.10), we can show that, for any $z \in F(S)$,

$$\|z_{n+1} - z\| \leq \|x_n - z\|$$

and so

$$\|z_{n+1} - z\| \leq \|x_n - z\| \leq \|z_n - z\| + \|x_n - z_n\|.$$

By (3.7) and Lemma 2.1, it follows that $\lim_{n\to\infty} \|z_n - z\|$ exists. From (3.8), we also have $\lim_{n\to\infty} \|x_n - z\|$ exists.

Suppose that $v \neq v^*$. Then, by Opial's property, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|x_{n_i} - v\| &< \liminf_{i\to\infty} \|x_{n_i} - v^*\| \\
&= \lim_{n\to\infty} \|x_n - v^*\| \\
&= \liminf_{j\to\infty} \|x_{n_j} - v^*\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| \\
&= \liminf_{i\to\infty} \|x_{n_i} - v\|,
\end{aligned}$$

which is a contradiction. Therefore, $v = v^*$ and, by Theorem 2.5, the sequence $\{x_n\}$ converges weakly to $v \in F(S)$. This completes the proof. □

If T is a nonspreading mapping and nonexpansive mapping, then $I - T$ is demiclosed at zero (see [1, 4]) and so we have the following:

Corollary 3.2 ([4]). *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by* (3.1)*. If*

$$\liminf_{n\to\infty} (1 - \alpha_n)(\alpha_n) > 0, \quad \sum_{n=1}^{\infty} (1 - \beta_n) < \infty,$$

then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S)$.

Proof. Since S is a (a, b)-monotone mapping with $b = \frac{1}{2}$, then $\frac{2b-1}{2a-1} = 0$. Thus Corollary 3.2 follow from Theorem 3.1. □

Corollary 3.3. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a mapping of C into itself which satisfies one of the following conditions:*

(1) *S is a hybrid mapping;*

(2) *S is a TY-mapping.*

Let T be a nonexpansive mapping of C into itself such that $F(S)\cap F(T)\neq\emptyset$ and $\{x_n\}$ be the sequence defined by (3.1)*. If*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n)>0,\quad \sum_{n=1}^{\infty}(1-\beta_n)<\infty,$$

then the sequence $\{x_n\}$ converges weakly to a point $v\in F(S)$.

Proof. Since S is a (a,b)-monotone mapping with $b=\frac{1}{2}$, then $\frac{2b-1}{2a-1}=0$. Thus Corollary 3.3 follow from Theorem 3.1. □

Corollary 3.4 ([8])**.** *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a (a,b)-monotone mapping mapping of C into itself such that $F(S)\neq\emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1\in C, \text{ chosen arbitrary,} \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)Sx_n \end{cases}\tag{3.12}$$

for all $n\geq 1$, where $\{\alpha_n\}\subset[0,1]$ with $\alpha_n>\frac{2b-1}{2a-1}$. If

$$\liminf_{n\to\infty}(1-\alpha_n)\Big(\alpha_n+\frac{1-2b}{2a-1}\Big)>0,$$

then the sequence $\{x_n\}$ converges weakly to a point $v\in F(S)$.

Proof. Setting $\beta_n=1$ for all $n\geq 1$ in Theorem 3.1, we obtain Corollary 3.4. □

Corollary 3.5 ([10])**.** *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself such that $F(S)\neq\emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1\in C, \text{ chosen arbitrary,} \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)Sx_n \end{cases}\tag{3.13}$$

for all $n\geq 1$, where $\{\alpha_n\}\subset[0,1]$. If $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n)>0$, then the sequence $\{x_n\}$ converges weakly to a point $v\in F(S)$.

Proof. Setting $\beta_n=1$ for all $n\geq 1$ in Corollary 3.2, we obtain Corollary 3.5. □

Next, we give an example of (a,b)-monotone mappings and nonexpansive mappings to illustrate Theorem 3.1.

Example 3.6. Consider a Hilbert space $H=\mathbb{R}^2$ with the usual inner product. Let $\phi:\mathbb{R}\times H\to H$, $S:H\to H$ and $T:H\to H$ be the mappings defined by

$$\phi(\omega,x)=(r\cos(\theta+\omega),r\sin(\theta+\omega)),$$

$$Sx=\frac{5}{4}\phi(\frac{3}{4}\pi,x)=\frac{5}{4}\Big(r\cos\big(\theta+\frac{3}{4}\pi\big),r\sin\big(\theta+\frac{3}{4}\pi\big)\Big)\tag{3.14}$$

and

$$T(a,b)=(-b,a)$$

for all $\omega\in\mathbb{R}$, $x=(r\cos\theta,r\sin\theta)\in H$ and $(a,b)\in H$, respectively. We know that, from [8], the mapping S defined by (3.14) is a $(4,3)$-monotone mapping, but it is not quai-nonexpansive, nonspreading, TY, hybrid and λ-hybrid. Clearly,

$F(S) = \{(0,0)\}$ and T is a nonexpansive mapping. For any fixed $x_1 \in H$, take the sequence $\{x_n\}$ defined in Theorem 3.1 with $\alpha_n = \frac{3}{4}$ for all $n \geq 1$ and $\beta_n = 1 - \frac{1}{2^n}$. Then, for each $n \geq 1$, we have

$$\begin{aligned} x_{n+1} &= \frac{3}{4}x_n + \frac{1}{4}\Big(\Big(1 - \frac{1}{2^n}\Big)Sx_n + \frac{1}{2^n}Tx_n\Big) \\ &= \frac{3}{4}x_n + \frac{1}{4}\Big(1 - \frac{1}{2^n}\Big)\Big[\frac{5\sqrt{2}}{8}\phi(\pi, x_n) + \frac{5\sqrt{2}}{8}\phi\Big(\frac{\pi}{2}, x_n\Big)\Big] + \frac{1}{2^{n+2}}Tx_n \end{aligned}$$

and hence

$$\begin{aligned} &\|x_{n+1}\| \\ &= \sqrt{\Big(\frac{3}{4} - \frac{5\sqrt{2}}{32}\Big(1 - \frac{1}{2^n}\Big)\Big)^2 + \Big(\frac{5\sqrt{2}}{32}\Big(1 - \frac{1}{2^n}\Big) + \frac{1}{2^{n+2}}\Big)^2}(\|x_n\|) \\ &= \sqrt{\frac{9}{16} + \frac{25}{256} - \frac{15\sqrt{2}}{64} + \Big(20\sqrt{2} - \frac{25}{2}\Big)\frac{1}{2^{n+6}} + \Big(1 + \frac{25}{16} - \frac{5\sqrt{2}}{4}\Big)\frac{1}{2^{2n+4}}}(\|x_n\|) \\ &= \sqrt{\frac{(169 - 60\sqrt{2})}{256} + \frac{40\sqrt{2} - 25}{2^{n+7}} + \frac{41 - 20\sqrt{2}}{2^{2n+8}}}(\|x_n\|) \\ &< \sqrt{\frac{85}{256} + \frac{40\sqrt{2} - 25}{2^{n+7}} + \frac{41 - 20\sqrt{2}}{2^{2n+8}}}(\|x_n\|) \\ &\leq M\|x_n\|, \end{aligned}$$

where $M = \sup\Big\{\sqrt{\frac{85}{256} + \frac{40\sqrt{2}-25}{2^{n+7}} + \frac{41-20\sqrt{2}}{2^{2n+8}}} : n \geq 1\Big\}$, which $M < 1$. Therefore, $\{x_n\}$ is bounded. Setting $z_{n+1} = \frac{3}{4}x_n + \frac{1}{4}Sx_n$ for all $n \geq 1$, as in the proof of Theorem 3.1, we can see that

$$\lim_{n\to\infty} \|x_n - Sx_n\|^2 = 0.$$

Therefore, by Theorem 2.4 and Theorem 2.5, we conclude that $x_n \rightharpoonup (0,0)$.

Theorem 3.7. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a (a,b)-monotone mapping of C into itself and T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$be the sequence as in Theorem 3.1. If*

$$\liminf_{n\to\infty}(1 - \alpha_n)\Big(\alpha_n + \frac{1 - 2b}{2a - 1}\Big) > 0, \quad \liminf_{n\to\infty}(1 - \beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S) \cap F(T)$.

Proof. Since (3.1) can be written as

$$x_{n+1} = \beta_n[\alpha_n x_n + (1 - \alpha_n)Sx_n] + (1 - \beta_n)[\alpha_n x_n + (1 - \alpha_n)Tx_n]$$

for all $n \geq 1$, putting $V_n = \beta_n[\alpha_n I + (1 - \alpha_n)S] + (1 - \beta_n)[\alpha_n I + (1 - \alpha_n)T]$, we have $x_{n+1} = Vx_n$ for all $n \geq 1$.

First, we show that sequence $\{x_n\}$ converges weakly to a point in $F(S)$. Let $u \in F(S) \cap F(T)$. Since

$$\begin{aligned} \|Sx - Tx\|^2 &= \|Sx - x + x - Tx\|^2 \\ &= \|Sx - x\|^2 + \|x - Tx\|^2 + 2\langle Sx - x, x - Tx\rangle \end{aligned} \tag{3.15}$$

for all $x \in C$, it follow from (2.2), (3.3) and (3.15) that

$$\begin{aligned} &\|V_n x_n - u\|^2 \\ = \;& \beta_n\|\alpha_n x_n + (1-\alpha_n)Sx_n - u\|^2 + (1-\beta_n)\|\alpha_n x_n + (1-\alpha_n)Tx_n - u\|^2 \\ & -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2 \\ = \;& \beta_n\big[\alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|Sx_n - u\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2\big] \\ & +(1-\beta_n)\big[\alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|Tx_n - u\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2\big] \\ & -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2 \\ \le \;& \beta_n\alpha_n\|x_n - u\|^2 + \beta_n(1-\alpha_n)\|Sx_n - u\|^2 - \beta_n\alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 \\ & +(1-\beta_n)\alpha_n\|x_n - u\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - u\|^2 \\ & -(1-\beta_n)\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 - \beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2 \\ = \;& \beta_n\alpha_n\|x_n - u\|^2 + (1-\beta_n)\|x_n - u\|^2 + \beta_n(1-\alpha_n)\|Sx_n - u\|^2 \\ & -\beta_n\alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 - (1-\beta_n)\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ & -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2 \\ \le \;& \beta_n\alpha_n\|x_n - u\|^2 + (1-\beta_n)\|x_n - u\|^2 + \beta_n(1-\alpha_n)\|x_n - u\|^2 \\ & -\beta_n(1-\alpha_n)\frac{1-2b}{2a-1}\|x_n - Sx_n\|^2 - \beta_n\alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 \\ & -(1-\beta_n)\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 - \beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2 \\ = \;& \|x_n - u\|^2 - \beta_n(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 \\ & -(1-\beta_n)\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 - \beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2 \\ \le \;& \|x_n - u\|^2 - \beta_n(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 \\ & -(1-\beta_n)\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ \le \;& \|x_n - u\|^2 \end{aligned} \tag{3.16}$$

and hence

$$0 \le \|x_n - u\|^2 - \|V_n x_n - u\|^2 = \|x_n - u\|^2 - \|x_{n+1} - u\|^2. \tag{3.17}$$

Taking $n \to \infty$ in the above inequality, by (3.9), we get

$$\lim_{n\to\infty}\big(\|x_n - u\|^2 - \|V_n x_n - u\|^2\big) = 0. \tag{3.18}$$

From (3.16), we can see that

$$\|V_n x_n - u\|^2 \le \|x_n - u\|^2 - \beta_n(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2$$

and hence

$$\begin{aligned} 0 &\le \|x_n-u\|^2-\Big[\|x_n-u\|^2-\beta_n(1-\alpha_n)\Big(\alpha_n+\frac{1-2b}{2a-1}\Big)\|x_n-Sx_n\|^2\Big] \\ &= \beta_n(1-\alpha_n)\Big(\alpha_n+\frac{1-2b}{2a-1}\Big)\|x_n-Sx_n\|^2 \\ &\le \|x_n-u\|^2-\|V_nx_n-u\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} 0 &\le (1-\beta_n)\beta_n(1-\alpha_n)\Big(\alpha_n+\frac{1-2b}{2a-1}\Big)\|x_n-Sx_n\|^2 \\ &\le (1-\beta_n)(\|x_n-u\|^2-\|V_nx_n-u\|^2). \end{aligned} \tag{3.19}$$

Since $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n+\frac{1-2b}{2a-1})>0$ and $\liminf_{n\to\infty}(1-\beta_n)(\beta_n)>0$, it follow from (3.18) and (3.19) that

$$\lim_{n\to\infty}\|x_n-Sx_n\|^2=0. \tag{3.20}$$

By the same argument in the proof of Theorem 3.1, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $v\in C$ and $v\in F(S)$.

Next, we show that v also an element in $F(T)$. From (3.16), for any $u\in F(S)\cap F(T)$, we see that

$$\begin{aligned} \|V_nx_n-u\|^2 &= \beta_n\|\alpha_nx_n+(1-\alpha_n)Sx_n-u\|^2+(1-\beta_n)\|\alpha_nx_n \\ &\quad +(1-\alpha_n)Tx_n-u\|^2 \\ &\quad -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n-Tx_n\|^2 \end{aligned} \tag{3.21}$$

By (2.2), (3.3), (3.21), the triangle property and the property of T, we have

$$\begin{aligned} &\|V_nx_n-u\|^2 \\ &= \beta_n\|\alpha_nx_n+(1-\alpha_n)Sx_n-u\|^2+(1-\beta_n)\|\alpha_nx_n+(1-\alpha_n)Tx_n-u\|^2 \\ &\quad -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n-Tx_n\|^2 \\ &\le \beta_n\big[\alpha_n\|x_n-u\|^2+(1-\alpha_n)\|Sx_n-u\|^2-\alpha_n(1-\alpha_n)\|Sx_n-x_n\|^2\big] \\ &\quad +(1-\beta_n)\big[\alpha_n\|x_n-u\|+(1-\alpha_n)\|Tx_n-u\|\big]^2 \\ &\quad -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n-Tx_n\|^2 \\ &\le \beta_n\big[\alpha_n\|x_n-u\|^2+(1-\alpha_n)\|Sx_n-u\|^2-\alpha_n(1-\alpha_n)\|Sx_n-x_n\|^2\big] \\ &\quad +(1-\beta_n)\big[\alpha_n\|x_n-u\|+(1-\alpha_n)\|x_n-u\|\big]^2 \\ &\quad -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n-Tx_n\|^2 \\ &= \beta_n\alpha_n\|x_n-u\|^2+\beta_n(1-\alpha_n)\|Sx_n-u\|^2-\beta_n\alpha_n(1-\alpha_n)\|Sx_n-x_n\|^2 \\ &\quad +(1-\beta_n)\|x_n-u\|^2-\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n-Tx_n\|^2 \\ &= \beta_n\alpha_n\|x_n-u\|^2+\beta_n(1-\alpha_n)\Big(\|x_n-u\|^2-\frac{1-2b}{2a-1}\|x_n-Sx_n\|^2\Big) \\ &\quad -\beta_n\alpha_n(1-\alpha_n)\|Sx_n-x_n\|^2+(1-\beta_n)\|x_n-u\|^2 \\ &\quad -\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n-Tx_n\|^2 \end{aligned} \tag{3.22}$$

$$= \|x_n - u\|^2 - \beta_n(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 - \beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2$$

it follows that

$$\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n - Tx_n\|^2 \le \|x_n - u\|^2 - \|V_n x_n - u\|^2 - \beta_n(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|x_n - Sx_n\|^2 \tag{3.23}$$

and hence

$$\beta_n(1-\beta_n)(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)\|Sx_n - Tx_n\|^2 \le \Big(\alpha_n + \frac{1-2b}{2a-1}\Big)(\|x_n-u\|^2 - \|V_n x_n - u\|^2) - \beta_n(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big)^2\|x_n - Sx_n\|^2 \tag{3.24}$$

Since $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n + \frac{1-2b}{2a-1}) > 0$ and $\liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0$, it follows from (3.18) and (3.20) that

$$\lim_{n\to\infty}\|Sx_n - Tx_n\|^2 = 0. \tag{3.25}$$

From the fact that

$$\begin{aligned} \|x_n - Tx_n\|^2 &= \|x_n - Sx_n + Sx_n - Tx_n\|^2 \\ &= \|x_n - Sx_n\|^2 + \|Sx_n - Tx_n\|^2 + 2\langle x_n - Sx_n + Sx_n - Tx_n\rangle \\ &\le \|x_n - Sx_n\|^2 + \|Sx_n - Tx_n\|^2 + 2\|x_n - Sx_n\|\|Sx_n - Tx_n\| \end{aligned} \tag{3.26}$$

By (3.20), (3.25) and (3.26), we obtain

$$\lim_{n\to\infty}\|x_n - Tx_n\|^2 = 0. \tag{3.27}$$

Since $\{x_{n_i}\}$ converges weakly to v, we have $v \in F(T)$. Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $v^* \in C$. We show that $v = v^*$.

Suppose that $v \neq v^*$. By Opial's property, we have

$$\begin{aligned} \liminf_{i\to\infty}\|x_{n_i} - v\| &< \liminf_{i\to\infty}\|x_{n_i} - v^*\| \\ &= \lim_{n\to\infty}\|x_n - v^*\| \\ &= \liminf_{k\to\infty}\|x_{n_k} - v^*\| \\ &< \liminf_{k\to\infty}\|x_{n_k} - v\| \\ &= \lim_{n\to\infty}\|x_n - v\| \\ &= \liminf_{i\to\infty}\|x_{n_i} - v\|, \end{aligned}$$

which is a contradiction and hence $v = v^*$. Therefore, we can conclude that the sequence $\{x_n\}$ converges weakly to $v \in F(S) \cap F(T)$. This completes the proof. □

Now, we establish an example of (a,b)-monotone mappings and nonexpansive mappings to illustrate Theorem 3.7.

Example 3.8. Let H, ϕ, S and T be same in as Example 3.6. Then $F(S) \cap F(T) = \{(0,0)\}$. For any fixed $x_1 \in H$, take the sequence $\{x_n\}$ defined in Theorem 3.1 with $\alpha_n = \frac{3}{4}$ and $\beta_n = \frac{1}{2}$ for all $n \geq 1$. Then we get

$$\begin{aligned} x_{n+1} &= \frac{3}{4}x_n + \frac{1}{4}\left(\frac{1}{2}Sx_n + \frac{1}{2}Tx_n\right) \\ &= \frac{3}{4}x_n + \frac{1}{8}\Big[\frac{5\sqrt{2}}{8}\phi(\pi, x_n) + \frac{5\sqrt{2}}{8}\phi\Big(\frac{\pi}{2}, x_n\Big)\Big] + \frac{1}{8}Tx_n \end{aligned}$$

for all $n \geq 1$ and hence

$$\begin{aligned} \|x_{n+1}\| &= \sqrt{\left(\frac{3}{4} - \frac{5\sqrt{2}}{64}\right)^2 + \left(\frac{5\sqrt{2}}{64} + \frac{1}{8}\right)^2}\,(\|x_n\|) \\ &= \sqrt{\frac{617 - 100\sqrt{2}}{1024}}\,(\|x_n\|) \\ &< \frac{11}{16}\|x_n\|. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. By the similar argument of Example 3.6 and the proof of Theorem 3.7, we can conclude that $x_n \rightharpoonup (0,0)$.

Corollary 3.9 ([4])**.** *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by* (3.1)*. If*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S) \cap F(T)$.

Corollary 3.10. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a mapping of C into itself which satisfies one of the following conditions:*

(1) *S is a hybrid mapping;*
(2) *S is a TY-mapping.*

Let T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence in C defined by (3.1)*. If*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S) \cap F(T)$.

4. Strong convergence theorems

In this section, we prove the approximation of common fixed points of (a, b)-monotone mappings and nonexpansive mappings satisfying the condition (A') in Hilbert spaces by using Moudafi's iterative scheme.

Theorem 4.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a (a,b)-monotone mapping of C into itself and T be a nonexpansive mapping of C into itself satisfying the condition (A') and $\mathcal{F} := F(S)\cap F(T) \neq \emptyset$.Let $\{x_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$be the sequence as in Theorem 3.1. If*

$$\liminf_{n\to\infty}(1-\alpha_n)\Big(\alpha_n + \frac{1-2b}{2a-1}\Big) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges strongly to a point $v \in F(S)\cap F(T)$.

Proof. From the inequalities (3.20) and (3.27), we obtain

$$\lim_{n\to\infty}\|x_n - Sx_n\|^2 = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|^2. \tag{4.1}$$

Moreover, by the similar argument of the proof of Theorem 3.1, we can show that

$$\|x_{n+1} - p\| \le \|x_n - p\| \tag{4.2}$$

for any $p \in F(S)\cap F(T)$.

On the other hand, by the condition (A') of S and T, we get

$$f(d(x_n,\mathcal{F})) \le \frac{1}{2}\big(\|x_n - Sx_n\| + \|x_n - Tx_n\|\big) \tag{4.3}$$

for all $n \ge 1$. Taking the infimum over all $p \in \mathcal{F}$ on both sides of (4.2), we can show that $\lim_{n\to\infty} d(x_n,\mathcal{F})$ exists.

Now, we claim that $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$. Suppose the contrary, choose $n_0 \ge 1$ such that $0 < \frac{k}{2} < d(x_n,\mathcal{F})$ for all $n \ge n_0$. Since f is nondecreasing, it follows from (4.1) and (4.3) that

$$0 < f(\frac{k}{2}) \le f(d(x_n,\mathcal{F})) \le \frac{1}{2}\big(\|x_n - Sx_n\| + \|x_n - Tx_n\|\big) \to 0$$

as $n \to \infty$, which is a contradiction. Therefore, $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$ and hence there exists $n_1 \ge 1$ such that

$$d(x_n,\mathcal{F}) \le \frac{\epsilon}{2} \tag{4.4}$$

for all $n \ge n_1$. Let $m,n \ge n_1$ and $p \in \mathcal{F}$. Then it follows from (4.2) that

$$\|x_n - x_m\| \le \|x_n - p\| + \|x_m - p\| \le 2\|x_{n_1} - p\|.$$

Taking the infimum over all $p \in \mathcal{F}$ on both sides of the above inequality, it follows from (4.4) that

$$\|x_n - x_m\| \le 2d(x_{n_1},\mathcal{F}) < \epsilon$$

for all $m,n \ge n_1$, which implies that $\{x_n\}$ is a Cauchy sequence. Suppose that $\lim_{n\to\infty} x_n = v$ for some $v \in H$. Since $\mathcal{F}$ is closed, we have $v \in \mathcal{F}$. Therefore, the sequence $\{x_n\}$ converges strongly to $v \in \mathcal{F}$. This completes the proof. □

Corollary 4.2. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and T be a nonexpansive mapping of C into itself satisfying the condition (A') and $\mathcal{F} := F(S)\cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by* (3.1)*. If*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ *converges strongly to a point* $v \in F(S) \cap F(T)$.

Corollary 4.3. *Let* H *be a Hilbert space and* C *be a nonempty closed convex subset of* H. *Let* S *be a mapping of* C *into itself which satisfies one of the following conditions:*

(1) S *is a hybrid mapping;*
(2) S *is a* TY*-mapping.*

Let T *be a nonexpansive mapping of* C *into itself satisfying the condition* (A') *and* $F(S) \cap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence in* C *defined by* (3.1). *If*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ *converges strongly to a point* $v \in F(S) \cap F(T)$.

Corollary 4.4. *Let* H *be a Hilbert space and* C *be a nonempty closed convex subset of* H. *Let* S *be a* (a, b)*-monotone mapping mapping of* C *into itself satisfying the condition* (A) *and* $F(S) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence defined as follows:*

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n \end{cases} \tag{4.5}$$

for all $n \geq 1$, *where* $\{\alpha_n\} \subset [0, 1]$. *If* $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n + \frac{1-2b}{2a-1}) > 0$, *then the sequence* $\{x_n\}$ *converges strongly to a point* $v \in F(S)$.

Proof. Putting $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 4.1, the conclusion follow from Theorem 4.1. □

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