Journal of Nonlinear and Convex Analysis
Volume 16, Number 9, 2015, 1861–1868

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

AN EXPONENTIAL ESTIMATES OF THE SOLUTION FOR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Wataru Takahashi on his 70th birthday

Abstract. The main aim of this paper is to discuss the exponential estimate of solution of the stochastic functional differential equations under the monotone condition. Furthermore, almost surely asymptotic estimates and pth moment continuous of the solution for these equations are given. More accurately, we shall estimate sample Lyapunov exponent almost surely.

1. Introduction

With the development of industrial technology, Stochastic systems including Brownian motion processes has played an important role in many areas of science and engineering for a long time. Since the white noise is mathematically represented by a formal derivative of a Brownian motion process, such stochastic system is based on various types of stochastic functional differential equations (SFDEs) of Itô type as stochastic model. After Itô introduced his stochastic calculus, the theory of SFDEs have been developed very quickly. SFDEs is the most fundamental concept in modern stochastic models.

In recent years, the existence and uniqueness theorem, exponential estimate, almost sure estimate, stability and approximation of the solution for SFDEs have attracted great attention (see, [1]-[4], [6]-[11] and references therein for details).

In this, we consider a class of stochastic equations depending on past and present values that involves derivatives with delays. Such equations historically have been referred to as *stochanstic fundtional differential equtions, or stochanstic fundtional differential equtions with delay,* it becomes apparent that the principle of causality is often only a first approximation to the true situaton and that a more realistic model would include some of the past states of the system (see, [2], [6], [7], [9]-[12] and references therein for details).

One of the classical and a subject for inquiry in the study of stochanstic fundtional differential equtions(SFDEs) is an existence and uniqueness theorem of the solution to SFDEs under some special conditions. For the work on the existence and uniqueness theorem of the NSFDEs with delay, in 2007 Mao [7] established the existence and uniqueness theorem of solutions of th e equations under uniform Lipschitz condition. In 2007, Wei *et al.* [11] studied the existence and uniqueness

2010 *Mathematics Subject Classification.* 60H10, 60H30.

Key words and phrases. Itô's formula, exponential estimate, almost surely asymptotic estimates, sample Lyapunov exponent.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2014046494).

of the solutions to SFDEs with infinite delay under non-linear growth condition. In 2008, under non-Lipschitz condition, Ren *et al.* [9] obtained the existence and uniqueness theorem and continuous dependence of the solutions to NSFDEs with infinite delay.

Further, in the study of the solution for the SFDEs, one question arises naturally: Does the p-th moment of the solution assure the solution for such SFDEs ? To the best of our knowledge, there are few results on this problem. It is also worth noting that the p-th moment of the solution for such SFDEs has not been fully investigated, which remains an interesting research topic.

Our study is essentially based on the [7] by Mao referring to the exponential estimate and almost surely asymptotic estimates of SFDEs. In fact, we improve Mao's results by finding sufficient conditions, which are easy to verity, guaranteeing exponential estimate and almost surely asymptotic estimates of the solutions of these equations.

2. Preliminary

Let $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\text{trace}(A^T A)}$. Assume that $B(t)$ is an m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$. Let $\mathcal{L}^p([a,b]; R^d)$ denote the family of R^d-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a \leq t \leq b}$ such that $\int_a^b |f(t)|^p dt < \infty$ a.s.. Let $BC((-\infty, 0]; R^d)$ denote the family of bounded continuous R^d-valued functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty < \theta \leq 0} |\varphi|$. Let $\mathcal{M}^2((-\infty, 0]; R^d)$ denote the family of $\mathcal{F}_{t_0}$-measurable, R^d-valued process $\varphi(t) = \varphi(t, \omega), t \in (-\infty, 0]$ such that $E \int_{-\infty}^0 |\varphi(t)|^2 dt < \infty$.

Consider a d-dimensional stochastic functional differential equations

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t) \quad \textit{on } \ t_0 \leq t \leq T, \tag{2.1}$$

where $x_t = \{x(t+\theta) : -\infty < \theta \leq 0\}$ can be regarded as a $BC((-\infty, 0]; R^d)$-value stochastic process, where $f : BC((-\infty, 0]; R^d) \times [t_0, T] \to R^d$ and $g : BC((-\infty, 0]; R^d) \times [t_0, T] \to R^{d \times m}$ be Borel measurable.

The first question is what is the solution of (2.1). More accurately, what is the smallest date of stochastic prcess $x(t)$ defined on $[t_0, T]$? On a moment's thought, we derive that initial data of stochastic process must define on $(-\infty, t_0]$, so, the initial value is followed:

$$\begin{aligned} &x_{t_0} = \xi = \{\xi(\theta) : -\infty \leq \theta \leq 0\} \quad \text{is an } \mathcal{F}_{t_0} - \text{measurable} \\ &BC([-\infty, 0]; R^d) - \text{value random variable such that } \xi \in \mathcal{M}^2((-\infty, 0]; R^d). \end{aligned}$$

In [9], the author presented a result stating that, for initial value $x_{t_0} = \xi$, there exists a unique solution $x(t), t_0 \leq t \leq T$ to the equation (2.1) under non-Lipschitz condition and non-linear growth condition. We reproduce the result here.

Theorem 2.1. *Assume that there exists a constant K and a concave function κ such that*

(i) *(non-Lipschitz condition) For any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that*

$$|f(\varphi, t) - f(\psi, t)|^2 \vee |g(\varphi, t) - g(\psi, t)|^2 \leq \kappa(\|\varphi - \psi\|^2),$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$.

(ii) *(non-linear growth condition) $f(0,t), g(0,t) \in L^2$ and for all $t \in [t_0, T]$, it follows that s*

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K,$$

where $K > 0$ is a constant. Then, there exist a unique solution to (2.1) with initial data.

3. Main results

The topic of our analysis is the equations (2.1) with initial data $x_{t_0} = \xi$. An $\{\mathcal{F}_t\}-$adapted process $x(t)$ with values in R^d is said to be the solution to equation (2.1) if it satisfies the initial condition and the corresponding stochastic integral equation holds a.s., i.e. for every $t \geq t_0$,

$$x(t) = \xi(0) + \int_{t_0}^{t} f(x_s, s)ds + \int_{t_0}^{t} g(x_s, s)dB(s) \quad a.s.$$

The basic existence and uniqueness theorem based on the Picard method of iterations requires the global(see, [1], [7], [9]-[11]). Moreover, there exists a unique a.s. continuous and adapted solution $x(t)$ to equation (2.1) satisfying $E|\sup_{t_0 \leq t \leq T} x(t)|^2 < \infty$ for every $T \geq t_0$ under the linear growth condition. Since our goal is to study exponential estimates and almost surely asymptotic estimates problems, we assume that there exists a unique solution $x(t)$ to equation (2.1) under non Lipschitz condition and non-linear growth condition. We also assume that all the Lebesgue and Itô integrals employed further are well defined.

In this section we shall give the exponential estimates for the solution of equation (2.1), namely

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t) \quad on \ t \in [t_0, \infty) \tag{3.1}$$

with initial data $x_{t_0} = \xi$. We assume that this equation has a unique global solution $x(t)$. We also impose the non-linear growth condition: For any $\varphi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that

$$|f(\varphi, t)|^2 \vee |g(\varphi, t)|^2 \leq \kappa(1 + \|\varphi\|^2), \tag{3.2}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$.

We start with following an exponential estimate.

Theorem 3.1. *Let $p \geq 2$ and $E\|\xi\| < \infty$. Assume that for all $(\varphi, t) \in BC((-\infty, 0]; R^d) \times [t_0, \infty)$, $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$, and $\{g(x(t), t)\} \in$*

$\mathcal{L}^2([t_0,T];R^d)$, *it follows that*

$$x^T f(\varphi,t) \vee \frac{p-1}{2}|g(\varphi,t)|^2 \le \kappa(1+\|\varphi\|^2), \tag{3.3}$$

where $\kappa(\cdot)$ *is a concave nondecreasing function from* R_+ *to* R_+ *such that* $\kappa(0) = 0, \kappa(u) > 0$ *for* $u > 0$. *Then for a pair of positive constants* a *and* b *such that* $\kappa(u) \le a + bu$, *we have*

$$E\Big(\sup_{-\infty<s\le t}|x(s)|^p\Big) \le \frac{3}{2}2^p[1+E\|\xi\|^p]\exp(c_1(t-t_0)) \tag{3.4}$$

for all $t \ge t_0$, *where* $c_1 = 4p(a+b)(1+16p/(p-1))$.

Proof. By Itô's formula, we can derive that for $t \ge t_0$,

$$\begin{aligned}[1+|x(t)|^2]^{p/2} \;=\;& [1+\|\xi\|^2]^{p/2} + p\int_{t_0}^t [1+|x(s)|^2]^{(p-2)/2}x^T(s)f(x_s,s)ds\\ &+\frac{p}{2}\int_{t_0}^t[1+|x(s)|^2]^{(p-2)/2}|g(x_s,s)|^2ds\\ &+\frac{p(p-2)}{2}\int_{t_0}^t[1+|x(s)|^2]^{(p-4)/2}|x^T(s)g(x_s,s)|^2ds\\ &+p\int_{t_0}^t[1+|x(s)|^2]^{(p-2)/2}x^T(s)g(x_s,s)dB(s).\end{aligned}$$

By the condition (3.3), it is easy to see that

$$\begin{aligned}[1+|x(t)|^2]^{\frac{p}{2}} \;\le\;& 2^{\frac{p-2}{2}}(1+\|\xi\|^p) + 2p\int_{t_0}^t[1+|x(s)|^2]^{\frac{p-2}{2}}\kappa(1+\|x_s\|^2)ds\\ &+p\int_{t_0}^t[1+|x(s)|^2]^{(p-2)/2}x^T(s)g(x_s,s)dB(s).\end{aligned} \tag{3.5}$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0)=0$, we can find a pair of positive constants a and b such that $\kappa(u) \le a+bu$ for all $u \ge 0$. Therefore

$$\begin{aligned}&E\Big(\sup_{t_0\le s\le t}[1+|x(s)|^2]^{\frac{p}{2}}\Big)\\ &\le 2^{\frac{p-2}{2}}[1+E\|\xi\|^p] + 2p(a+b)E\int_{t_0}^t[1+\|x_s\|^2]^{\frac{p}{2}}ds\\ &\quad +pE\Big(\sup_{t_0\le s\le t}\int_{t_0}^s[1+|x(r)|^2]^{\frac{p-2}{2}}x^T(r)g(x_r,r)dB(r)\Big).\end{aligned} \tag{3.6}$$

On the other hand, by the Burkholder-Davis-Gundy inequality(see, [7], Theorem 1.7.3), we derive that

$$\begin{aligned}&pE\Big(\sup_{t_0\le s\le t}\int_{t_0}^s[1+|x(r)|^2]^{\frac{p-2}{2}}x^T(r)g(x_r,r)dB(r)\Big)\\ &\le 4\sqrt{2}pE\Big(\int_{t_0}^t[1+|x(s)|^2]^{p-2}|x^T(s)g(x_s,s)|^2ds\Big)^{\frac{1}{2}}\end{aligned}$$

$$\leq 4\sqrt{2}pE\left\{\left(\sup_{t_0\leq s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\right)\int_{t_0}^{t}[1+|x(s)|^2]^{\frac{p-4}{2}}|x(s)|^2|g(x_s,s)|^2ds\right\}^{\frac{1}{2}}$$
$$\leq \frac{1}{2}E\left(\sup_{t_0\leq s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\right)+16p^2E\int_{t_0}^{t}[1+|x(s)|^2]^{\frac{p-4}{2}}|x(s)|^2|g(x_s,s)|^2ds$$
$$\leq \frac{1}{2}E\left(\sup_{t_0\leq s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\right)+\frac{32p^2}{p-1}(a+b)E\int_{t_0}^{t}[1+\|x_s\|^2]^{\frac{p}{2}}ds.$$

Substituting this into (3.6) yields that

$$E\Big(\sup_{t_0\leq s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\Big)\leq 2^{\frac{p}{2}}[1+E\|\xi\|^p]+c_1E\int_{t_0}^{t}[1+\|x_s\|^2]^{\frac{p}{2}}ds, \tag{3.7}$$

where $c_1=4p(a+b)(1+16p/(p-1))$. Note that

$$E\Big(\sup_{-\infty<s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\Big)\leq 2^{\frac{p-2}{2}}[1+E\|\xi\|^p]+E\Big(\sup_{t_0\leq s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\Big),$$

It then follows from (3.7) that

$$E\Big(\sup_{-\infty<s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\Big)\leq \frac{3}{2}2^p[1+E\|\xi\|^p]+c_1\int_{t_0}^{t}E\Big(\sup_{-\infty<s\leq t}[1+\|x_s\|^2]^{\frac{p}{2}}\Big)ds.$$

An application of the Gronwall inequality implies that

$$E\Big(\sup_{-\infty<s\leq t}[1+|x(s)|^2]^{\frac{p}{2}}\Big)\leq \frac{3}{2}2^p[1+E\|\xi\|^p]e^{c_1(t-t_0)}. \tag{3.8}$$

and the desired inequality follows. The proof is compliet. □

Let us now turn to consider the case of $0<p<2$. This is rather easy if we note that the Hölder inequality implies

$$E|x(t)|^p\leq (E|x(t)|^2)^{\frac{p}{2}}.$$

In other words, the estimate for $E|x(t)|^p$ can be done via the estimate for the second moment. For instance, we have the following corollary.

Corollary 3.2. *Let $0<p<2$ and $E\|\xi\|<\infty$. Assume that for all $(\varphi,t)\in BC((-\infty,0];R^d)\times[t_0,\infty)$, $\{f(x(t),t)\}\in\mathcal{L}^1([t_0,T];R^d)$, and $\{g(x(t),t)\}\in\mathcal{L}^2([t_0,T];R^d)$, it follows that*

$$x^Tf(\varphi,t)\vee\frac{1}{2}|g(\varphi,t)|^2\leq\kappa(1+\|\varphi\|^2), \tag{3.9}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from R_+ to R_+ such that $\kappa(0)=0,\kappa(u)>0$ for $u>0$. Then for a pair of positive constants a and b such that $\kappa(u)\leq a+bu$, we have

$$E\Big(\sup_{-\infty<s\leq t}|x(s)|^p\Big)\leq 3[1+E\|\xi\|^2]^{\frac{p}{2}}\exp(c_2(t-t_0)) \tag{3.10}$$

for all $t\geq t_0$, where $c_2=4p(a+b)(1+16p)$.

We now consider a nonlinear growth condition

$$|f(\varphi,t)|^2 \vee |g(\varphi,t)|^2 \le \kappa_1(1+\|\varphi\|^2) \tag{3.11}$$

for all $(\varphi,t) \in BC((-\infty,0];R^d) \times [t_0,\infty)$ with $\kappa_1(\cdot)$ is a concave nondecreasing function from R_+ to R_+ such that $\kappa_1(0)=0, \kappa_1(u)>0$ for $u>0$.

In fact, using (3.11) and the elementary inequality $2ab \le a^2+b^2$ one can derive that the condition (3.3) is satisfied with $\kappa(u) = u\sqrt{\kappa_1(u)} \vee \frac{p-1}{2}\kappa_1(u)$.

As an application of Theorem 3.1 we give one of the important properties of the solution.

Theorem 3.3. *Let $p \ge 2$ and and $E\|\xi\| < \infty$. Assume that the nonlinear growth condition (3.11) for all $(\varphi,t) \in BC((-\infty,0];R^d) \times [t_0,\infty)$. Then*

$$E|x(t)-x(s)|^p \le \gamma(t)(t-s)^{\frac{p}{2}},$$

where

$$\gamma(t) = \Big[\frac{1}{2}c_3(2\alpha)^{\frac{p}{2}} + 3c_3 2^{p-2}(2\beta)^{\frac{p}{2}}[1+E\|\xi\|^p]e^{c_1(t-t_0)}\Big],$$

$c_3 = (2(t-t_0))^{\frac{p}{2}} + \frac{1}{2}(2p(p-1))^{\frac{p}{2}}$, *and α and β are a pair of positive constants such that $\kappa_1(u) \le \alpha+\beta u$ for all $u \ge 0$. In particular, the pth moment of the solution is continuous.*

Proof. Applying the elementary inequality $|a+b|^p \le 2^{p-1}(|a|^p+|b|^p)$, it is easy to see that

$$E|x(t)-x(s)|^p \le 2^{p-1}E\Big|\int_s^t f(x_r,r)\,\mathrm{d}r\Big|^p + 2^{p-1}E\Big|\int_s^t g(x_r,r)\,\mathrm{d}B(r)\Big|^p.$$

Using the Hölder's inequality, the moment inequality ([7], Theorem 1.7.1), and the condition (3.11), one can show that

$$\begin{aligned} E|x(t)-x(s)|^p &\le (2(t-s))^{p-1}E\int_s^t |f(x_r,r)|^p dr \\ &\quad + \frac{1}{2}(2p(p-1))^{\frac{p}{2}}(t-s)^{\frac{p-2}{2}}E\int_s^t |g(x_r,r)|^p\,dr \\ &\le c_3(t-s)^{\frac{p-2}{2}}E\int_s^t (\kappa_1(1+\|x_r\|^2))^{\frac{p}{2}}\,dr, \end{aligned}$$

where $c_3 = (2(t-t_0))^{\frac{p}{2}} + \frac{1}{2}(2p(p-1))^{\frac{p}{2}}$. Given that $\kappa_1(\cdot)$ is concave and $\kappa_1(0)=0$, we can find a pair of positive constants α and β such that $\kappa_1(u) \le \alpha+\beta u$ for all $u \ge 0$. So we have

$$E|x(t)-x(s)|^p \le \frac{1}{2}c_3(2\alpha)^{\frac{p}{2}}(t-s)^{\frac{p}{2}} + \frac{1}{2}c_3(2\beta)^{\frac{p}{2}}(t-s)^{\frac{p-2}{2}}\int_s^t E(1+\|x_r\|^2)^{\frac{p}{2}}\,dr.$$

Substituting this into (3.8) yields that

$$E|x(t)-x(s)|^p \le \Big(\frac{1}{2}c_3(2\alpha)^{\frac{p}{2}} + 3c_3 2^{p-2}(2\beta)^{\frac{p}{2}}[1+E\|\xi\|^p]e^{c_1(t-t_0)}\Big)(t-s)^{\frac{p}{2}},$$

which is required inequality. $\square$

In view of Theorem 3.1, we know that the pth moment of the solution satisfies

$$E|x(t)|^p \le \frac{3}{2}2^p[1 + E\|\xi\|^p]\exp(c_1(t - t_0))$$

for all $t \ge t_0$. This means that the pth moment will grow at most exponentially with exponent $4p(a+b)(1+16p/(p-1))$. This can also be expressed as

$$\limsup_{t\to\infty} \frac{1}{t}\log(E|x(t)|^p) \le 4p(a+b)\Big(1 + \frac{16p}{(p-1)}\Big). \tag{3.12}$$

The left-hand side of (3.12) is called the *pth moment Lyapunov exponent,* and (3.12) shows that the pth moment Lyapunov exponent should not be greater than $4p(a+b)(1+16p/(p-1))$. In next, we shall establish the asymptotic estimate for the solution almost surely. More accurately, we shall estimate

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)|$$

almost surely, which is called the *sample Lyapunov exponent.*

Theorem 3.4. *Under the condition (3.9) the sample Lyapunov exponent of the solution of equation (3.1) should not be greater than* $264(a+b)$, *that is*

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \le 264(a+b)$$

almost surely.

Proof. For each $n = 1, 2, \dots,$ it follows from Theorem 3.1 (taking $p = 2$) that

$$E\Big(\sup_{t_0+n-1\le t\le t_0+n} |x(t)|^2\Big) \le \mu e^{\nu},$$

where $\mu = 6(1 + E\|\xi\|^2)$, and $\nu = 264(a+b)$. Hence, for arbitrary $\epsilon > 0$,

$$P\left\{\omega : \sup_{t_0+n-1\le t\le t_0+n} |x(t)|^2 > e^{(\nu+\epsilon)n}\right\} \le \mu e^{-\epsilon n}.$$

The Borel-Cantelli lemma now yields that for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega)$ such that

$$\sup_{t_0+n-1\le t\le t_0+n} |x(t)|^2 \le e^{(\nu+\epsilon)n}$$

whenever $n \ge n_0$. Consequently, for almost all $\omega \in \Omega$, if $t_0 + n - 1 \le t \le t_0 + n$ and $n \ge n_0$,

$$\frac{1}{t}\log|x(t)| \le \frac{(\nu+\epsilon)n}{2(t_0+n-1)}.$$

Thus

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \le \frac{\nu+\epsilon}{2} = 264(a+b) + \frac{\epsilon}{2}$$

almost surely. Since ϵ is arbitrary, the assertion must hold. □

References

[1] Y. J. Cho, S. S. Dragomir and Y.-H. Kim, *A note on the existence and uniqueness of the solutions to SFDEs*, J. Inequal. Appl. 2012:126 (2012), 1–16.

[2] T. E. Govindan, *Stability of mild solution of stochastic evolution equations with variable delay*, Stochastic Anal. Appl. **21** (2003), 1059–1077.

[3] D. Henderson and P. Plaschko, *Stochastic Differential Equations in Science and Engineering*, World Scientific Publishing, Hackensack, NJ, 2006.

[4] Y.-H. Kim, *An estimate on the solutions for stochastic functional differential equations*, J. Appl. Math. and Informatics **29** (2011), 1549–1556.

[5] V. B. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Kluwer Academic Publishers, Dordrecht, 1992.

[6] X. Li and X. Fu, *Stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks*, J. Comput. Appl. Math. **234** (2010), 407–417.

[7] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Publication Chichester, UK, 2007.

[8] X. Mao, Y. Shen, and C. Yuan, *Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching*, Stochastic process. Appl. **118** (2008), 1385–1406.

[9] Y. Ren, S. Lu, and N. Xia, *Remarks on the existence and uniqueness of the solutions to stochastic functional differential equations with infinite delay*, J. Comput. Appl. Math. **220** (2008), 364–372.

[10] Y. Ren and N. Xia, *Existence, uniqueness and stability of the solutions to neutral stochastic functional differential equations with infinite delay*, Appl. Math. Comput. **210** (2009), 72–79.

[11] F. Wei and K. Wang, *The existence and uniqueness of the solution for stochastic functional differential equations with infinite delay*, J. Math. Anal. Appl. **331** (2007), 516–531.

[12] F. Wei and Y. Cai, *Existence, uniqueness and stability of the solution to neutral stochastic functional differential equations with infinite delay under non-Lipschitz conditions*, Advances in Difference Equations 2013:151 (2013).

Manuscript received June 30, 2014
revised October 2, 2014

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