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MULTIVALUED NONEXPANSIVE MAPPINGS WITH AN ALMOST CONVEX DISPLACEMENT FUNCTION

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Dedicated to Professor Wataru Takahashi on the occasion of his 70^{th} birthday

Abstract. Is is shown that every nonexpansive set-valued mapping in a uniformly convex Banach space has a displacement function $H_T(x) := \sup\{\text{dist}(x, y) : y \in T(x)\}$ which is, in some sense, convex.

1. Introduction

If C is a nonempty subset of a Banach space X, and $T : C \to X$ is a mapping, one can immediately define the *displacement function* associated to T, that is, the function $J_T : C \to [0, \infty)$ defined as $J_T(x) := \|x - T(x)\|$. Since $x \in C$ is a zero of J_T if and only if x is a fixed point of T, one can understand that the study of the properties of the (nonlinear) functional J_T in order to derive fixed point results for T has a long tradition in Fixed Point Theory.

In particular many authors have payed attention to classes of (single-valued) mappings for which J_T is, in some sense, convex. For example in 1969 L.P. Belluce and W.A. Kirk [2] proved that if C is nonempty, weakly compact and convex, a continuous mapping $T : C \to C$ has a fixed point in C provided that

$$\inf_{x \in C} J_T(x) = 0, \tag{1.1}$$

and J_T is *midpoint convex* in the sense that

$$J_T\Big(\frac{x+y}{2}\Big) \le \frac{J_T(x) + J_T(y)}{2} \tag{1.2}$$

holds for every $x, y \in C$.

Later on, in 1971, A. Montagnana and A. Vignoli ([13], Theorem 1) gave a similar result, but asking that J_T satisfies (1.1) and it is *quasiconvex*, that is, such that for every $x, y \in C$,

$$J_T\Big(\frac{x+y}{2}\Big) \le \max\{J_T(x), J_T(y)\}. \tag{1.3}$$

Similar results can be found in the paper due to J. Daneš [5] published in 1970. Of course, midpoint convex mappings are quasiconvex but the converse is not true. Passing by, we may point out that if $T : C \to C$ is nonexpansive (i.e., 1 Lipschitzian), then it satisfies condition (1.1).

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In 1998 in [6], (see also [4]), a more general fixed point theorem based on the convexity of J_T was given. With more precision, the main result in [6] is the following. Let C be a weakly compact and convex subset of X. Let $T : C \to C$ be a continuous mapping satisfying (1.1) and such that for every $x, y \in C$,

$$J_T(\lambda x + (1-\lambda)y) \leq \alpha\left(\max\{J_T(x), J_T(y)\}\right), \tag{1.4}$$

where $\alpha : [0, \infty) \to [0, \infty)$ is any continuous strictly increasing function with $\alpha(0) = 0$. Then, T has a fixed point in C.

Continuous mappings satisfying (1.4) were called α-almost convex in [6]. It turns out that continuous quasiconvex mappings are α-almost convex, but the converse is not true. In fact, the class of mappings satisfying (1.4) is quite large and it contains the strict contractions of C, some Kannan type selfmappings of C, as well as the so called Γ *type mappings* on C, among others.

Recall that if Γ denotes the set of all the strictly increasing convex continuous functions $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\gamma(0) = 0$, according to R.E. Bruck [3], a mapping $T : C \to X$ is said to be of type Γ if there exists $\gamma \in \Gamma$ such that, for all $x, y \in C$ and $c \in [0, 1]$

$$\gamma\left(\|cT(x) + (1-c)T(y) - T(cx + (1-c)y)\|\right) \leq \|x - y\| - \|T(x) - T(y)\|. \tag{1.5}$$

It is obvious that every Γ-type mapping is nonexpansive, and the set of the fixed points of a Γ-type mapping is always convex. Immediate examples of Γ-type mappings are the nonexpansive linear and affine mappings.

A relevant result due to R. Bruck, (see [3] again) is that if the Banach space $(X, \|.\|)$ is uniformly convex, and C is a nonempty closed convex and bounded subset of X, then there exists a function $\gamma \in \Gamma$ (depending only on $(X, \|.\|)$ and the diameter of C) such that every nonexpansive mapping $T : C \to X$ is of Γ type with respect to the function γ. Moreover, in 1989, M.A. Khamsi (see [9]), proved that this property in fact characterizes the uniformly convex Banach spaces.

Bruck's result implies in particular that, in uniformly convex Banach spaces, (single-valued) nonexpansive mappings are α-almost convex with $\alpha(t) = t + \gamma^{-1}(2t)$ (see [6] for details).

To develop a similar approach for set-valued mappings, a suitable definition of the displacement function is required. In fact, two choices are possible, namely $J_T(x) := d(x, Tx)$ and $H_T(x) := H(\{x\}, T(x))$. Here H denotes the Hausdorff distance in X and it is supposed that T is a multivalued mapping defined on C and taking nonempty bounded closed values. Note that the zeroes of J_T are usually called fixed points of T, while the zeroes of H_T, that is the points x for which $T(x) = \{x\}$, are often called stationary points.

Notice that Caristi type fixed point theorems for set-valued mappings in terms of the function J_T were given by Mizogouchi and Takahashi very early in [12].

In this paper we will give a quite natural definition of Γ-type set-valued mappings, although we will present an example of a (set-valued) nonexpansive mapping in a uniformly convex Banach which fails to be a Γ-type mapping. This shows that a generalization, word by word, of Bruck's results to the set-valued case is not possible.

However we will show also that, roughly speaking, a Bruck's type result for set valued mappings still holds.

With a bit more precision we will show that every nonexpansive set-valued mapping in a uniformly convex Banach space is strongly convex, or, in other words, it has an α-almost convex displacement function $H(x) := \sup\{\mathrm{dist}(x, y) : y \in T(x)\}$. This kind of convexity for set-valued mappings is closely related with the similar concepts considered in Ko [10] or Sach and Yen [14].

2. Notations and preliminaries

If C is a nonempty closed convex subset of a Banach space $(X, \|.\|)$, by a set-valued mapping we usually understand a map $T : C \to \mathcal{P}_{bc}(X)$, where $\mathcal{P}_{bc}(X)$ is the set of all nonempty bounded and closed subsets of X.

Recall that on $\mathcal{P}_{bc}(X)$ one can define the *Hausdorff metric* H induced by the norm of X in the following way. For $M, N \in \mathcal{P}_{bc}(X)$

$$H(M, N) := \max\{\beta(M, N), \beta(N, M)\}$$

where

$$\beta(M, N) := \sup\{\mathrm{dist}(x, N) : x \in M\}.$$

Sometimes the following alternative expression is useful

$$H(M, N) := \inf\{\varepsilon > 0 : M \subset N_\varepsilon, N \subset M_\varepsilon\},$$

where $A_\varepsilon := \{x \in X : \mathrm{dist}(x, A) < \varepsilon\}$ denotes the so called open expansion of the nonempty set $A \subset X$. It is well known that $(\mathcal{P}_{bc}(X), H)$ is a complete metric space.

If $M \subset X$ we will use $\mathcal{P}_{bc}(M)$ and $\mathcal{P}_{kv}(M)$ to denote the set of all nonempty bounded and closed subsets of M, and the set of all nonempty compact convex subsets of M, respectively.

Given a set-valued mapping $T : C \longrightarrow \mathcal{P}_{bc}(X)$, we can consider the associate displacement functions $J_T : C \longrightarrow \mathbb{R}$ and $H_T : C \longrightarrow \mathbb{R}$ respectively defined by

$$J_T(x) := \mathrm{dist}(x, T(x)) = \inf\{\|x - y\| : y \in T(x)\} = \beta(\{x\}, T(x)),$$

and

$$H_T(x) := H(\{x\}, T(x)) = \beta(T(x), \{x\}) = \sup\{\|x - y\| : y \in T(x)\}.$$

If $x, y \in C$, for simplicity we will write $m_T(x, y)$ instead of $\max\{J_T(x), J_T(y)\}$, and $M_T(x, y)$ instead of $\max\{H_T(x), H_T(y)\}$.

According to [11], for a continuous strictly increasing function $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$, we say that the set-valued mapping $T : C \longrightarrow \mathcal{P}_{bc}(X)$ is α-almost convex if for all $x, y \in C$ and all $\lambda \in [0, 1]$ one has that

$$J_T(\lambda x + (1 - \lambda)y) \le \alpha(m_T(x, y)).$$

In the same way, we say that the set-valued mapping $T : C \longrightarrow \mathcal{P}_{bc}(X)$ is said to be strongly α-almost convex if for all $x, y \in C$ and all $\lambda \in [0, 1]$

$$H_T(\lambda x + (1 - \lambda)y) \le \alpha(M_T(x, y)).$$

When the previous definitions are fulfilled with a $\alpha(t) = rt$, for some $r > 0$, we will say that T is r-almost convex (resp. strongly r-almost convex).

Notice that the set of all the fixed points of an α-almost convex mapping is convex. In the same way, the set of all the stationary points of a strongly α-almost convex mapping is convex

If the mapping T is α-almost convex (or strongly α-almost convex) and single valued (i.e. $T(x)$ is a singleton for every $x \in C$), then the corresponding single-valued mapping is α-almost convex in the sense of [6].

Finally we recall that a set-valued mapping $T : C \longrightarrow \mathcal{P}_{bc}(X)$ is called *nonexpansive* whenever for all $x, y \in C$

$$H(T(x), T(y)) \leq \|x - y\|.$$

3. Main result

In a uniformly convex Banach space every nonexpansive (single valued) mapping $T : C \to X$ is a Γ-type mapping, and hence an α-almost convex mapping (in the sense of [6]). We will show in this section that an analogous result holds for the set-valued case.

Theorem 3.1. *Let $(X, \|.\|)$ be a uniformly convex Banach space. Let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \longrightarrow \mathcal{P}_{bc}(X)$ be a nonexpansive mapping. Then there exists a continuous strictly increasing function $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ such that T is strongly α-almost convex. The function α only depends on the diameter of C and on the modulus of convexity of $(X, \|.\|)$.*

Proof. Let δ the modulus of convexity of $(X, \|.\|)$. It is well known that

$$\eta := \delta^{-1} : [0, 1] \longrightarrow [0, 2]$$

is a continuous strictly increasing function with $\eta(0) = 0$ and $\eta(1) = 2$.

Let us define the function $a : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$a(\varepsilon) := \sup \left\{ (r + \varepsilon)\eta\left(\frac{\varepsilon}{r + \varepsilon}\right) : r \in [0, \operatorname{diam}(K)] \right\}.$$

Given $r > 0$, the real function $\varphi(t) := \frac{t}{r+t}$, $(t \geq 0)$, satisfies that $\varphi(0) = 0$ and, for $t > 0$,

$$\varphi'(t) = \frac{r}{(r + t)^2} \geq 0.$$

Therefore, if $0 \leq \varepsilon_1 \leq \varepsilon_2 < \infty$,

$$(r + \varepsilon_1)\eta\left(\frac{\varepsilon_1}{r + \varepsilon_1}\right) \leq (r + \varepsilon_1)\eta\left(\frac{\varepsilon_2}{r + \varepsilon_2}\right).$$

We easily derive that $0 = a(0) \leq a(\varepsilon_1) \leq a(\varepsilon_2)$. It is unclear whether a is a continuous function.

We now extend the function η as follows: Let $\tilde{\eta} : [0, \infty) \to [0, \infty)$ given by

$$\tilde{\eta}(t) := \begin{cases} \eta(t) & 0 \leq t \leq 1 \\ 2t & 1 < t. \end{cases}$$

As η is continuous and $\eta(1) = 2$ it is obvious that $\tilde{\eta}$ is continuous and strictly increasing.

Now let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be the function given by

$$\alpha(\varepsilon) := \max\{\eta(1)\psi(\varepsilon), (\varepsilon + \mathrm{diam}(C))\tilde{\eta}(\psi(\varepsilon))\},$$

where $\psi(\varepsilon) := \max\{\varepsilon, \sqrt{\varepsilon}\}$.

If $0 < \varepsilon < 1$ in [15], p. 447 one can see that

$$a(\varepsilon) \leq \max\{\sqrt{\varepsilon}\eta(1), (\mathrm{diam}(C) + \varepsilon)\eta(\sqrt{\varepsilon})\} = \alpha(\varepsilon).$$

On the other hand, if $0 \leq r \leq \mathrm{diam}(C)$,

$$(1 + r)\eta\left(\frac{1}{1 + r}\right) \leq (1 + \mathrm{diam}(C))\eta(1).$$

Taking the supremum of the right hand side of this inequality we obtain

$$a(1) \leq (1 + \mathrm{diam}(C))\eta(1) = \alpha(1).$$

Finally, for $\varepsilon > 1$ and for all $r \in [0, \mathrm{diam}(C)]$ we have

$$(r + \varepsilon)\tilde{\eta}\left(\frac{\varepsilon}{r + \varepsilon}\right) \leq (\varepsilon + \mathrm{diam(C)})\tilde{\eta}(\varepsilon) = \alpha(\varepsilon).$$

Taking again the supremum of the right hand side of this inequality when $r \in [0, \mathrm{diam}(C)]$ we obtain $a(\varepsilon) \leq \alpha(\varepsilon)$.

In summary, for all $\varepsilon \in \mathbb{R}^+$,

$$2\varepsilon \leq a(\varepsilon) \leq \alpha(\varepsilon),$$

and the function α is strictly increasing, continuous and satisfies $\alpha(0) = 0$.

The inequality $H_T(cx + (1 - c)y) \leq \alpha(M_T(x, y))$ trivially holds for the cases $c = 0$, $c = 1$ and $x = y$. Now let x, y be two different points of C and $c \in (0, 1)$. Let $\sigma := M_T(x, y)$.

We claim that, for every $z \in T(cx + (1 - c)y)$ either

$$\left\| x - \frac{cx + (1 - c)y - z}{2} \right\| \geq \|x - (cx + (1 - c)y)\| \tag{3.1}$$

or

$$\left\| y - \frac{cx + (1 - c)y - z}{2} \right\| \geq \|y - (cx + (1 - c)y)\|. \tag{3.2}$$

Otherwise we would have the following contradiction:

$$\begin{aligned} \|x - y\| &\leq \left\| x - \tfrac{cx+(1-c)y-z}{2} \right\| + \left\| \tfrac{cx+(1-c)y-z}{2} - y \right\| \\ &< \|x - (cx + (1 - c)y)\| + \|y - (cx + (1 - c)y)\| \\ &= (1 - c)\|x - y\| + c\|x - y\| = \|x - y\|. \end{aligned}$$

Let us suppose that $z \in T(cx + (1 - c)y)$ satisfies inequality (3.1). Since T is nonexpansive one has

$$\left.\begin{array}{l} \|x - (cx + (1 - c)y)\| =: r \leq \sigma + r \\ \|x - z\| \leq H(\{x\}, T(cx + (1 - c)y)) \leq H(\{x\}, T(x)) + r \leq \sigma + r \\ \left\| x - \tfrac{cx+(1-c)y-z}{2} \right\| \geq \|x - (cx + (1 - c)y)\| \end{array}\right\}$$

Therefore, from a well known property of the function η, (see for example, [15]),

$$\|cx + (1-c)y - z\| \le (r+\sigma)\eta\left(\frac{\sigma}{r+\sigma}\right).$$

If inequality (3.2) holds for z, we easily get the same conclusion. As the right hand side of the above inequality does not depend on z,

$$\sup\{\|cx + (1-c)y - z\| : z \in T(cx + (1-c)y)\} \le (r+\sigma)\eta\left(\frac{\sigma}{r+\sigma}\right)$$

or, in other words

$$\begin{aligned} H_T(cx + (1-c)y) &= H(\{cx + (1-c)y\}, T(cx + (1-c)y)) \\ &\le (r+\sigma)\eta\left(\frac{\sigma}{r+\sigma}\right) \\ &\le a(\sigma) \le \alpha(\sigma). \end{aligned}$$

□

Example 3.2. Let $(X, \|.\|)$ be the space $\mathbb{R}^2$ endowed with the ordinary Euclidean norm. Let $T : B_X \to \mathcal{P}_{bc}(X)$ be the mapping given by

$$T(x_1, x_2) := (\{x_1\} \times [-1, 1]) \cap B_X,$$

where $B_X := \{x \in \mathbb{R}^2 : \|x\| \le 1\}$. The set of the stationary points of this mapping is just

$$\{(-1, 0), (1, 0)\}$$

which is not convex. Hence T can not be nonexpansive. Otherwise, T would be strongly α-almost convex, and hence the set of the stationary points of T would be convex, which it is no true.

The requirement of uniform convexity for $(X, \|.\|)$ in the above theorem is not superfluous. This is clear when the following example is considered.

Example 3.3. Let $(X, \|.\|)$ be the Banach space $(\mathbb{R}^2, \|.\|_\infty)$ where $\|.\|_\infty$ is the standard sup norm. If

$$C := \{(x_1, x_2) :\in \mathbb{R}^2 : |x_1| + |x_2| \le 1\},$$

let $T : C \longrightarrow \mathcal{P}_{kv}(C)$ be the mapping given by

$$T((x_1, x_2)) := \{x_1\} \times [|x_1| - 1, 1 - |x_1|].$$

On can see that $T(x)$ is the largest vertical segment included in C which contains the point x. Hence the set of fixed points of T is C, while the set of stationary points of T is $S(T) = \{(-1, 0), (1, 0)\}$, which is non convex. (Even disconnected).

Let H_∞ be the Hausdorff metric associated to the metric $d_\infty(x, y) = \|x - y\|_\infty$. First let us point out that, if $A = \{x\} \times [a, b] \subset \mathbb{R}^2$, then it is straigtforward to check that, given $\varepsilon > 0$,

$$A_\varepsilon := \{y \in \mathbb{R}^2 : \mathrm{d}_\infty(y, A) < \varepsilon\} = (x - \varepsilon, x + \varepsilon) \times (a - \varepsilon, b + \varepsilon).$$

We claim that

$$H_\infty(T((x_1, x_2)), T((y_1, y_2))) \le |x_1 - y_1|.$$

Indeed, let $\varepsilon' > \varepsilon := |x_1 - y_1|$.

As $|y_1| - |x_1| \leq |x_1 - y_1| < \varepsilon'$, then

$$\left.\begin{array}{l} |y_1| - \varepsilon' - 1 < |x_1| - 1 \\ 1 - |x_1| < 1 + \varepsilon' - |y_1| \end{array}\right\} \Rightarrow [|x_1| - 1, 1 - |x_1|] \subset (|y_1| - 1 - \varepsilon', 1 - |y_1| + \varepsilon').$$

This inclusion together with the fact that

$$x_1 \in (y_1 - \varepsilon', y_1 + \varepsilon')$$

yield

$$T((x_1, x_2)) := \{x_1\} \times [|x_1| - 1, 1 - |x_1|] \subset (y_1 - \varepsilon', y_1 + \varepsilon') \times (|y_1| - 1 - \varepsilon', 1 - |y_1| + \varepsilon')$$

Thus, for all $\varepsilon' > \varepsilon$,

$$T((x_1, x_2)) \subset (T((y_1, y_2)))_{\varepsilon'}$$

and, by the same argument,

$$T((y_1, y_2)) \subset (T((x_1, x_2)))_{\varepsilon'}.$$

Therefore, for all $\varepsilon' > \varepsilon$

$$H_\infty(T((x_1, x_2)), T((y_1, y_2))) \leq \varepsilon'$$

which proves the claim.

As a direct consequence we have that T is $\|.\|_\infty$-nonexpansive, that is

$$H_\infty(T((x_1, x_2)), T((y_1, y_2))) \leq \varepsilon = |x_1 - y_1| \leq \|(x_1, x_2) - (y_1, y_2)\|_\infty.$$

Nevertheless, the mapping T can not be strongly α-almost convex in this space $(\mathbb{R}^2, \|.\|_\infty)$. Otherwise the set of the stationary points of T would be convex. Hence the requirement of the uniform convexity in Theorem 3.1 can not be dropped.

Proposition 10.2 in the book by Goebel and Kirk [7] reads

Suppose K is a bounded closed and convex subset of a uniformly convex Banach space X, and suppose $T : K \to X$ is a nonexpansive mapping which satisfies

$$\inf\{\|x - T(x)\| : x \in K\} = 0.$$

Then T has a fixed point.

The following corollary is an extension (in some sense) of this result to set-valued mappings.

Corollary 3.4. *Suppose C is a bounded closed and convex subset of a uniformly convex Banach space $(X, \|.\|)$, and suppose $T : C \to \mathcal{P}_{bc}(X)$ is a nonexpansive mapping which satisfies*

$$\inf\{H_T(x) : x \in C\} = 0. \tag{3.3}$$

Then T has a stationary point.

Proof. If T is nonexpansive, since $(X, \|.\|)$ is uniformly convex then from the above theorem we can assure that T is strongly α-almost convex, and, of course Lipschitzian. These properties along with condition (3.3) yield the conclusion by a direct application of Corollary 7 of [11]. $\square$

Remark 3.5. Regarding Theorem 3.1, the question of whether it holds for another class of Banach spaces beyond the uniformly convex ones naturally arises.

4. Γ TYPE SET VALUED MAPPINGS

A natural way of defining Γ-type set-valued mappings could be the following

Definition 4.1. A mapping $T : C \to \mathcal{P}_{bc}(X)$ is said to be of type Γ if there exists $\gamma \in \Gamma$ such that, for all $x, y \in C$ and $c \in [0,1]$

$$\gamma\left(H(cT(x) + (1-c)T(y), T(cx + (1-c)y))\right) \leq \|x - y\| - H(T(x), T(y)).$$

If T is, in particular, single valued this concept of Γ-type is the same as the corresponding one defined by Bruck.

Example 4.2. Let $(X, \|\cdot\|)$ be a Banach space and let $T : B_X \longrightarrow \mathcal{P}_{bc}(B_X)$ be the multivalued mapping given by

$$T(x) := B[0_X, \|x\|].$$

Here and henceforth $B[0_X, r]$ stands for the closed ball of $(X, \|.\|)$ centered at 0_X and with radius $r > 0$, and $B_X := B[0_X, 1]$. Let us recall that, for $r, s > 0$, $B[0_X, r] + B[0_X, s] = B[0_X, r + s]$ and $H(B[0_X, r], B[0_X, s]) = |r - s|$. Therefore, if $x, y \in B_X$, (we may suppose $\|x\| \leq \|y\|$), one has.

$$\begin{aligned}
\|x - y\| - H(T(x), T(y)) &= \|x - y\| - \big|\|x\| - \|y\|\big| \\
&= \|x - y\| - \|y\| + \|x\| \\
&\geq c\left(\|x - y\| - \|y\| + \|x\|\right) \\
&= c\|x\| + (1-c)\|y\| - \|y\| + c\|y - x\| \\
&\geq c\|x\| + (1-c)\|y\| - \big|\, \|y\| + c\|y - x\| \,\big| \\
&\geq c\|x\| + (1-c)\|y\| - \|y + c(y - x)\| \\
&= H(cT(x) + (1-c)T(y), T(cx + (1-c)y)).
\end{aligned}$$

Thus, T is a Γ-type set valued mapping with respect to the function $\gamma(t) = t$.

Remark 4.3. The above definition can be the most direct extension of the single valued case. Nevertheless, one can point out that, if x, y are fixed points of such a set valued Γ-type mapping T, that is, if $x \in T(x)$, $y \in T(y)$, then it does not follow that the right hand side of the above inequality is equal to 0, although this is true whenever x, y are stationary points. Then, the set of the all the stationary points of a Γ-type mapping $T : C \to \mathcal{P}_{bc}(X)$ is convex.

In the same way as the single-valued case, the class of Γ-type set-valued mappings is included in the class of the *strongly α-almost convex* mappings.

Proposition 4.4. *If $T : C \longrightarrow \mathcal{P}_{bc}(X)$ is a Γ-type set valued mapping, then T is strongly α-almost convex (with respect to the real function α given by $\alpha(t) := t + \gamma^{-1}(2t)$).*

Proof. There exists a function $\gamma \in \Gamma$ such that, for all $x, y \in C$ and $c \in [0,1]$,

$$\begin{aligned}
&\gamma\left(H(cT(x) + (1-c)T(y), T(cx + (1-c)y))\right) \\
\leq\ &\|x - y\| - H(T(x), T(y))
\end{aligned}$$

$$\begin{aligned} &\leq H(\{x\},T(x))+H(T(x),T(y))+H(T(y),\{y\})-H(T(x),T(y)) \\ &\leq 2\max\{H(\{x\},T(x)),H(T(y),\{y\})\} \\ &= 2M_T(x,y). \end{aligned}$$

Thus,

$$H(cT(x)+(1-c)T(y),T(cx+(1-c)y)) \leq \gamma^{-1}(2M_T(x,y)).$$

Applying this inequality together with Corollary 1.18 of [8], it follows that

$$\begin{aligned} M_T(cx+(1-c)y) &= H(\{cx+(1-c)y\},T(cx+(1-c)y)) \\ &\leq H(\{cx+(1-c)y\},cT(x)+(1-c)T(y)) \\ &\qquad + H(cT(x)+(1-c)T(y),T(cx+(1-c)y)) \\ &\leq H(c\{x\}+(1-c)\{y\},cT(x)+(1-c)T(y))+\gamma^{-1}(2M_T(x,y)) \\ &\leq cH(\{x\},T(x))+(1-c)H(\{y\},T(y))+\gamma^{-1}(2M_T(x,y)) \\ &\leq \max\{H(\{x\},T(x)),H(\{y\},T(y))\}+\gamma^{-1}(2M_T(x,y)) \\ &= M_T(x,y)+\gamma^{-1}(2M_T(x,y)). \end{aligned}$$

As $\gamma \in \Gamma$, the function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\alpha(t) := t + \gamma^{-1}(2t)$ is continuous and strictly increasing. We have seen that

$$M_T(cx+(1-c)y) \leq \alpha(M_T(x,y))$$

for all $x, y \in C$ and $c \in [0,1]$, that is, we have seen that T is strongly α-almost convex in C. □

Corollary 4.5. *Suppose C is a nonempty weakly compact subset of a Banach space $(X, \|.\|)$, and suppose $T : C \to \mathcal{P}_{bc}(X)$ is a Γ-type mapping which satisfies condition (3.3). Then T has a stationary point.*

Proof. Indeed Γ-type mappings are nonexpansive and, from Proposition 4.4 are strongly α-almost convex too. The conclusion follows again from Corollary 7 of [11]. □

5. On the failure of Bruck's property in the set valued case

At this point still one could ask whether the conclusion of Bruck's Theorem, which establish that in uniformly convex Banach spaces nonexpansive mappings are Γ type mappings, remains true for nonexpansive set-valued mappings in uniformly convex Banach spaces. The aim of this section is to show that the answer to this question is negative, at least whenever the definition of Γ-type mapping is stated as in this paper.

Example 5.1 (Hwei-Mei Ko, 1972, [10])**.** Let $(X, \|.\|)$ be the (uniformly convex) Banach space $(\mathbb{R}^2, \|.\|_2)$, where $\|.\|_2$ is the ordinary Euclidean norm. Let $C = [0,1] \times [0,1] \subset X$ and let $T : C \to \mathcal{P}_{kv}(C)$ be the mapping given by

$$T((x_1,x_2)) := \overline{\text{co}}\,(\{(0,0),(x_1,0),(0,x_2)\}).$$

Note that $T((x_1,x_2))$ is the triangle with vertices $(0,0),(x_1,0),(0,x_2)$, ant that, if $x_1x_2 = 0$, then $T((x_1,x_2))$ is a degenerate triangle.

It is clear that T has compact convex values and, according with [10], T is nonexpansive. The (nonconvex) set of fixed points of T is $W = \{(x_1, x_2) \in C : x_1 x_2 = 0\}$. Moreover $(0, 0)$ is the unique stationary point of T.

Let us observe that, for $x = (1, 1)$ and $y = (1, 0)$ one has

$$T(x) = \mathrm{co}\,(\{(0,0),(1,0),(0,1)\})$$

and

$$T(y) = \mathrm{co}\,(\{(0,0),(1,0)\}) = [0,1] \times \{0\}.$$

Therefore, it is easy to check that

$$H(T(x), T(y)) = 1 = \|x - y\|_2.$$

But

$$H\left(\frac{1}{2}T(x) + \frac{1}{2}T(y), T\left(\frac{1}{2}x + \frac{1}{2}y\right)\right) \neq 0.$$

Otherwise $\frac{1}{2}T(x) + \frac{1}{2}T(y) = T\left(\frac{1}{2}x + \frac{1}{2}y\right)$, a contradiction because

$$\left(\frac{1}{2}, \frac{1}{2}\right) \notin T\left(\frac{1}{2}x + \frac{1}{2}y\right) = T\left(\left(1, \frac{1}{2}\right)\right) = \mathrm{co}\left(\left\{(0,0)(1,0), \left(0, \frac{1}{2}\right)\right\}\right)$$

whereas

$$\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(0,1) + \frac{1}{2}(1,0) \in \frac{1}{2}T(x) + \frac{1}{2}T(y).$$

If T were a Γ-type mapping (w.r.t. $\gamma \in \Gamma$), we would have the following contradiction

$$0 < \gamma\left(H\left(\frac{1}{2}T(x) + \frac{1}{2}T(y), T\left(\frac{1}{2}x + \frac{1}{2}y\right)\right)\right) \leq \|x - y\|_2 - H(T(x), T(y)) = 0.$$

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