Journal of Nonlinear and Convex Analysis
Volume 16, Number 9, 2015, 1847–1859

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONVERGENCE BALL OF NEWTON'S METHOD FOR GENERALIZED EQUATION AND UNIQUENESS OF THE SOLUTION

JINHUA WANG

Dedicated to Wataru Takahashi on the occasion of his 70th birth day

Abstract. In the present paper, we consider a generalized Newton method for a generalized equation

$$0 \in F(x) + T(x),$$

where F is Fréchet differentiable and T is set-valued and maximal monotone. Under a generalized L-average Lipschitz condition, we establish an estimation of convergence ball for the generalized Newton method. Moreover, we also get an estimation of uniqueness ball for a solution of the inclusion problem. As applications, we obtain Kantorovich type theorem under the classical Lipschitz condition, convergence results under the γ-condition, and Smale's point estimate theory. Our results extend some corresponding results in [22].

1. Introduction

Newton's method is one of the most important methods known for solving systems of nonlinear equations when they are continuously differentiable. It has been studied and used extensively (see [12, 19–21, 23–25] and the references therein). One of the famous results on Newton's method is the well-known Kantorovich's theorem (*cf.* [12]), which provides a criterion ensuring the quadratic convergence of Newton's method under a mild condition on a proper open metric ball of the initial point x_0. Another important result on Newton's method is Smale's point estimate theory (i.e., α-theory and γ-theory) in [19, 20], where the rules to judge an initial point x_0 to be an approximate zero were established, depending on the information of the analytic nonlinear operator at this initial point and at a solution x^*, respectively. The γ-condition for nonlinear operators in Banach spaces was first introduced and explored by Wang [25] for the study of Smale's point estimate theory, which extended the corresponding results in [19].

Newton's method has been extensively studied and developed to solve problems arisen from various areas, such as abstract inequality problems in Banach space (see [17]), variational inequality problems (cf. [3, 4, 6, 7, 9–11, 14, 15, 22]), multiobjective optimization problems (see [8]).

In the present paper, our interests are focused on extending Newton's method for generalized inclusion problems. Let H be a real Hilbert space with inner product

2010 *Mathematics Subject Classification.* Primary 49J40, 65J05.

Key words and phrases. Newton's method, generalized equation, generalized Lipschitz condition, convergence ball, uniqueness ball.

The research is partially supported by the National Natural Science Foundation of China (grant 11371325) and by Zhejiang Provincial Natural Science Foundation of China (grant LY13A010011).

$\langle \cdot, \cdot \rangle$. Let $T : H \rightrightarrows H$ be a (set-valued) maximal monotone operator. Let $F : H \to H$ be a Fréchet differentiable function. Consider the generalized equation:

$$\text{(1.1)} \qquad \text{Find } x^* \in H \quad \text{such that} \quad 0 \in F(x^*) + T(x^*).$$

Such problems have been studied by [17, 22] and have important applications in the physical and engineering sciences and in many other fields (cf. [2, 5, 13]). For example, let $f : H \to (-\infty, +\infty]$ be a differentiable convex function. Let C be a nonempty closed convex subset of H, and let $N_C(x)$ be the normal cone of C at x. If $F = \nabla f$ and $T(\cdot) = N_C(\cdot)$, then problem (1.1) is reduced to the minimization problem:

$$\min_{x \in C} f(x),$$

or equivalently the variational inequality problem:

$$\text{Find } x^* \in C \quad \text{such that} \quad \langle c - x^*, F(x^*) \rangle \geq 0 \quad \text{for each } c \in C.$$

Recall that the generalized Newton method for inclusion problem (1.1) is given as follows (cf. [22]): Let $x_0 \in H$ be given. Have $x_0, x_1, \ldots, x_n$. Define x_{n+1} such that

$$0 \in F(x_n) + F'(x_n)(x_{n+1} - x_n) + T(x_{n+1}).$$

In [22], Uko established convergence results for the generalized Newton method under classical Lipschitz condition. In sprit of Smale's point estimate theory [19,20] and Wang's work in [23], the purpose of the present paper is to continue the study of the generalized Newton method for (1.1) under more generalized Lipschitz condition. Under a generalized L-average Lipschitz condition, we give an estimation of convergence ball for the generalized Newton method. Moreover, we also get an estimation of uniqueness ball for the solution of (1.1). As applications, we obtain Kantorovich type theorem under the classical Lipschitz condition, convergence results under the γ-condition, and Smale's point estimate theory. Hence, our results extend some corresponding results in [22].

The remainder of the paper is organized as follows. In Section 2, some notions, notations and preliminaries are provided. In Section 3, estimation of convergence ball is established under a generalized L-average Lipschitz condition, while in Section 4, we present an estimation of uniqueness ball of the solution of (1.1). In the final section, as applications, we get Kantorovich type theorem under the classical Lipschitz condition, convergence results under the γ-condition, and Smale's point estimate theory.

2. Notions and preliminaries

Let $x \in H$, and let $r > 0$. As usual, we use $\mathbf{B}(x, r)$ and $\overline{\mathbf{B}(x, r)}$ to denote, respectively, the open metric ball and the closed metric ball at x with radius r, that is,

$$\mathbf{B}(x, r) := \{y \in H |\ \|x - y\| < r\} \quad \text{and} \quad \overline{\mathbf{B}(x, r)} := \{y \in H |\ \|x - y\| \leq r\}.$$

Recall that a bounded linear operator $G : H \to H$ is called a positive operator if G is self-conjugate and $\langle Gx, x \rangle \geq 0$ for each $x \in H$ (cf. [18, p. 313]).

Lemma 2.1. *Let G be a positive operator. Then the following conclusions hold:*

(i) $\|G^2\| = \|G\|^2$.

(ii) *If G^{-1} exists, then G^{-1} also is a positive operator.*

Proof. Note by definition that G^2 is a positive operator and

$$\|G^2\| = \sup_{\|x\|=1} \langle G^2x, x\rangle = \sup_{\|x\|=1} \langle Gx, Gx\rangle = \sup_{\|x\|=1} \|Gx\|^2 = (\sup_{\|x\|=1} \|Gx\|)^2 = \|G\|^2.$$

Let $x, y \in H$. Since G is self-conjugate, one has

$$\langle G^{-1}x, y\rangle = \langle G^{-1}x, GG^{-1}y\rangle = \langle GG^{-1}x, G^{-1}y\rangle = \langle x, G^{-1}y\rangle.$$

Hence, G^{-1} is self-conjugate. Observe further that

$$\langle G^{-1}x, x\rangle = \langle G^{-1}x, G(G^{-1}x)\rangle \geq 0.$$

Thus, it follows that G^{-1} also is a positive operator. □

Lemma 2.2. *Let G be a positive operator. Suppose that G^{-1} exists. Then*

$$\langle Gx, x\rangle \geq \frac{\|x\|^2}{\|G^{-1}\|} \quad \text{for each } x \in H. \tag{2.1}$$

Proof. Let $x \in H$. Since G is a positive operator, we have from [18, p. 313] that there exists a positive operator $G^{\frac{1}{2}}$ such that $G^{\frac{1}{2}}G^{\frac{1}{2}} = G$. As G^{-1} exists, it follows that $(G^{\frac{1}{2}})^{-1}$ exists. Observe from Lemma 2.1 that $(G^{\frac{1}{2}})^{-1}$ is a positive operator and

$$\|(G^{\frac{1}{2}})^{-1}\|^2 = \|(G^{\frac{1}{2}})^{-1}(G^{\frac{1}{2}})^{-1}\| = \|G^{-1}\|.$$

Thus, it follows that

$$\|x\|^2 = \|(G^{\frac{1}{2}})^{-1}G^{\frac{1}{2}}x\|^2 \leq \|(G^{\frac{1}{2}})^{-1}\|^2\|G^{\frac{1}{2}}x\|^2 = \|G^{-1}\|\langle Gx, x\rangle.$$

Hence, (2.1) is seen to hold. □

Let $T : H \rightrightarrows H$ be a set-valued operator. The domain $\mathrm{dom}T$ of T is defined as $\mathrm{dom}T := \{x \in H|\ T(x) \neq \emptyset\}$. Below, we recall notions of monotonicity for set-valued operators (see [1, 26] for details).

Definition 2.3. Let $T : H \rightrightarrows H$ be a set-valued operator. T is said to be

(a) *monotone* if the following condition holds for any $x, y \in \mathrm{dom}T$:

$$\langle u - v, y - x\rangle \geq 0 \quad \text{for each } u \in T(y) \text{ and } v \in T(x); \tag{2.2}$$

(b) *maximal monotone* if it is monotone and the following implication holds for any $x, u \in H$:

$$\langle u - v, x - y\rangle \geq 0 \ \text{ for each } y \in \mathrm{dom}T \text{ and } v \in T(y) \Longrightarrow x \in \mathrm{dom}T \text{ and } u \in T(x). \tag{2.3}$$

Throughout the whole paper, let R be a positive constant and $L(\cdot)$ be a nonnegative nondecreasing integrable function on $[0, R)$ satisfying

$$\int_0^R L(s)\mathrm{d}s \geq 1.$$

A generalized Lipschitz condition with L-average has been introduced in [23]. Below, we extend generalized Lipschitz condition with L-average for operators on Hilbert spaces which is slightly different from that in [23]. Throughout the whole paper, for any bounded linear operator $G : H \to H$, we always adopt the convention that $\widehat{G} := \frac{1}{2}(G + G^*)$ where G^* is the conjugate operator of G. Clearly, $\widehat{G}$ is a self-conjugate operator.

Definition 2.4. Let $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists, and let $r > 0$. Then $\|\widehat{F'(\bar{x})}^{-1}\|F'$ is said to satisfy

(a) the center Lipschitz condition with L average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$ if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x) - F'(\bar{x})\| \leq \int_0^{\|x-\bar{x}\|} L(u)\mathrm{d}u \quad \text{for each } x \in \mathbf{B}(\bar{x}, r).$$

(b) the radius Lipschitz condition with L average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$ if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x) - F'(x^\tau)\| \leq \int_{\tau\|x-\bar{x}\|}^{\|x-\bar{x}\|} L(u)\mathrm{d}u \quad \text{for each } x \in \mathbf{B}(\bar{x}, r), 0 \leq \tau \leq 1,$$

where $x^\tau = x^* + \tau(x - x^*)$.

Let $r_0 > 0$ be such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1. \tag{2.4}$$

Lemma 2.5. *Let $r < r_0$. Let $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}$ is a positive operator and $\widehat{F'(\bar{x})}^{-1}$ exists. Suppose that $\|\widehat{F'(\bar{x})}^{-1}\|F'$ satisfies the center Lipschitz condition with L average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$. Then, for each $x \in \mathbf{B}(\bar{x}, r)$, $\widehat{F'(x)}$ is a positive operator and $\widehat{F'(x)}^{-1}$ exists. Moreover,*

$$\|\widehat{F'(x)}^{-1}\| \leq \frac{\|\widehat{F'(\bar{x})}^{-1}\|}{1 - \displaystyle\int_0^{\|x-\bar{x}\|} L(u)\mathrm{d}u}. \tag{2.5}$$

Proof. Note that

$$\|\widehat{F'(x)} - \widehat{F'(\bar{x})}\| \leq \|F'(x) - F'(\bar{x})\|.$$

Hence, it follows that

$$\begin{aligned} \|\widehat{F'(\bar{x})}^{-1}\| \cdot \|\widehat{F'(x)} - \widehat{F'(\bar{x})}\| &\leq \|\widehat{F'(\bar{x})}^{-1}\| \cdot \|F'(x) - F'(\bar{x})\| \\ &\leq \int_0^{\|x-\bar{x}\|} L(u)\mathrm{d}u \\ &< \int_0^{r_0} L(u)\mathrm{d}u \\ &= 1. \end{aligned} \tag{2.6}$$

Thus, by the Banach Lemma, $\widehat{F'(x)}^{-1}$ exists and (2.5) holds. Observe from (2.6) that

$$\|\widehat{F'(x)} - \widehat{F'(\bar{x})}\| \leq \frac{1}{\|\widehat{F'(\bar{x})}^{-1}\|}. \tag{2.7}$$

Let $y \in H$. Then, it follows from (2.7) that

$$\langle (\widehat{F'(\bar{x})} - \widehat{F'(x)})y, y\rangle \le \|\widehat{F'(\bar{x})} - \widehat{F'(x)}\|\|y\|^2 \le \frac{1}{\|\widehat{F'(\bar{x})}^{-1}\|}\|y\|^2,$$

which implies that

$$\langle \widehat{F'(\bar{x})}y, y\rangle - \frac{1}{\|\widehat{F'(\bar{x})}^{-1}\|}\|y\|^2 \le \langle \widehat{F'(x)}y, y\rangle. \tag{2.8}$$

Note by Lemma 2.2 that

$$\langle \widehat{F'(\bar{x})}y, y\rangle \ge \frac{1}{\|\widehat{F'(\bar{x})}^{-1}\|}\|y\|^2.$$

Combining this with (2.8) yields that

$$\langle \widehat{F'(x)}y, y\rangle \ge 0.$$

Hence, it follows that $\widehat{F'(x)}$ is a positive operator. □

3. Convergence ball

Let $T : H \rightrightarrows H$ be a (set-valued) maximal monotone operator. Let $F : H \to H$ be a Fréchet differentiable function. Consider the generalized equation:

$$\text{Find } x^* \in H \quad \text{such that} \quad 0 \in F(x^*) + T(x^*). \tag{3.1}$$

Newton's method for inclusion problem (3.1) is given as follows:

Algorithm 3.1. Let $x_0 \in H$ be given. Have $x_0, x_1, \ldots, x_k$. Define x_{k+1} such that

$$0 \in F(x_k) + F'(x_k)(x_{k+1} - x_k) + T(x_{k+1}). \tag{3.2}$$

Remark 3.2. Fix n. If there exists a constant $c > 0$ such that

$$\langle F'(x_k)y, y\rangle \ge c\|y\|^2 \quad \text{for each } y \in H, \tag{3.3}$$

then there exists a unique point x_{n+1} such that (3.2) holds because T is maximal monotone (see [22, Lemma 2.2]). Hence, if for each n, there exists a constant $c > 0$ such that (3.3) holds, then the sequence generated by (3.3) is well defined.

Recall that $r_0 > 0$ is such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1.$$

Let $\bar{r} > 0$ be such that

$$\frac{\int_0^{\bar{r}} L(u)u\mathrm{d}u}{\bar{r}\left(1 - \int_0^{\bar{r}} L(u)\mathrm{d}u\right)} \le 1. \tag{3.4}$$

Clearly, $\bar{r} \le r_0$.

Theorem 3.3. *Let $r \le \bar{r}$. Suppose that x^* is a solution of (3.1) such that $F'(x^*)$ is a positive operator (not necessary self-conjugate) and $\widehat{F'(x^*)}^{-1}$ exists. Suppose further that $\|\widehat{F'(x^*)}^{-1}\|F'$ satisfies the radius Lipschitz condition with L average at x^* on $\mathbf{B}(x^*, r)$. Let $x_0 \in \mathbf{B}(x^*, r)$. Then, the sequence $\{x_k\}$ generated by Newton's method (3.2) with initial point x_0 is well defined and*

$$\|x_k - x^*\| \le \lambda^{2^k-1}\|x_0 - x^*\| \quad \text{for each } k = 0, 1, 2, \ldots, \tag{3.5}$$

where

$$\lambda = \frac{\displaystyle\int_0^{\rho(x_0)} L(u)u\mathrm{d}u}{\rho(x_0)\left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)\mathrm{d}u\right)} < 1 \tag{3.6}$$

and $\rho(x_0) = \|x_0 - x^*\|$.

Proof. Since the function $t \to \frac{\int_0^t L(u)u\mathrm{d}u}{t^2}$ is nondecreasing (see [24, Lemma 2.2]), it follows that

$$\begin{aligned} \lambda &= \frac{\displaystyle\int_0^{\rho(x_0)} L(u)u\mathrm{d}u}{\rho(x_0)^2\left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)\mathrm{d}u\right)}\rho(x_0) \\ &\le \frac{\displaystyle\int_0^{\bar{r}} L(u)u\mathrm{d}u}{\bar{r}^2\left(1 - \displaystyle\int_0^{\bar{r}} L(u)\mathrm{d}u\right)}\rho(x_0) \\ &\le \frac{\rho(x_0)}{\bar{r}} \\ &< 1. \end{aligned}$$

Below we will show that (3.5) holds for each $k = 0, 1 \ldots$, by induction. Clearly, it is trivial in the case when $k = 0$. Now assume that (3.5) holds for k. Then, (3.5) implies $x_k \in \mathbf{B}(x^*, r)$ and

$$\|x_k - x^*\| < r \le \bar{r} \le r_0.$$

This, together with the assumption that $F'(x^*)$ is a positive operator and $\widehat{F'(x^*)}^{-1}$ exists, implies that Lemma 2.5 is applicable to concluding that $\widehat{F'(x_k)}$ is a positive operator and

$$\|\widehat{F'(x_k)}^{-1}\| \le \frac{\|\widehat{F'(x^*)}^{-1}\|}{1 - \displaystyle\int_0^{\|x_k - x^*\|} L(u)\mathrm{d}u}. \tag{3.7}$$

Hence, it follows from Lemma 2.2 that for each $y \in H$

$$\langle \widehat{F'(x_k)}y, y\rangle \ge \frac{\|y^2\|}{\|\widehat{F'(x_k)}^{-1}\|}.$$

Note that $\langle F'(x_k)y, y\rangle = \langle \widehat{F'(x_k)}y, y\rangle$. Thus $F'(x_k)$ satisfies (3.3). Then, by Remark 3.2, we have that x_{k+1} is well defined. Consequently, to complete the proof, it remains to verify that (3.5) holds for $k+1$. Since x^* is a solution of (3.1), we have that

$$0 \in F(x^*) + T(x^*).$$

Observe further from (3.2) that

$$0 \in F(x_k) + F'(x_k)(x_{k+1} - x_k) + T(x_{k+1}).$$

As T is maximal monotone, it follows that

$$\langle F(x_k) - F(x^*) + F'(x_k)(x^* - x_k) + F'(x_k)(x_{k+1} - x^*), x^* - x_{k+1}\rangle \geq 0,$$

and so

$$\langle F(x_k) - F(x^*) + F'(x_k)(x^* - x_k), x^* - x_{k+1}\rangle \geq \langle F'(x_k)(x^* - x_{k+1}), x^* - x_{k+1}\rangle. \tag{3.8}$$

Since $\widehat{F'(x_k)}$ is a positive operator and $\widehat{F'(x_k)}^{-1}$ exists, one get from Lemma 2.2 that

$$\frac{\|x^* - x_{k+1}\|^2}{\|\widehat{F'(x_k)}^{-1}\|} \leq \langle \widehat{F'(x_k)}(x^* - x_{k+1}), x^* - x_{k+1}\rangle. \tag{3.9}$$

Observe further that

$$\langle \widehat{F'(x_k)}(x^* - x_{k+1}), x^* - x_{k+1}\rangle = \langle F'(x_k)(x^* - x_{k+1}), x^* - x_{k+1}\rangle.$$

This, together with (3.9) and (3.8), yields that

$$\|x^* - x_{k+1}\| \leq \|\widehat{F'(x_k)}^{-1}\| \cdot \|F(x_k) - F(x^*) + F'(x_k)(x^* - x_k)\|. \tag{3.10}$$

Note that

$$\begin{aligned} &F(x_k) - F(x^*) + F'(x_k)(x^* - x_k) \\ &= \int_0^1 F'(x^* + t(x_k - x^*))(x_k - x^*)\mathrm{d}t + F'(x_k)(x^* - x_k) \\ &= \int_0^1 [F'(x_k) - F'(x^* + t(x_k - x^*))](x^* - x_k)\mathrm{d}t. \end{aligned}$$

Combining this with (3.10) and (3.7) yields that

$$\begin{aligned} \|x^* - x_{k+1}\| &\leq \frac{\|\widehat{F'(x^*)}^{-1}\|}{1 - \int_0^{\|x_k - x^*\|} L(u)\mathrm{d}u} \int_0^1 \|F'(x_k) - F'(x^* + t(x_k - x^*))\| \|x_k - x^*\|\mathrm{d}t \\ &\leq \frac{1}{1 - \int_0^{\|x_k - x^*\|} L(u)\mathrm{d}u} \int_0^1 \int_{t\|x_k - x^*\|}^{\|x_k - x^*\|} L(u)\mathrm{d}u \|x_k - x^*\|\mathrm{d}t \qquad (3.11) \\ &= \frac{\int_0^{\|x_k - x^*\|} L(u)u\mathrm{d}u}{1 - \int_0^{\|x_k - x^*\|} L(u)\mathrm{d}u} \end{aligned}$$

As

$$\frac{\int_0^{\|x_k-x^*\|} L(u)u\mathrm{d}u}{1-\int_0^{\|x_k-x^*\|} L(u)\mathrm{d}u} = \frac{\int_0^{\|x_k-x^*\|} L(u)u\mathrm{d}u}{\|x_k-x^*\|^2(1-\int_0^{\|x_k-x^*\|} L(u)\mathrm{d}u)}\|x_k-x^*\|^2$$
$$\leq \frac{\int_0^{\|x_0-x^*\|} L(u)u\mathrm{d}u}{\|x_0-x^*\|^2(1-\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u)}\|x_k-x^*\|^2$$
$$\leq \lambda\frac{\|x_k-x^*\|^2}{\|x_0-x^*\|}$$
$$\leq \lambda^{2^{k+1}-1}\|x_0-x^*\|$$

This, together with (3.11), implies that (3.5) holds for $k+1$. □

4. Uniqueness ball of the solution

This section is devoted to the uniqueness ball of the solution of (3.1). Let $\hat{r}>0$ be such that

$$\frac{1}{\hat{r}}\int_0^{\hat{r}} L(u)(\hat{r}-u)\mathrm{d}u \leq 1. \tag{4.1}$$

Theorem 4.1. *Let $r\leq \hat{r}$. Suppose that x^* is a solution of* (3.1) *such that $F'(x^*)$ is a positive operator (not necessary self-conjugate) and $\widehat{F'(x^*)}^{-1}$ exists. Suppose further that $\|\widehat{F'(x^*)}^{-1}\|F'$ satisfies the center Lipschitz condition with L average at x^* on $\mathbf{B}(x^*,r)$. Then, x^* is the unique solution of* (3.1) *on $\mathbf{B}(x^*,r)$.*

Proof. Since x^* is a solution of (3.1), one has

$$0\in F(x^*)+T(x^*).$$

Assume on the contrary that y^* is another solution of (3.1). Then, it follows that

$$0\in F(x^*)+T(x^*).$$

As T is maximal monotone, we get

$$\langle F(y^*)-F(x^*),x^*-y^*\rangle\geq 0,$$

which implies that

$$\langle F(y^*)-F(x^*)-F'(x^*)(y^*-x^*)+F'(x^*)(y^*-x^*),x^*-y^*\rangle\geq 0$$

and so

$$\langle F(y^*)-F(x^*)-F'(x^*)(y^*-x^*),x^*-y^*\rangle\geq\langle F'(x^*)(x^*-y^*),x^*-y^*\rangle. \tag{4.2}$$

Since $F'(x^*)$ is a positive operator and $\widehat{F'(x^*)}^{-1}$ exists, we apply Lemma 2.2 to get that

$$\langle F'(x^*)(x^*-y^*),x^*-y^*\rangle=\langle\widehat{F'(x^*)}(x^*-y^*),x^*-y^*\rangle\geq\frac{\|x^*-y^*\|^2}{\|\widehat{F'(x^*)}^{-1}\|}. \tag{4.3}$$

Observe further that

$$F(y^*)-F(x^*)-F'(x^*)(y^*-x^*) = \int_0^1 F'(x^*+t(y^*-x^*))(y^*-x^*)\mathrm{d}t-F'(x^*)(y^*-x^*).$$

Combining this with (4.2) and (4.3) yields that

$$\begin{aligned}\|y^*-x^*\| &\le \int_0^1 \|\widehat{F'(x^*)}^{-1}\|\|F'(x^*+t(y^*-x^*))-F'(x^*)\|\|y^*-x^*\| \\ &\le \int_0^1\int_0^{t\|y^*-x^*\|} L(u)\mathrm{d}u\mathrm{d}t\|y^*-x^*\| \\ &= \tfrac{1}{\|y^*-x^*\|}\int_0^{\|y^*-x^*\|} L(u)(\|y^*-x^*\|-u)\mathrm{d}u\|y^*-x^*\| \\ &< \tfrac{1}{\hat{r}}\int_0^{\hat{r}} L(u)(\hat{r}-u)\mathrm{d}u\|y^*-x^*\| \\ &\le \|y^*-x^*\|,\end{aligned}$$

where the third strict inequality holds because of the fact that the function $t \to \frac{1}{t}\int_0^t L(u)(t-u)\mathrm{d}u$ is increasing monotonically (cf. [23]) and $\|y^*-x^*\| < \hat{r}$. This implies that $y^* = x^*$. □

5. Applications

This section is devoted to the application of our previous results for some special cases such as the classical Lipschitz condition and the γ-condition.

5.1. **The classical Lipschitz condition.** Let $L > 0$ be a constant, and let $r > 0$. Let $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists. Then $\|\widehat{F'(\bar{x})}^{-1}\|F'$ is said to satisfy:

(i) the center Lipschitz condition with L on $\mathbf{B}(\bar{x}, r)$ if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x)-F'(\bar{x})\| \le L\|x-\bar{x}\| \quad \text{for each } x \in \mathbf{B}(\bar{x}, r).$$

(ii) the radius Lipschitz condition with L on $\mathbf{B}(\bar{x}, r)$ if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x)-F'(x^\tau)\| \le L(1-\tau)\|x-\bar{x}\| \quad \text{for each } x \in \mathbf{B}(\bar{x}, r), 0 \le \tau \le 1,$$

where $x^\tau = x^* + \tau(x-x^*)$.

Since $L(\cdot) \equiv L$, it follows from (3.4), (3.6) and (4.1) that

$$\bar{r} = \frac{2}{3L}, \quad \lambda = \frac{L\|x_0-x^*\|}{2(1-L\|x_0-x^*\|)} \text{ and } \hat{r} = \frac{2}{L}.$$

Hence, the following two corollaries follow directly from Theorems 3.3 and 4.1.

Corollary 5.1. *Let $r \le \frac{2}{3L}$. Suppose that x^* is a solution of (3.1) such that $F'(x^*)$ is a positive operator (not necessary self-conjugate) and $\widehat{F'(x^*)}^{-1}$ exists. Suppose further that $\|\widehat{F'(x^*)}^{-1}\|F'$ satisfies the radius Lipschitz condition with L on $\mathbf{B}(x^*, r)$. Let $x_0 \in \mathbf{B}(x^*, r)$. Then, the sequence $\{x_k\}$ generated by Newton's method (3.2) with initial point x_0 is well defined and*

$$\|x_k-x^*\| \le \lambda^{2^k-1}\|x_0-x^*\| \quad \textit{for each } k = 0, 1, 2, \ldots,$$

where

$$\lambda = \frac{L\|x_0 - x^*\|}{2(1 - L\|x_0 - x^*\|)}.$$

Corollary 5.2. *Let $r \le \frac{2}{L}$. Suppose that x^* is a solution of* (3.1) *such that $F'(x^*)$ is a positive operator (not necessary self-conjugate) and $\widehat{F'(x^*)}^{-1}$ exists. Suppose further that $\|\widehat{F'(x^*)}^{-1}\|F'$ satisfies the center Lipschitz condition with L on $\mathbf{B}(x^*, r)$. Then, x^* is the unique solution of* (3.1) *on $\mathbf{B}(x^*, r)$.*

5.2. **The γ-condition.** Let $r > 0$ and $\gamma > 0$ be such that $\gamma r \le 1$. In this subsection, we always assume that $F : H \to H$ is a C^2 function. The γ-conditions for operators in Banach space were first presented by Wang [25] for the study of Smale's point estimate theory. Below, it's an analogue of γ-condition for operators, which is slightly different from the one given in [25].

Definition 5.1. Let $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists. F is said to satisfy γ-condition at $\bar{x}$ in $\mathbf{B}(\bar{x}, r)$, if

$$\|\widehat{F'(\bar{x})}^{-1}\| \cdot \|F''(x)\| \le \frac{2\gamma}{(1 - \gamma\|x - \bar{x}\|)^3} \quad \text{for each } x \in \mathbf{B}(\bar{x}, r). \tag{5.1}$$

The following proposition shows that the γ-condition implies the radius Lipschitz condition with L average, where the function L is defined by

$$L(s) := \frac{2\gamma}{(1 - \gamma s)^3}, \quad \forall s \in [0, r). \tag{5.2}$$

Proposition 5.1. *Let $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists. Suppose that F satisfies γ-condition at $\bar{x}$ in $\mathbf{B}(\bar{x}, r)$. Then $\|\widehat{F'(\bar{x})}^{-1}\|F'$ satisfies the radius Lipschitz condition with L average at $\bar{x}$ on $\mathbf{B}(\bar{x}, r)$, where L is given by* (5.2)*.*

Proof. Let $x \in \mathbf{B}(\bar{x}, r)$, $\tau \in [0, 1]$ and $x^\tau = \bar{x} + \tau(x - \bar{x})$. Then

$$F'(x) - F'(x^\tau) = \int_\tau^1 F''(\bar{x} + s(x - \bar{x}))(x - \bar{x})\mathrm{d}s.$$

Hence, it follows

$$\begin{aligned}
\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x) - F'(x^\tau)\| &\le \int_\tau^1 \|\widehat{F'(\bar{x})}^{-1}\|\|F''(\bar{x} + s(x - \bar{x}))\|\|x - \bar{x}\|\mathrm{d}s \\
&\le \int_\tau^1 \frac{2\gamma}{(1 - \gamma s\|x - \bar{x}\|)^3}\mathrm{d}s \\
&= \int_{\tau\|x-\bar{x}\|}^{\|x-\bar{x}\|} \frac{2\gamma}{(1 - \gamma u)^3}\mathrm{d}u.
\end{aligned}$$

Thus, the conclusion follows. □

Let L be given by (5.2). Then, it follows from (3.4), (3.6) and (4.1) that

$$\bar{r} = \frac{5 - \sqrt{17}}{4\gamma}, \quad \lambda = \frac{4\|x_0 - x^*\|}{1 - 4\gamma\|x_0 - x^*\| + 2(\gamma\|x_0 - x^*\|)^2} \text{ and } \hat{r} = \frac{1}{2\gamma}.$$

Hence, the following two corollaries follow directly from Proposition 5.1, Theorems 3.3 and 4.1.

Corollary 5.3. *Let* $r \le \frac{5-\sqrt{17}}{4\gamma}$. *Suppose that* x^* *is a solution of* (3.1) *such that* $F'(x^*)$ *is a positive operator (not necessary self-conjugate) and* $\widehat{F'(x^*)}^{-1}$ *exists. Suppose further that* F *satisfies* γ*-condition at* x^* *in* $\mathbf{B}(x^*, r)$. *Let* $x_0 \in \mathbf{B}(x^*, r)$. *Then, the sequence* $\{x_k\}$ *generated by Newton's method* (3.2) *with initial point* x_0 *is well defined and*

$$\|x_k - x^*\| \le \lambda^{2^k-1}\|x_0 - x^*\| \quad \textit{for each } k = 0, 1, 2, \ldots,$$

where

$$\lambda = \frac{4\|x_0 - x^*\|}{1 - 4\gamma\|x_0 - x^*\| + 2(\gamma\|x_0 - x^*\|)^2}.$$

Corollary 5.4. *Let* $r \le \frac{1}{2\gamma}$. *Suppose that* x^* *is a solution of* (3.1) *such that* $F'(x^*)$ *is a positive operator (not necessary self-conjugate) and* $\widehat{F'(x^*)}^{-1}$ *exists. Suppose further that* F *satisfies* γ*-condition at* x^* *in* $\mathbf{B}(x^*, r)$. *Then,* x^* *is the unique solution of* (3.1) *on* $\mathbf{B}(x^*, r)$.

5.3. **Analytic cases.** Let x^* be a solution of (3.1). In this subsection, we assume that F is analytic at x^*. Let x^* be such that $\widehat{F'(x^*)}^{-1}$ exists. Define

$$\gamma(F, x^*) := \|\widehat{F'(x^*)}^{-1}\| \sup_{k\ge 2} \left\| \frac{F^k(x^*)}{k!} \right\|^{\frac{1}{k-1}}.$$

Also we adopt the convention that $\gamma(F, x^*) = \infty$ if $\widehat{F'(x^*)}$ is not invertible. Note that this definition is justified, and in the case when $\widehat{F'(x^*)}$ is invertible, by analyticity, $\gamma(F, x^*)$ is finite. The following lemma shows that if F is analytic , then F satisfies the γ-condition. Its proof is easy and so is omitted here (see also [23, 24]).

Lemma 5.2. *Let* $\gamma := \gamma(F, x^*)$. *Let* $0 < r \le \frac{1}{\gamma}$. *Then* F *satisfies* γ*-condition at* x^* *in* $\mathbf{B}(x^*, r)$.

Hence, the following two corollaries follow directly from Lemma 5.2, Corollaries 5.3 and 5.4.

Corollary 5.5. *Let* $r \le \frac{5-\sqrt{17}}{4\gamma}$. *Suppose that* x^* *is a solution of* (3.1) *such that* $F'(x^*)$ *is a positive operator (not necessary self-conjugate) and* $\widehat{F'(x^*)}^{-1}$ *exists. Let* $x_0 \in \mathbf{B}(x^*, r)$. *Then, the sequence* $\{x_k\}$ *generated by Newton's method* (3.2) *with initial point* x_0 *is well defined and*

$$\|x_k - x^*\| \le \lambda^{2^k-1}\|x_0 - x^*\| \quad \textit{for each } k = 0, 1, 2, \ldots,$$

where

$$\lambda = \frac{4\|x_0 - x^*\|}{1 - 4\gamma\|x_0 - x^*\| + 2(\gamma\|x_0 - x^*\|)^2}.$$

Proof. Since $r = \frac{5-\sqrt{17}}{4\gamma} < \frac{1}{\gamma}$, it follows from Lemma 5.2 that F satisfies the γ-condition at x^* in $\mathbf{B}(x^*, r)$. Thus, Corollary 5.3 is applicable and the conclusions follow. □

Corollary 5.6. *Let* $r \le \frac{1}{2\gamma}$. *Suppose that* x^* *is a solution of* (3.1) *such that* $F'(x^*)$ *is a positive operator (not necessary self-conjugate) and* $\widehat{F'(x^*)}^{-1}$ *exists. Then,* x^* *is the unique solution of* (3.1) *on* $\mathbf{B}(x^*, r)$.

References

[1] F. E. Browder, *ulti-valued monotone nonlinear mappings and duality mappings in Banach spaces*, Trans. Am. Math. Soc. **118** (1965), 338–351
[2] M. Chipot, *Variational Inequalities and Flaw in Porous Media*, Springer, New York, 1984.
[3] A. L. Dontchev, *Local convergence of the Newton method for generalized equation*, C.R.A.S Paris, Series I, **322** (1996), 327–331.
[4] A. L. Dontchev, *Uniform convergence of the Newton method for Aubin continuous maps*, Serdica Math. J. **22** (1996), 385–398.
[5] G. Duvuat and J. L. Lions, *Inequalities m Physics and Mechanics*, Springer, Berlin, 1976.
[6] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.
[7] M. C. Ferris and J. S. Pang, *Engineering and economic applications of complementarity problems*, SIAM Rev. **39** (1997), 669–713.
[8] J. Fliege, L. M. Grana Drummond and S. Svaiter, *Newton's method for multiobjective optimization,* SIAM J. OPTIM. **20** (2009), 602–626.
[9] D. Fu, L. Niu and Z. Wang, *Extensions of the Newton-Kantorovich Theorem to Variational Inequality Problems,* $http : //math.nju.edu.cn/\ zywang/paper/Kantorovich_V IP.pdf$
[10] M. H. Geoffroy, S. Hilout and A. Piétrus, *Stability of a cubically convergent method for generalized equations,* Set-Valued Anal. **14** (2006), 41–54.
[11] S. Haliout, C. Jean-Alexis and A. Pietrus, *A semilocal convergence of the secant-type method for solving a generalized equations*, Possitivity **10** (2006), 673–700.
[12] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Oxford: Pergamon, 1982.
[13] J.S. Pang and D. Chart, Iterative methods for variational and complementarity problems, Math Program. **24** (1982), 284–313.
[14] A. Pietrus, *Does Newton's method for set-valued maps converges uniformly in mild differentiability context?*, Rev. Colombiana Mat. **32** (2000), 49–56.
[15] A. Pietrus, *Generalized equations under mild differentiability conditions*, Rev. R. Acad. Cienc. Exact. Fis. Nat. **94** (2000), 15–18.
[16] S. M. Robinson, *Extension of Newton's method to nonlinear functions with values in a cone*, Numer. Math. **9** (1972), 341–347.
[17] S. M. Robinson, *Strongly regular generalized equations*, Math Oper Res **5** (1980), 43–62.
[18] W. Rudin, *Functional Analysis*, McGraw-Hill, Inc, 1973.
[19] S. Smale, *Newton's method estimates from data at one point,* The Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics, R. Ewing, K. Gross and C. Martin (eds), New York: Springer, 1986, pp. 185–196.
[20] S. Smale, *Complexity theory and numerical analysis,* Acta Numer **6** (1997), 523–551.
[21] J. F. Traub and H. Wozniakowski, *Convergence and complexity of Newton iteration*, J. Assoc. Comput. Math. **29** (1979), 250–258.
[22] L. U. Uko, *Generalized equations and the generalized Newton method,* Math. Program. **73** (1996), 251–268.
[23] X. H. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA J. Numer. Anal. **20** (2000), 123–134.
[24] X. H. Wang, *Convergence of Newton's method and inverse function theorem in Banach space*, Math. of Comput. **225** (1999), 169–186.
[25] X. H. Wang and D. F. Han, *Criterion* α *and Newton's method,* Chinese J. Numer. Appl. Math. **19** (1997), 96–105.
[26] E. Zeidler, *Nonlinear Functional Analysis and Applications II*, Nonlinear Monotone Operators, Springer, Berlin, 1990.

Manuscript received June 30, 2014
revised July 31, 2014

JINHUA WANG
Department of Mathematics, Zhejiang University of Technology, Hangzhou 310032, P R China
E-mail address: wjh@zjut.edu.cn