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SOME IDENTITIES OF q-BERNOULLI POLYNOMIALS UNDER SYMMETRY GROUP S_3

TAEKYUN KIM, DMITRY V. DOLGY, AND DAE SAN KIM

Dedicated to Professor W. Takahashi on the occasion of his 70th birthday

Abstract. In this paper, we give some new identities of Carlitz q-Bernoulli polynomials under symmetry group S_3. The derivatives of identities are based on the q-Volkenborn integral expression of the generating function for the Carlitz q-Bernoulli polynomials and the q-Volkenborn integral equations that can be expressed as the exponential generating functions for the q-power sums.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$ and let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-number of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the q-Volkenborn integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)\, q^x, \quad (\text{see [13]}). \tag{1.1}$$

In (1.1), we note that

$$qI_q(f_1) - I_q(f) = \frac{q-1}{\log q} f'(0) + (q-1) f(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$.

In general, one derives

$$q^n I_q(f_n) - I_q(f) = \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l)\, q^l + (q-1) \sum_{l=0}^{n-1} f(l)\, q^l, \tag{1.3}$$

where $f_n(x) = f(x+n)$, $(n \geq 0)$, (see [15, 16, 27]).

It is well known that the Bernoulli numbers are given by

$$B_0 = 1, \quad (B+1)^n - B_n = \delta_{1,n}, \quad (\text{see [1–27]}) \tag{1.4}$$

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with the usual convention about replacing B^n by B_n.

The Bernoulli polynomials are defined by

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \quad (\text{see [29,30]}). \tag{1.5}$$

In [4,5], L. Carlitz considered a q-analogue of Bernoulli numbers as follows :

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.6}$$

with the usual convention about replacing β_q^i by $\beta_{i,q}$.

He also defined q-Bernoulli polynomials as follows :

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l}, \quad (\text{see [4,5]}). \tag{1.7}$$

From (1.6), Carlitz derived the following equation :

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l+1}{[l+1]_q}, \tag{1.8}$$

and

$$\beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}, \quad (\text{see [4]}). \tag{1.9}$$

Carlitz q-Bernoulli numbers and polynomials are also given by q-Volkenborn integrals on $\mathbb{Z}_p$ due to T. Kim [13] :

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n,q} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} e^{[y]_q t} d\mu_q(y) \\ &= \sum_{m=0}^{\infty} q^m e^{[m]_q t} (1 - q - q^m t), \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_q(y) \\ &= \sum_{m=0}^{\infty} q^m e^{[x+m]_q t} \left(1 - q - q^{x+m} t\right). \end{aligned} \tag{1.11}$$

The purpose of this paper is to give some new identities of Carlitz q-Bernoulli polynomials under symmetry group S_3. The derivations of identities are based on the q-Volkenborn integral expression of the generating function for the Carlitz q-Bernoulli polynomials and the q-Volkenborn integrals equations that can be expressed as the exponential generating functions for the q-power sums.

2. Some identities of Carlitz q-Bernoulli polynomials

By (1.7), we easily get

$$\beta_{n,q}(x+y) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q}(y) [x]_q^{n-l} = \sum_{l=0}^{n} \binom{n}{l} q^{(n-l)x} \beta_{n-l,q}(y) [x]_q^{l}. \tag{2.1}$$

On the other hand, Carlitz also introduced the expression of q-Bernoulli polynomials $\beta_{n,q}^{(h,k)}(x)$ as follows :

$$\beta_{n,q}^{(h,k)}(x) = \frac{1}{(1-q)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{jx} \frac{(j+h)_k}{[j+h]_{q,k}}, \tag{2.2}$$

where

$$[j+h]_{q,k} = [j+h]_q [j+h-1]_q \cdots [j+h-k+1]_q, \tag{2.3}$$

and

$$(j+h)_k = (j+h)(j+h-1)\cdots(j+h-k+1), \quad (\text{see } [5,13]). \tag{2.4}$$

T. Kim [13] obtained the Witt-type formula for $\beta_{n,q}^{(h,k)}(x)$ as follows :

$$\beta_{n,q}^{(h,k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^{k}(h-l)y_l} [x+y_1+\cdots+y_k]_q^n \, d\mu_q(y_1)\cdots d\mu_q(y_k). \tag{2.5}$$

If $k=1$, then $\beta_{n,q}^{(h,1)}(x)$ will be simply denoted by $\beta_{n,q}^{(h)}(x)$ so that

$$\beta_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)y} [x+y]_q^n \, d\mu_q(y). \tag{2.6}$$

The following simple facts will be used over and over again :

$$[a+b]_q = [a]_q + q^a [b]_q. \tag{2.7}$$

By (2.7), we easily see that

$$[a+b+c]_q = [a]_q + q^a [b]_q + q^{a+b} [c]_q, \tag{2.8}$$

and

$$[ab]_q = [a]_q [b]_{q^a}. \tag{2.9}$$

First, we will consider the following triple integral which is obviously invariant under any permutations of w_1, w_2, w_3.

So the expression obtained from this after integration will also be invariant under any permutations of w_1, w_2, w_3. This observation is simple enough but it is the philosophy that underlies this paper.

$$I = \int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2+y_3)]_q t} \tag{2.10}$$

$$\times\, d\mu_{q^{w_2w_3}}\left(x_1\right) d\mu_{q^{w_1w_3}}\left(x_2\right) d\mu_{q^{w_1w_2}}\left(x_3\right).$$

It is easy to show that

$$\begin{aligned}(2.11)\qquad & \left[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3\left(y_1+y_2+y_3\right)\right]_q \\ =& \left[w_2w_3\right]_q \left[x_1+w_1y_1\right]_{q^{w_2w_3}} + q^{w_2w_3(x_1+w_1y_1)} \left[w_1w_3\right]_q \left[x_2+w_2y_2\right]_{q^{w_1w_3}} \\ &+ q^{w_2w_3(x_1+w_1y_1)+w_1w_3(x_2+w_2y_2)} \left[w_1w_2\right]_q \left[x_3+w_3y_3\right]_{q^{w_1w_2}}.\end{aligned}$$

So the integrand is

$$\begin{aligned}(2.12)\qquad & e^{\left[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2+y_3)\right]_q t} \\ =& e^{\left[w_2w_3\right]_q \left[x_1+w_1y_1\right]_{q^{w_2w_3}} t} e^{q^{w_2w_3(x_1+w_1y_1)} \left[w_1w_3\right]_q \left[x_2+w_2y_2\right]_{q^{w_1w_3}} t} \\ & \times e^{q^{w_2w_3(x_1+w_1y_1)+w_1w_3(x_2+w_2y_2)} \left[w_1w_2\right]_q \left[x_3+w_3y_3\right]_{q^{w_1w_2}} t} \\ =& \sum_{n=0}^{\infty} \sum_{k+l+m=n} \binom{n}{k,l,m} \left[w_2w_3\right]_q^k \left[w_1w_3\right]_q^l \left[w_1w_2\right]_q^m q^{w_1w_2w_3(l+m)y_1} \\ & \times q^{w_1w_2w_3my_2} q^{w_2w_3(l+m)x_1} \left[x_1+w_1y_1\right]_{q^{w_2w_3}}^k q^{w_1w_3mx_2} \\ & \times \left[x_2+w_2y_2\right]_{q^{w_1w_3}}^l \left[x_3+w_3y_3\right]_{q^{w_1w_2}}^m \frac{t^n}{n!}.\end{aligned}$$

Thus the integral in (2.10) is given by

$$\begin{aligned}(2.13)\qquad I =& \sum_{n=0}^{\infty} \left\{ \sum_{k+l+m=n} \binom{n}{k,l,m} \left[w_2w_3\right]_q^k \left[w_1w_3\right]_q^l \left[w_1w_2\right]_q^m q^{w_1w_2w_3(l+m)y_1} \right. \\ & \times q^{w_1w_2w_3my_2} \int_{\mathbb{Z}_p} q^{w_2w_3(l+m)x_1} \left[x_1+w_1y_1\right]_{q^{w_2w_3}}^k d\mu_{q^{w_2w_3}}\left(x_1\right) \\ & \times \int_{\mathbb{Z}_p} q^{w_1w_3mx_2} \left[x_2+w_2y_2\right]_{q^{w_1w_3}}^l d\mu_{q^{w_1w_3}}\left(x_2\right) \\ & \left. \times \int_{\mathbb{Z}_p} \left[x_3+w_3y_3\right]_{q^{w_1w_2}}^m d\mu_{q^{w_1w_2}}\left(x_3\right) \right\} \frac{t^n}{n!} \\ =& \sum_{n=0}^{\infty} \left\{ \sum_{k+l+m=n} \binom{n}{k,l,m} \left[w_2w_3\right]_q^k \left[w_1w_3\right]_q^l \right. \\ & \times \left[w_1w_2\right]_q^m q^{w_1w_2w_3(l+m)y_1} q^{w_1w_2w_3my_2} \\ & \left. \times \beta_{k,q^{w_2w_3}}^{(l+m+1)}\left(w_1y_1\right) \beta_{l,q^{w_1w_3}}^{(m+1)}\left(w_2y_2\right) \beta_{m,q^{w_1w_2}}\left(w_3y_3\right) \right\} \frac{t^n}{n!}.\end{aligned}$$

Thus, by (2.13), we get the following theorem.

Theorem 2.1. *Let w_1, w_2, w_3 be any positive integers, n any nonnegative integer. Then the following expression is invariant under any permutation of w_1, w_2, w_3 so that it gives us six symmetries :*

$$\sum_{k+l+m=n} \binom{n}{k,l,m} \left[w_2w_3\right]_q^k \left[w_1w_3\right]_q^l \left[w_1w_2\right]_q^m q^{w_1w_2w_3(l+m)y_1}$$

$$\times q^{w_1w_2w_3my_2}\beta_{k,q^{w_2w_3}}^{(l+m+1)}(w_1y_1)\,\beta_{l,q^{w_1w_3}}^{(m+1)}(w_2y_2)\,\beta_{m,q^{w_1w_2}}(w_3y_3)$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_3]_q^k[w_2w_3]_q^l[w_1w_2]_q^m q^{w_1w_2w_3(l+m)y_1}$$

$$\times q^{w_1w_2w_3my_2}\beta_{k,q^{w_1w_3}}^{(l+m+1)}(w_2y_1)\,\beta_{l,q^{w_2w_3}}^{(m+1)}(w_1y_2)\,\beta_{m,q^{w_1w_2}}(w_3y_3)$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_3]_q^k[w_1w_2]_q^l[w_2w_3]_q^m q^{w_1w_2w_3(l+m)y_1}$$

$$\times q^{w_1w_2w_3my_2}\beta_{k,q^{w_1w_3}}^{(l+m+1)}(w_2y_1)\,\beta_{l,q^{w_1w_2}}^{(m+1)}(w_3y_2)\,\beta_{m,q^{w_2w_3}}(w_1y_3)$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}[w_2w_3]_q^k[w_1w_2]_q^l[w_1w_3]_q^m q^{w_1w_2w_3(l+m)y_1}$$

$$\times q^{w_1w_2w_3my_2}\beta_{k,q^{w_2w_3}}^{(l+m+1)}(w_1y_1)\,\beta_{l,q^{w_1w_2}}^{(m+1)}(w_3y_2)\,\beta_{m,q^{w_1w_3}}(w_2y_3)$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_2]_q^k[w_2w_3]_q^l[w_1w_3]_q^m q^{w_1w_2w_3(l+m)y_1}$$

$$\times q^{w_1w_2w_3my_2}\beta_{k,q^{w_1w_2}}^{(l+m+1)}(w_3y_1)\,\beta_{l,q^{w_2w_3}}^{(m+1)}(w_1y_2)\,\beta_{m,q^{w_1w_3}}(w_2y_3)$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}[w_1w_2]_q^k[w_1w_3]_q^l[w_2w_3]_q^m q^{w_1w_2w_3(l+m)y_1}$$

$$\times q^{w_1w_2w_3my_2}\beta_{k,q^{w_1w_2}}^{(l+m+1)}(w_3y_1)\,\beta_{l,q^{w_1w_3}}^{(m+1)}(w_2y_2)\,\beta_{m,q^{w_2w_3}}(w_1y_3).$$

We define, for nonnegative integers $n, m, w, T_{n,m}(w|q)$ as

$$T_{n,m}(w|q)=\sum_{i=0}^{w}q^{ni}[i]_q^m. \tag{2.14}$$

In particular, for $w=0$, we have

$$T_{n,m}(0|q)=\begin{cases}1, & \text{if } m=0\\ 0, & \text{if } m>0,\end{cases} \tag{2.15}$$

and

$$T_{n,0}(w|q)=\begin{cases}w+1 & \text{if } n=0\\ [w+1]_{q^n} & \text{if } n>0.\end{cases} \tag{2.16}$$

From (1.3), we have

$$\begin{aligned}(q^{w_1w_2})^{w_3}&\int_{\mathbb{Z}_p}e^{[w_1w_2(x+w_3)]_qt}d\mu_{q^{w_1w_2}}(x)\\ &-\int_{\mathbb{Z}_p}e^{[w_1w_2x]_qt}d\mu_{q^{w_1w_2}}(x)\\ =&t[w_1w_2]_q\sum_{i=0}^{w_3-1}q^{2w_1w_2i}e^{[w_1w_2i]_qt}\end{aligned} \tag{2.17}$$

$$
\begin{aligned}
&+(q-1)\left[w_1w_2\right]_q \sum_{i=0}^{w_3-1} q^{w_1w_2i} e^{[w_1w_2i]_q t} \\
&=\sum_{m=0}^{\infty} T_{2,m}\left(w_3-1|q^{w_1w_2}\right)\left[w_1w_2\right]_q^{m+1} \frac{t^{m+1}}{m!} \\
&+(q-1)\sum_{m=0}^{\infty} T_{1,m}\left(w_3-1|q^{w_1w_2}\right)\left[w_1w_2\right]_q^{m+1} \frac{t^{m}}{m!}.
\end{aligned}
$$

Thus, we have the following lemma.

Lemma 2.2. *For $w_1, w_2, w_3 \geq 1$, we have*

$$
\begin{aligned}
&q^{w_1w_2w_3} \int_{\mathbb{Z}_p} e^{[w_1w_2(x+w_3)]_q t} d\mu_{q^{w_1w_2}}(x) - \int_{\mathbb{Z}_p} e^{[w_1w_2x]_q t} d\mu_{q^{w_1w_2}}(x) \\
&=\sum_{m=0}^{\infty} \frac{[w_1w_2]_q^m}{m!} t^m \int_{\mathbb{Z}_p} \left(q^{w_1w_2w_3}\left[x+w_3\right]_{q^{w_1w_2}}^m - [x]_{q^{w_1w_2}}^m\right) d\mu_{q^{w_1w_2}}(x) \\
&=\sum_{m=0}^{\infty} T_{2,m}\left(w_3-1|q^{w_1w_2}\right)\left[w_1w_2\right]_q^{m+1} \frac{t^{m+1}}{m!} \\
&+(q-1)\sum_{m=0}^{\infty} T_{1,m}\left(w_3-1|q^{w_1w_2}\right)\left[w_1w_2\right]_q^{m+1} \frac{t^{m}}{m!} \\
&=t\left[w_1w_2\right]_q \sum_{i=0}^{w_3-1} q^{2w_1w_2i} e^{[w_1w_2i]_q t} + (q-1)\left[w_1w_2\right]_q \sum_{i=0}^{w_3-1} q^{w_1w_2i} e^{[w_1w_2i]_q t}.
\end{aligned}
$$

Now, we consider the following difference of triple integrals.

$$
\begin{aligned}
I_1 =& q^{w_1w_2w_3} \int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2+1)]_q t} \\
&\times d\mu_{q^{w_2w_3}}(x_1)\, d\mu_{q^{w_1w_3}}(x_2)\, d\mu_{q^{w_1w_2}}(x_3) \\
&- \int_{\mathbb{Z}_p^3} e^{[w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(y_1+y_2)]_q t} \\
&\times d\mu_{q^{w_2w_3}}(x_1)\, d\mu_{q^{w_1w_3}}(x_2)\, d\mu_{q^{w_1w_2}}(x_3),
\end{aligned}
\tag{2.18}
$$

which is obviously invariant under any permutations of w_1, w_2, w_3.

We put

$$
a = a(x_1) = q^{w_2w_3(x_1+w_1y_1)}, \quad b = b(x_2) = q^{w_1w_3(x_2+w_2y_2)}. \tag{2.19}
$$

Then, by (2.18), we get

$$
\begin{aligned}
I_1 =& \sum_{k,l=0}^{\infty} \left[w_2w_3\right]_q^k \left[w_1w_3\right]_q^l \frac{t^{k+l}}{k!l!} \int_{\mathbb{Z}_p^2} a^l \left[x_1+w_1y_1\right]_{q^{w_2w_3}}^k \left[x_2+w_2y_2\right]_{q^{w_1w_3}}^l \\
&\times \left\{ \sum_{m=0}^{\infty} \frac{\left[w_1w_2\right]_q^m (abt)^m}{m!} \right.
\end{aligned}
\tag{2.20}
$$

$$\times \int_{\mathbb{Z}_p} \left(q^{w_1 w_2 w_3} [x_3 + w_3]_{q^{w_1 w_2}}^m - [x_3]_{q^{w_1 w_2}}^m \right) d\mu_{q^{w_1 w_2}}(x_3) \Bigg\}$$
$$\times\, d\mu_{q^{w_2 w_3}}(x_1)\, d\mu_{q^{w_1 w_3}}(x_2).$$

From Lemma 2.2, the inner sum is

$$\sum_{m=0}^{\infty} \frac{[w_1 w_2]_q^{m+1} (abt)^{m+1}}{m!} T_{2,m}(w_3 - 1 | q^{w_1 w_2}) \tag{2.21}$$
$$+ (q-1) \sum_{m=0}^{\infty} \frac{[w_1 w_2]_q^{m+1} (abt)^m}{m!} T_{1,m}(w_3 - 1 | q^{w_1 w_2}).$$

Thus, by (2.20) and (2.21), we get

$$I_1 = \sum_{k,l,m=0}^{\infty} [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} \frac{t^{k+l+m+1}}{k!l!m!} T_{2,m}(w_3 - 1 | q^{w_1 w_2}) \tag{2.22}$$
$$\times \int_{\mathbb{Z}_p} a^{l+m+1} [x_1 + m_1 y_1]_{q^{w_2 w_3}}^k d\mu_{q^{w_2 w_3}}(x_1)$$
$$\times \int_{\mathbb{Z}_p} b^{m+1} [x_2 + w_2 y_2]_{q^{w_1 w_3}}^l d\mu_{q^{w_1 w_3}}(x_2)$$
$$+ (q-1) \sum_{k,l,m=0}^{\infty} [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} \frac{t^{k+l+m}}{k!l!m!}$$
$$\times T_{1,m}(w_3 - 1 | q^{w_1 w_2}) \int_{\mathbb{Z}_p} a^{l+m} [x_1 + w_1 y_1]_{q^{w_2 w_3}}^k d\mu_{q^{w_2 w_3}}(x_1)$$
$$\times \int_{\mathbb{Z}_p} b^m [x_2 + w_2 y_2]_{q^{w_1 w_3}}^l d\mu_{q^{w_1 w_3}}(x_2).$$

Recovering $a = q^{w_2 w_3 (x_1 + w_1 y_1)}$ and $b = q^{w_1 w_3 (x_2 + w_2 y_2)}$, (2.18) can be rewritten as:

$$I_1 = \sum_{n=0}^{\infty} \Bigg\{ \sum_{k+l+m=n} \binom{n}{k,l,m} [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} \tag{2.23}$$
$$\times T_{2,m}(w_3 - 1 | q^{w_1 w_2}) q^{w_1 w_2 w_3 (l+m+1) y_1} q^{w_1 w_2 w_3 (m+1) y_2}$$
$$\times \int_{\mathbb{Z}_p} q^{w_2 w_3 (l+m+1) x_1} [x_1 + w_1 y_1]_{q^{w_2 w_3}}^k d\mu_{q^{w_2 w_3}}(x_1)$$
$$\times \int_{\mathbb{Z}_p} q^{w_1 w_3 (m+1) x_2} [x_2 + w_2 y_2]_{q^{w_1 w_3}}^l d\mu_{q^{w_1 w_3}}(x_2) \Bigg\} \frac{t^{n+1}}{n!}$$
$$+ (q-1) \sum_{n=0}^{\infty} \Bigg\{ \sum_{k+l+m=n} \binom{n}{k,l,m}$$
$$\times [w_2 w_3]_q^k [w_1 w_3]_q^l [w_1 w_2]_q^{m+1} T_{1,m}(w_3 - 1 | q^{w_1 w_2})$$
$$\times q^{w_1 w_2 w_3 (l+m) y_1} q^{w_1 w_2 w_3 m y_2}$$

$$\times \int_{\mathbb{Z}_p} q^{w_2 w_3 (l+m) x_1} \left[x_1 + w_1 y_1\right]_{q^{w_2 w_3}}^k d\mu_{q^{w_2 w_3}}(x_1)$$

$$\left.\times \int_{\mathbb{Z}_p} q^{w_1 w_3 m x_2} \left[x_2 + w_2 y_2\right]_{q^{w_1 w_3}}^l d\mu_{q^{w_1 w_3}}(x_2)\right\} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{k+l+m=n} \binom{n}{k,l,m} \beta_{k,q^{w_2 w_3}}^{(l+m+2)}(w_1 y_1) \right.$$

$$\times \beta_{l,q^{w_1 w_3}}^{(m+2)}(w_2 y_2)\, T_{2,m}(w_3 - 1|q^{w_1 w_2})\, q^{w_1 w_2 w_3 (l+m+1) y_1}$$

$$\left. \times q^{w_1 w_2 w_3 (m+1) y_2} \left[w_2 w_3\right]_q^k \left[w_1 w_3\right]_q^l \left[w_1 w_2\right]_q^{m+1} \right\} \frac{t^{n+1}}{n!}$$

$$+ (q-1) \sum_{n=0}^{\infty} \left\{ \sum_{k+l+m=n} \binom{n}{k,l,m} \beta_{k,q^{w_2 w_3}}^{(l+m+1)}(w_1 y_1) \right.$$

$$\times \beta_{l,q^{w_1 w_3}}^{(m+1)}(w_2 y_2)\, T_{1,m}(w_3 - 1|q^{w_1 w_2})$$

$$\left. \times q^{w_1 w_2 w_3 (l+m) y_1} q^{w_1 w_2 w_3 m y_2} \left[w_2 w_3\right]_q^k \left[w_1 w_3\right]_q^l \left[w_1 w_2\right]_q^{m+1} \right\} \frac{t^n}{n!}.$$

Separating the term corresponding to $n = 0$ from the second term and after rearranging, we get :

$$I_1 = (q-1)\left[w_1 w_2 w_3\right]_q + \sum_{n=1}^{\infty} \left\{ \sum_{k+l+m=n-1} \binom{n}{k,l,m} \right. \tag{2.24}$$

$$\times \beta_{k,q^{w_2 w_3}}^{(l+m+2)}(w_1 y_1)\, \beta_{l,q^{w_1 w_3}}^{(m+2)}(w_2 y_2)\, T_{2,m}(w_3 - 1|q^{w_1 w_2})$$

$$\times q^{w_1 w_2 w_3 (l+m+1) y_1} q^{w_1 w_2 w_3 (m+1) y_2} \left[w_2 w_3\right]_q^k \left[w_1 w_3\right]_q^l \left[w_1 w_2\right]_q^{m+1}$$

$$+ (q-1) \sum_{k+l+m=n} \binom{n}{k,l,m} \beta_{k,q^{w_2 w_3}}^{(l+m+1)}(w_1 y_1)$$

$$\times \beta_{l,q^{w_1 w_3}}^{(m+1)}(w_2 y_2)\, T_{1,m}(w_3 - 1|q^{w_1 w_2})$$

$$\times q^{w_1 w_2 w_3 (l+m) y_1} q^{w_1 w_2 w_3 m y_2}$$

$$\left. \times \left[w_2 w_3\right]_q^k \left[w_1 w_3\right]_q^l \left[w_1 w_2\right]_q^{m+1} \right\} \frac{t^n}{n!}.$$

As this expression is invariant under any permutations in w_1, w_2, w_3, we get the following theorem.

Theorem 2.3. *Let $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1$, $w_2 \geq 1$, $w_3 \geq 1$. Then, for any positive integer n, the following expressions*

$$\sum_{k+l+m=n-1} \binom{n}{k,l,m} \beta_{k,q^{w_{\sigma(2)} w_{\sigma(3)}}}^{(l+m+2)} \left(w_{\sigma(1)} y_1\right)$$

$$\times \beta_{l,q^{w_{\sigma(1)} w_{\sigma(3)}}}^{(m+2)} \left(w_{\sigma(2)} y_2\right) T_{2,m} \left(w_{\sigma(3)} - 1|q^{w_{\sigma(1)} w_{\sigma(2)}}\right)$$

$$\times q^{w_1 w_2 w_3 (l+m+1) y_1} q^{w_1 w_2 w_3 (m+1) y_2}$$

$$\times \left[w_{\sigma(2)}w_{\sigma(3)}\right]_q^k \left[w_{\sigma(1)}w_{\sigma(3)}\right]_q^l \left[w_{\sigma(1)}w_{\sigma(2)}\right]_q^{m+1}$$
$$+ (q-1) \sum_{k+l+m=n} \binom{n}{k,l,m} \beta^{(l+m+1)}_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}} \left(w_{\sigma(1)}y_1\right)$$
$$\times \beta^{(m+1)}_{l,q^{w_{\sigma(1)}w_{\sigma(3)}}} \left(w_{\sigma(2)}y_2\right) T_{1,m}\left(w_{\sigma(3)}-1|q^{w_{\sigma(1)}w_{\sigma(2)}}\right)$$
$$\times q^{w_1w_2w_3(l+m)y_1} q^{w_1w_2w_3my_2} \left[w_{\sigma(2)}w_{\sigma(3)}\right]_q^k \left[w_{\sigma(1)}w_{\sigma(3)}\right]_q^l \left[w_{\sigma(1)}w_{\sigma(2)}\right]_q^{m+1}$$

are the same for any $\sigma \in S_3$.

Remark 2.4. We can get interesting identities by letting $w_3 = 1$ or by letting $w_2 = w_3 = 1$, in view of (2.15). However, writing those down requires much space. So we omit it.

With the same $a = q^{w_2w_3(x_1+w_1y_1)}$, $b = q^{w_1w_3(x_2+w_2y_2)}$ as in (2.18), I_1 can be rewritten as

$$I_1 = \int_{\mathbb{Z}_p^2} e^{[w_2w_3]_q [x_1+w_1y_1]_{q^{w_2w_3}} t} e^{[w_1w_3]_q [x_2+w_2y_2]_{q^{w_1w_3}} (at)} \times \left\{ \int_{\mathbb{Z}_p} \left(q^{w_1w_2w_3} e^{[w_1w_2(x_3+w_3)]_q (abt)} - e^{[w_1w_2x_3]_q (abt)} \right) d\mu_{q^{w_1w_2}}(x_3) \right\} \times d\mu_{q^{w_2w_3}}(x_1)\, d\mu_{q^{w_1w_3}}(x_2). \tag{2.25}$$

From Lemma 2.2 and (2.25), we note that the inner integral is equal to

$$abt\,[w_1w_2]_q \sum_{i=0}^{w_3-1} q^{2w_1w_2i} e^{[w_1w_2i]_q abt} + (q-1)\,[w_1w_2]_q \sum_{i=0}^{w_3-1} q^{w_1w_2i} e^{[w_1w_2i]_q abt}. \tag{2.26}$$

Thus, by (2.25) and (2.26), we get

$$\begin{aligned} I_1 =& t\,[w_1w_2]_q \sum_{i=0}^{w_3-1} q^{2w_1w_2i} \int_{\mathbb{Z}_p^2} abe^{[w_2w_3]_q [x_1+w_1y_1]_{q^{w_2w_3}} t} \\ &\times e^{[w_1w_3]_q \left[x_2+w_2y_2+\frac{w_2}{w_3}i\right]_{q^{w_1w_3}} at} d\mu_{q^{w_2w_3}}(x_1)\, d\mu_{q^{w_1w_3}}(x_2) \\ &+ (q-1)\,[w_1w_2]_q \sum_{i=0}^{w_3-1} q^{w_1w_2i} \int_{\mathbb{Z}_p^2} e^{[w_2w_3]_q [x_1+w_1y_1]_{q^{w_2w_3}} t} \\ &\times e^{[w_1w_3]_q \left[x_2+w_2y_2+\frac{w_2}{w_3}i\right]_{q^{w_1w_3}} at} d\mu_{q^{w_2w_3}}(x_1)\, d\mu_{q^{w_1w_3}}(x_2) \\ =& t\,[w_1w_2]_q \sum_{i=0}^{w_3-1} q^{2w_1w_2i} \sum_{k,l=0}^{\infty} \frac{t^{k+l}}{k!l!} q^{w_1w_2w_3(l+1)y_1} q^{w_1w_2w_3y_2} [w_2w_3]_q^k \\ &\times [w_1w_3]_q^l \int_{\mathbb{Z}_p} q^{w_2w_3(l+1)x_1} [x_1+w_1y_1]_{q^{w_2w_3}}^k d\mu_{q^{w_2w_3}}(x_1) \end{aligned} \tag{2.27}$$

$$\times \int_{\mathbb{Z}_p} q^{w_1 w_3 x_2} \left[x_2 + w_2 y_2 + \frac{w_2}{w_3} i\right]^l_{q^{w_1 w_3}} d\mu_{q^{w_1 w_3}}(x_2)$$

$$+ (q-1)[w_1 w_2]_q \sum_{i=0}^{w_3-1} q^{w_1 w_2 i} \sum_{k,l=0}^{\infty} \frac{t^{k+l}}{k!l!} q^{w_1 w_2 w_3 l y_1} [w_2 w_3]_q^k$$

$$\times [w_1 w_3]_q^l \int_{\mathbb{Z}_p} q^{w_2 w_3 l x_1} [x_1 + w_1 y_1]^k_{q^{w_2 w_3}} d\mu_{q^{w_2 w_3}}(x_1)$$

$$\times \int_{\mathbb{Z}_p} \left[x_2 + w_2 y_2 + \frac{w_2}{w_3} i\right]^l_{q^{w_1 w_3}} d\mu_{q^{w_1 w_3}}(x_2)$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \beta^{(n-k+2)}_{k,q^{w_2 w_3}}(w_1 y_1) q^{w_1 w_2 w_3 (n-k+1) y_1} \right.$$

$$\times q^{w_1 w_2 w_3 y_2} [w_1 w_2]_q [w_2 w_3]_q^k$$

$$\left. \times [w_1 w_3]_q^{n-k} \sum_{i=0}^{w_3-1} q^{2 w_1 w_2 i} \beta^{(2)}_{n-k, q^{w_1 w_3}} \left(w_2 y_2 + \frac{w_2}{w_3} i\right) \right\} \frac{t^{n+1}}{n!}$$

$$+ (q-1) \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \beta^{(n-k+1)}_{k,q^{w_2 w_3}}(w_1 y_1) q^{w_1 w_2 w_3 (n-k) y_1} \right.$$

$$\times [w_1 w_2]_q [w_2 w_3]_q^k [w_1 w_3]_q^{n-k}$$

$$\left. \times \sum_{i=0}^{w_3-1} q^{w_1 w_2 i} \beta_{n-k, q^{w_1 w_3}} \left(w_2 y_2 + \frac{w_2}{w_3} i\right) \right\} \frac{t^n}{n!}.$$

Separating the term corresponding to $n = 0$ from the second sum and after rearranging, we obtain

$$I_1 = (q-1)[w_1 w_2 w_3]_q + \sum_{n=1}^{\infty} \left\{ n \sum_{k=0}^{n-1} \binom{n-1}{k} \beta^{(n-k+1)}_{k,q^{w_2 w_3}}(w_1 y_1) q^{w_1 w_2 w_3 (n-k) y_1} \right.$$

$$\times q^{w_1 w_2 w_3 y_2} [w_1 w_2]_q [w_2 w_3]_q^k [w_1 w_3]_q^{n-1-k} \sum_{i=0}^{w_3-1} q^{2 w_1 w_2 i}$$

$$\times \beta^{(2)}_{n-1-k, q^{w_1 w_3}} \left(w_2 y_2 + \frac{w_2}{w_3} i\right) + (q-1) \sum_{k=0}^{n} \binom{n}{k} \beta^{(n-k+1)}_{k,q^{w_2 w_3}}(w_1 y_1) q^{w_1 w_2 w_3 (n-k) y_1}$$

$$\left. \times [w_1 w_2]_q [w_2 w_3]_q^k [w_1 w_3]_q^{n-k} \sum_{i=0}^{w_3-1} q^{w_1 w_2 i} \beta_{n-k, q^{w_1 w_3}} \left(w_2 y_2 + \frac{w_2}{w_3} i\right) \right\} \frac{t^n}{n!}.$$

As this expression is invariant under any permutations in w_1, w_2, w_3, we have the following results.

Theorem 2.5. *Let $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1$, $w_2 \geq 1$, $w_3 \geq 1$. Then, for any positive integer n, the following expressions*

$$
\begin{aligned}
& n \sum_{k=0}^{n-1} \binom{n-1}{k} \beta^{(n-k+1)}_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}} \left(w_{\sigma(1)} y_1\right) q^{w_1 w_2 w_3 (n-k) y_1} \\
& \times q^{w_1 w_2 w_3 y_2} \left[w_{\sigma(1)} w_{\sigma(2)}\right]_q \left[w_{\sigma(2)} w_{\sigma(3)}\right]_q^k \left[w_{\sigma(1)} w_{\sigma(3)}\right]_q^{n-1-k} \\
& \times \sum_{i=0}^{w_{\sigma(3)}-1} q^{2 w_{\sigma(1)} w_{\sigma(2)} i} \beta^{(2)}_{n-1-k, q^{w_{\sigma(1)} w_{\sigma(3)}}} \left(w_{\sigma(2)} y_2 + \frac{w_{\sigma(2)}}{w_{\sigma(3)}} i\right) \\
& + (q-1) \sum_{k=0}^{n} \binom{n}{k} \beta^{(n-k+1)}_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}} \left(w_{\sigma(1)} y_1\right) q^{w_1 w_2 w_3 (n-k) y_1} \\
& \times \left[w_{\sigma(1)} w_{\sigma(2)}\right]_q \left[w_{\sigma(2)} w_{\sigma(3)}\right]_q^k \left[w_{\sigma(1)} w_{\sigma(3)}\right]_q^{n-k} \\
& \times \sum_{i=0}^{w_{\sigma(3)}-1} q^{w_{\sigma(1)} w_{\sigma(2)} i} \beta_{n-k, q^{w_{\sigma(1)} w_{\sigma(3)}}} \left(w_{\sigma(2)} y_2 + \frac{w_{\sigma(2)}}{w_{\sigma(3)}} i\right)
\end{aligned}
$$

are all the same for any $\sigma \in S_3$.

Remark 2.6. Using (2.14) and by specializing $w_3 = 1$ or $w_2 = w_3 = 1$, we can obtain many interesting identities. However, as this requires much space, we will omit those.

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Taekyun Kim
Department of Mathematics, Tianjin Polytechnic University, Tianjin, China; and Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

Dmitry V. Dolgy
Institute of Mathematics and Computer Sciences, Far Eastern Federal University, Vladivostok 690060, Russia
E-mail address: d_dol@mail.ru

Dae San Kim
Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr