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ITERATIVE METHODS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS, FIXED POINT PROBLEMS AND MINIMIZATION PROBLEMS

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This paper is dedicated to Professor Wataru Takahashi on the occasion of his 70th Birthday

Abstract. In this paper, we propose new implicit and explicit iterative methods for finding a common element of the set of solutions of a generalized mixed equilibrium problem and the set of fixed points of a k-strictly pseudocontractive mapping in Hilbert spaces. We establish the strong convergence of the proposed iterative algorithms to a common point of two sets, which is a solution of a certain variational inequality. As a direct consequence, we obtain the unique minimum-norm common point of two sets.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H, and let $S : C \to C$ be a self-mapping on C. We denote by $Fix(S)$ the set of fixed points of S and by P_C the metric projection of H onto C.

Let $A : C \to H$ be a nonlinear mapping, let $\varphi : C \to \mathbb{R}$ be a function, and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

Then, we consider the following generalized mixed equilibrium problem (for short, GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{1.1}$$

which was introduced by Peng and Yao [23] recently (also see [6, 14, 17, 34]). The set of solutions of the problem (1.1) is denoted by $\Omega := GMEP(\Theta, \varphi, A)$. Here some special cases of the problem (1.1) are stated as follows:

If $\varphi = 0$, then the problem (1.1) reduced the following generalized equilibrium problem (for short GEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C, \tag{1.2}$$

which was studied by Takahashi and Takahashi [27]. The set of solutions of the problem (1.2) is denoted by $GEP(\Theta, A)$.

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If $A = 0$, then the problem (1.1) reduces the following mixed equilibrium problem (for short, MEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by Ceng and Yao [7] (see also [32]). The set of solutions of the problem (1.3) is denoted by $MEP(\Theta, \varphi)$.

If $\varphi = 0$ and $A = 0$, then the problem (1.1) reduces the following equilibrium problem (for short, EP) of finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of the problem (1.4) is denoted by $EP(\Theta)$.

If $\varphi = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$, the problem (1.1) reduces the following variational inequality problem (for short, VIP) of finding $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of the problem (1.5) is denoted by $VIP(C, A)$.

The problem GMEP (1.1) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, minmax problems, Nash equilibrium problems in noncooperative games and others; see, e.q., [4, 7, 9, 10].

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall that a mapping $T : C \to H$ is said to be *k-strictly pseudo-contractive* if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C.$$

Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, S is nonexpansive ($i.e.$, $\|Tx-Ty\| \leq \|x-y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. The mapping T is also said to be pseudocontractive if $k = 1$ and T is said to be strongly pseudocontractive if there exists a constant $\nu \in (0, 1)$ such that $T - \nu I$ is pseudocontractive. Clearly, the class of k-strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. Also we remark that the class of strongly pseudocontractive mappings is independent of the class of k-strictly pseudocontractive mappings (see [2, 3]). Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudocontractive mappings, see, for example, [1, 8, 12, 16, 21, 22, 35] and the references therein.

Recently, in order to study the GMEP(1.1), GEP(1.2), MEP(1.3), EP(1.4) and VIP(1.5), respectively, coupled with the fixed point problem, many authors have introduced some iterative methods for finding a common element of the set of solutions of the GMEP(1.1), GEP(1.2), MEP(1.3), EP(1.4) and VIP(1.5), respectively, and the set of fixed points of a countable family of nonexpansive mappings or strictly pseudocontractive mappings; see [7,14,18,24,26,27,32,33] and the references therein.

On the other hand, in 2001, Yamada [31] introduced the hybrid iterative method for the nonexpansive mapping to solve a variational inequality related to a Lipschitzian and strongly monotone operator. Since then, by combining the iterative method of Yamada [31] with ones of Marino and Xu [19], Tien [28, 29] and Ceng et al. [5] provided the general iterative methods for finding a fixed point of the nonexpansive mapping, which is a solution of a certain variational inequality related to a Lipschitzian and strongly monotone operator. Cho et al. [8] and Jung [12, 13] gave the general iterative methods for finding a fixed point of the k-strictly pseudocontractive mapping, which is a solution of a certain variational inequality.

Motivated by the recent works of [5, 13–15, 29, 33], in this paper, we introduce new implicit and explicit iterative methods for finding a common element of the set of the solutions of the GMEP(1.1) and the set of fixed points of a k-strictly pseudocontractive mapping T. Then we establish the strong convergence of the proposed iterative algorithms to a common point of two sets, which is a solution of a certain variational inequality. As a direct consequence, we find the unique solution of the quadratic minimization problem

$$\|x^*\|^2 = \min\{\|x\|^2 : x \in \Omega \cap Fix(T)\},$$

where Ω is the set of the GMEP(1.1) and $Fix(T)$ is the fixed point set of T.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

Recall that

(i) a mapping $V : C \to H$ is called *l-Lipschitzian* if there exists a constant $l \geq 0$

$$\|Vx - Vy\| \leq l\|x - y\|, \quad \forall x,\ y \in C;$$

(ii) a mapping $A : C \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x,\ y \in C;$$

(iii) a mapping $F : C \to H$ is called *ρ-Lipschitzian* and *η-strongly monotone* if there exist constants $\rho > 0$ and $\eta > 0$ such that

$$\|Fx - Fy\| \leq \rho\|x - y\| \text{ and } \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x,\ y \in C.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and P_C satisfies

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2$$

for every $x,\ y \in H$. Moreover, $P_C(x)$ is characterized by the properties:

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$$

and

$$u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \geq 0 \ \ \forall x \in H,\ y \in C. \tag{2.1}$$

In the context of the variational inequality problem for a nonlinear mapping A, this implies that

$$u \in VIP(C, A) \Longleftrightarrow u = P_C(u - \lambda Au) \quad \forall \lambda > 0.$$

For solving the generalized mixed equilibrium problem (1.1), mixed equilibrium problem (1.2), and equilibrium problem (1.3) for a bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x,\ y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x, y);$$

(A4) for each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

We can prove the following lemma by using the same method as in [17, 34], and so we omit its proof.

Lemma 2.1. *Let C be a nonempty closed convex subset of H. Let Θ be a bifunction form $C \times C$ to $\mathbb{R}$ satisfies* (A1)–(A4), *and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Let $A : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x \rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_r : H \to C$ as follows:

$$K_r(x) = \left\{ u \in C : \Theta(u, y) + \langle Au, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x \rangle \geq 0,\ \forall y \in C \right\}$$

for all $x \in H$ and $r > 0$. Then, the following hold:

(1) *For each $x \in H$, $K_r(x) \neq \emptyset$;*
(2) *K_r is single-valued;*
(3) *K_r is firmly nonexpansive, that is, for any $x,\ y \in H$,*

$$\|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y \rangle;$$

(4) *$Fix(K_r) = GMEP(\Theta, \varphi, A)$;*
(5) *$GMEP(\Theta, \varphi, A)$ is closed and convex.*

We need the following lemmas for the proof of our main results.

Lemma 2.2 ([35]). *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudocontractive mapping on C. Then the following hold:*

(i) *The fixed point set $Fix(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*
(ii) *$Fix(P_C T) = Fix(T)$.*
(iii) *If we define a mapping $S : C \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $Fix(T) = Fix(S)$.*

Lemma 2.3 ([30]). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \xi_n)s_n + \xi_n \delta_n, \quad \forall n \geq 1,$$

where $\{\xi_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\xi_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \xi_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \xi_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([25]). *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a real Banach space* E*, and let* $\{\gamma_n\}$ *be a sequence in* $[0,1]$ *which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n)z_n$ *for all* $n \geq 1$ *and*

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.5 ([11, Demiclosedness principle]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*, and let* $S : C \to C$ *be a nonexpansive mapping. Then, the mapping* $I - S$ *is demiclosed. That is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup x^*$ *and* $(I-S)x_n \to y$*, then* $(I-S)x = y$.

The following lemmas can be easily proven, and therefore, we omit their proofs.

Lemma 2.6. *Let* $V : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$*, and let* $F : H \to H$ *be a* ρ*-Lipschitzian and* η*-strongly monotone mapping with constants* $\rho > 0$ *and* $\eta > 0$*. Then for* $0 \leq \gamma l < \mu\eta$,

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in H.$$

That is, $\mu F - \gamma V$ *is strongly monotone with constant* $\mu\eta - \gamma l$.

Finally, we need the following lemma. We can refer to [13, 31] for the proof.

Lemma 2.7. *Let* H *be a real Hilbert space* H*. Let* $F : H \to H$ *be a* ρ*-Lipschizian and* η*-strongly monotone mapping with constants* $\rho > 0$ *and* $\eta > 0$*. Let* $0 < \mu < \frac{2\eta}{\rho^2}$ *and* $0 < t < \varsigma \leq 1$*. Then* $G := \varsigma I - t\mu F : H \to H$ *is a contraction with contractive constant* $\varsigma - t\tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.

3. Iterative algorithms

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed convex subset of H;
- Θ is a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4);
- $A : C \to H$ is a continuous monotone mapping;
- $V : C \to H$ is l-Lipschitzian with constant $l \in [0, \infty)$;
- $F : C \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- Constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- K_r is a mapping defined as in Lemma 2.1 for $r > 0$;
- $\Omega := GMEP(\Theta, \varphi, A)$ is the set of solutions of the GMEP (1.1);

- $T : C \to C$ is a k-strictly pseudocontractive mapping for $k \in [0, 1)$ such that $Fix(T) \cap \Omega \neq \emptyset$;
- $T_t : C \to C$ is a mapping defined by $T_t x = \lambda_t x + (1 - \lambda_t)Tx$, $t \in (0, 1)$, for $0 \leq k \leq \lambda_t \leq \lambda < 1$ and $\lim_{t\to 0} \lambda_t = \lambda$;
- $T_n : C \to C$ is a mapping defined by $T_n x = \lambda_n x + (1 - \lambda_n)Tx$ for $0 \leq k \leq \lambda_n \leq \lambda < 1$ and $\lim_{n\to\infty} \lambda_n = \lambda$;
- P_C is a metric projection of H onto C.

By Lemma 2.2 (iii), T_t and T_n are nonexpansive and $Fix(T) = Fix(T_t) = Fix(T_n)$.

In this section, we introduce the following algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = T_t P_C[t\gamma V x_t + (I - t\mu F)K_r(x_t)]. \tag{3.1}$$

We prove strong convergence of $\{x_t\}$ as $t \to 0$ to a point $\widetilde{x}$ of $Fix(T) \cap \Omega$ which is a solution of the following variational inequality:

$$\langle(\mu F - \gamma V)q, p - q\rangle \geq 0, \quad \forall p \in Fix(T) \cap \Omega. \tag{3.2}$$

We also propose the following explicit algorithm which generates a sequence in an explicit way:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T_n P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)K_r(x_n)], \quad \forall n \geq 0, \tag{3.3}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and $x_0 \in C$ is an arbitrary initial guess.

Consider the following mapping Q_t on C defined by

$$Q_t x = T_t P_C[t\gamma V x + (I - t\mu F)K_r(x)].$$

By Lemmas 2.1, 2.2 and 2.7, we have

$$\begin{aligned}\|Q_t x - Q_t y\| &\leq t\gamma\|V x - V y\| + \|(I - t\mu F)K_r(x) - (I - t\mu F)K_r(y)\| \\ &\leq t\gamma l\|x - y\| + (1 - t\tau)\|K_r(x) - K_r(y)\| \\ &\leq t\gamma l\|x - y\| + (1 - t\tau)\|x - y\| \\ &= (1 - t(\tau - \gamma l))\|x - y\|.\end{aligned}$$

Since $0 < 1-t(\tau-\gamma l) < 1$, Q_t is a contraction. Therefore, by the Banach contraction principle, Q_t has a unique fixed point $x_t \in C$, which uniquely solves the fixed point equation

$$x_t = T_t P_C[t\gamma V x_t + (I - t\mu F)K_r(x_t)].$$

Now, we establish the strong convergence of the net $\{x_t\}$ generated by (3.1) and show the existence of the $q \in Fix(T) \cap \Omega$, which solves the variational inequality (3.2).

Theorem 3.1. *The net $\{x_t\}$ defined via* (3.1) *converges strongly, as $t \to 0$, to a point $q \in Fix(T) \cap \Omega$, which solves the variational inequality* (3.2).

Proof. First, we can show easily the uniqueness of a solution of the variational inequality (3.2). In fact, noting that $0 \leq \gamma l < \tau$ and $\mu\eta \geq \tau \Leftrightarrow \rho \geq \eta$, it follows from Lemma 2.8 that

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2.$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \leq \gamma l < \tau \leq \mu\eta$. So the variational inequality (3.2) has only one solution. Below we use $q \in Fix(T) \cap \Omega$ to denote the unique solution of the variational inequality (3.2).

Now, we divide the proof into several steps

Step 1. We show that $\{x_t\}$ is bounded. To this end, let $u_t = K_r(x_t)$ for all $t \in (0,1)$. Take $p \in Fix(T) \cap \Omega$. Then, from $Fix(T) = Fix(T_t)$ by Lemma 2.2 (iii), it follows that $p = T_t p = T_t P_C p$, and it is clear that $p = K_r(p)$. Since K_r is nonexpansive, we have that

$$\|u_t - p\|^2 = \|K_r(x_t) - K_r(p)\|^2 \leq \|x_t - p\|^2, \tag{3.4}$$

that is, $\|u_t - p\| \leq \|x_t - p\|$. Also it follows from (3.1) and Lemma 2.7 that

$$\begin{aligned} \|x_t - p\| &= \|T_t P_C[t\gamma V x_t + (I - t\mu F)u_t] - T_t P_C[t\gamma V p + (I - t\mu F)p] \\ &\quad + T_t P_C[t\gamma V p + (I - t\mu F)p] - T_t P_C p\| \\ &\leq \|t\gamma V x_t + (I - t\mu F)u_t - (t\gamma V p + (I - t\mu F)p\| \\ &\quad + \|t\gamma V p + (I - t\mu F)p - p\| \\ &\leq t\gamma\|V x_t - V p\| + (1 - t\tau)\|u_t - p\| + t\|\gamma V p - \mu F p\| \\ &\leq t\gamma l\|x_t - p\| + (1 - t\tau)\|x_t - p\| + t(\gamma\|V p\| + \mu\|\mu F p\|). \end{aligned} \tag{3.5}$$

So, we have that

$$\|x_t - p\| \leq \frac{\gamma\|Vp\| + \mu\|\mu F p\|}{\tau - \gamma l}.$$

Thus, $\{x_t\}$ is bounded and we also obtain that $\{u_t\}$, $\{V x_t\}$, and $\{F u_t\}$ are bounded.

Step 2. We show that $\lim_{t\to 0} \|x_t - u_t\| = 0$. In fact, from (3.4) and (3.5), we have

$$\begin{aligned} (1 - t\gamma l)^2\|x_t - p\|^2 &\leq [(1 - t\tau)\|u_t - p\| + t(\gamma\|Vp\| + \mu\|Fp\|)]^2 \\ &= (1 - t\tau)^2\|u_t - p\|^2 + t^2(\gamma\|Vp\| + \mu\|\mu F p\|)^2 \\ &\quad + 2(1 - t\tau)t\|u_t - p\|(\gamma\|Vp\| + \mu\|Fp\|) \\ &\leq \|u_t - p\|^2 + tM, \end{aligned} \tag{3.6}$$

where $M = \sup\{t(\gamma\|Vp\| + \mu\|Fp\|) + 2\|u_t - p\|(\gamma\|Vp\| + \mu\|Fp\|)\}$. By Lemma 2.1, we obtain

$$\begin{aligned} \|u_t - p\|^2 &= \|K_r(x_t) - K_r(p)\|^2 \\ &\leq \langle K_r x_t - K_r p, x_t - p\rangle \\ &= \langle u_t - p, x_t - p\rangle \\ &= \frac{1}{2}(\|x_t - p\|^2 + \|u_t - p\|^2 - \|x_t - u_t\|), \end{aligned}$$

which implies that

$$\|u_t - p\|^2 \leq \|x_t - p\|^2 - \|x_t - u_t\|^2. \tag{3.7}$$

By (3.6) and (3.7), we have

$$(1 - t\gamma l)^2\|x_t - p\|^2 \leq \|x_t - p\|^2 - \|x_t - u_t\|^2 + tM.$$

It follows that

$$\|x_t - u_t\|^2 \le t(2\gamma l - t\gamma^2 l^2)\|x_t - p\|^2 + tM.$$

This together with Step 1 implies that

$$\lim_{t\to 0}\|x_t - u_t\| = 0.$$

Step 3. We show that $\lim_{t\to 0}\|x_t - T_t x_t\| = 0$. Indeed, from Step 2, we have

$$\begin{aligned}\|x_t - T_t x_t\| &= \|T_t P_C[t\gamma V x_t + (I - t\mu F)u_t] - T_t P_C x_t\| \\ &\le \|t\gamma V x_t + (I - t\mu F)u_t - x_t\| \\ &\le \|u_t - x_t\| + t(\gamma\|V x_t\| + \mu\|F u_t\|) \to 0 \quad (\text{as } t \to 0).\end{aligned}$$

Step 4. We show that $\{x_t\}$ is relatively norm compact as $t \to 0$. To this end, let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $u_n := u_{t_n}$. From Step 3, we have

$$\|x_n - T_{t_n} x_n\| \to 0 \quad (\text{as } n \to \infty).$$

By (3.1), we deduce

$$\begin{aligned}&\|x_t - p\|^2 \\ &= \|T_t P_C[t\gamma V x_t + (I - t\mu F)u_t] - T_t P_C p\|^2 \\ &\le \|t\gamma V x_t + (I - t\mu F)u_t] - p\|^2 \\ &= \|(I - t\mu F)u_t - (I - t\mu F)p - t(\mu F - \gamma V)p + t\gamma(V x_t - V p)\|^2 \\ &= \|(I - t\mu F)u_t - (I - t\mu F)p\|^2 \\ &\quad - 2t[\langle(\mu F - \gamma V)p, u_t - p\rangle - t\langle(\mu F - \gamma V)p, \mu F u_t - \mu F p\rangle] \\ &\quad + 2t\gamma[\langle V x_t - V p, u_t - p\rangle - t\langle V x_t - V p, \mu F u_t - \mu F p\rangle] \\ &\quad - 2t^2\gamma\langle(\mu F - \gamma V)p, V x_t - V p\rangle + t^2\|(\mu F - \gamma V)p\|^2 + t^2\gamma^2\|V x_t - V p\|^2 \\ &\le (1 - t\tau)^2\|u_t - p\|^2 - 2t\langle(\mu F - \gamma V)p, u_t - p\rangle + 2t\gamma l\|x_t - p\|\|u_t - p\| \\ &\quad + 2t^2\|(\mu F - \gamma V)p\|(\|\mu F u_t\| + \|\mu F p\|) \\ &\quad + 2t^2\gamma l\|x_t - p\|(\|\mu F u_t\| + \|\mu F p\|) + 2t^2 l\|(\mu F - \gamma V)p\|\|x_t - p\| \\ &\quad + t^2(\|(\mu F - V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2) \\ &= [1 - 2t\tau + t^2\tau^2]\|u_t - p\|^2 - 2t\langle(\mu F - \gamma V)p, u_t - p\rangle \\ &\quad + 2t\gamma l\|x_t - p\|\|u_t - p\| + 2t^2\|(\mu F - \gamma V)p\|(\|(\mu F)u_t\| + \|\mu F p\|) \\ &\quad + 2t^2\gamma l\|x_t - p\|(\|\mu F u_t\| + \|\mu F p\|) + 2t^2 l\|(\mu F - \gamma V)p\|\|x_t - p\| \\ &\quad + t^2(\|(\mu F - V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2) \\ &\le (1 - 2t\tau)\|u_t - p\|^2 - 2t\langle(\mu F - \gamma V)p, u_t - p\rangle + t\gamma l(\|x_t - p\|^2 + \|u_t - p\|^2) \\ &\quad + t^2 M,\end{aligned}$$

where

$$M = \sup\{\tau^2\|u_t - p\|^2 + 2(\|(\mu F - \gamma V)p\| + \gamma l\mu\|x_t - p\|)(\|Fu_t\| + \|Fp\|) + 2l\|(\mu F - \gamma V)p\|\|x_t - p\| + \|(\mu F - \gamma V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2\}.$$

Hence, for small enough t, we obtain

$$\begin{aligned}\|x_t - p\|^2 &\le \frac{1 - 2t\tau + t\gamma l}{1 - t\gamma l}\|u_t - p\|^2 - \frac{2t}{1 - t\gamma l}\langle(\mu F - \gamma V)p, u_t - p\rangle + \frac{t^2}{1 - t\gamma l}M \\ &\le \frac{1 - 2t\tau + t\gamma l}{1 - t\gamma l}\|x_t - p\|^2 - \frac{2t}{1 - t\gamma l}\langle(\mu F - \gamma V)p, u_t - p\rangle + \frac{t^2}{1 - t\gamma l}M.\end{aligned}$$

It follows that

$$\|x_t - p\|^2 \le \frac{1}{\tau - \gamma l}\langle(\mu F - \gamma V)p, p - u_t\rangle + \frac{tM}{2(\tau - \gamma l)}. \tag{3.8}$$

In particular,

$$\|x_n - p\|^2 \le \frac{1}{\tau - \gamma l}\langle(\mu F - \gamma V)p, p - u_n\rangle + \frac{t_n M}{2(\tau - \gamma l)}. \tag{3.9}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $q \in C$. We show that $q \in Fix(T)$. To this end, define $S : C \to C$ by $Sx = \lambda x + (1 - \lambda)Tx$, $\forall x \in C$. Then S is nonexpansive with $Fix(S) = Fix(T)$ by Lemma 2.2 (iii). Notice that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - T_{t_n}x_n\| + \|T_{t_n}x_n - x_n\| \\ &= (\lambda - \lambda_{t_n})\|x_n - Tx_n\| + \|T_{t_n}x_n - x_n\| \\ &= \frac{\lambda - \lambda_{t_n}}{1 - \lambda_{t_n}}\|x_n - T_{t_n}x_n\| + \|T_{t_n}x_n - x_n\| \\ &= \frac{1 + \lambda - 2\lambda_{t_n}}{1 - \lambda_{t_n}}\|x_n - T_{t_n}x_n\|.\end{aligned}$$

By Step 3 and $\lambda_{t_n} \to \lambda$ as $n \to \infty$, we have $\|Sx_n - x_n\| \to 0$ as $n \to \infty$, and $q \in Fix(S)$ from Lemma 2.5. So, we get $q \in Fix(T)$ by Lemma 2.2 (iii).

Now, we show that $q \in \Omega$. By $u_n = K_r(x_n)$, we know that

$$\Theta(u_n, y) + \langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \ge \Theta(y, u_n), \quad \forall y \in C. \tag{3.10}$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1 - t)q$. Since $y \in C$ and $q \in C$, we have $y_t \in C$. So, from (3.10), we have

$$\begin{aligned}\langle y_t - u_n, Ay_t\rangle &\ge \langle y_t - u_n, Ay_t\rangle - \varphi(y_t) + \varphi(u_n) - \langle y_t - u_n, Au_n\rangle \\ &\quad - \left\langle y_t - u_n, \frac{u_n - x_n}{r}\right\rangle + \Theta(y_t, u_n) \\ &= \langle y_t - u_n, Ay_t - Au_n\rangle - \varphi(y_t) + \varphi(u_n) \\ &\quad - \left\langle y_t - u_n, \frac{u_n - x_n}{r}\right\rangle + \Theta(y_t, u_n).\end{aligned}$$

Since $\|u_n - x_n\| \to 0$ by Step 2, we have $\frac{u_n - x_n}{r} \to 0$ and $u_n \rightharpoonup q$. Moreover, from the monotonicity of A, we have $\langle y_t - u_n, Ay_t - Au_n\rangle \geq 0$. So, from (A4) and the weak lower semicontinuity of φ, it follows that

$$\langle y_t - q, Ay_t\rangle \geq -\varphi(y_t) + \varphi(q) + \Theta(y_t, q) \quad \text{as } n \to \infty. \tag{3.11}$$

By (A1), (A4) and (3.11), we also obtain

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, q) + t\varphi(y) + (1-t)\varphi(q) - \varphi(y_t) \\ &\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - q, Ay_t\rangle \\ &= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - q, Ay_t\rangle, \end{aligned}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - q, Ay_t\rangle. \tag{3.12}$$

Letting $t \to 0$ in (3.12), we have for each $y \in C$

$$\Theta(q, y) + \langle Aq, y - q\rangle + \varphi(y) - \varphi(q) \geq 0.$$

This implies that $q \in \Omega$. Therefore, $q \in Fix(T) \cap \Omega$.

We substitute q for p in (3.9) to obtain

$$\|x_n - q\|^2 \leq \frac{1}{\tau - \gamma l}\langle(\mu F - \gamma V)q, q - u_n\rangle + \frac{t_n M}{2(\tau - \gamma l)}. \tag{3.13}$$

Note that $u_n \rightharpoonup q$. This facts and the inequality (3.13) imply that $x_n \to q$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

Step 5. We show that q solves the variational inequality (3.2). In fact, taking the limit in (3.9) as $n \to \infty$, we get

$$\|q - p\|^2 \leq \frac{1}{\tau - \gamma l}\langle(\mu F - \gamma V)p, p - q\rangle, \quad \forall p \in Fix(T) \cap \Omega.$$

In particular, q solves the following variational inequality

$$q \in Fix(T) \cap \Omega, \quad \langle(\mu F - \gamma V)p, p - q\rangle \geq 0, \quad p \in Fix(T) \cap \Omega,$$

or the equivalent dual variational inequality (see [20])

$$q \in Fix(T) \cap \Omega, \quad \langle(\mu F - \gamma V)q, p - q\rangle \geq 0, \quad p \in Fix(T) \cap \Omega. \tag{3.14}$$

Step 6. We show that the entire net $\{x_t\}$ converges strongly to q. To this end, let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ and assume $x_{n_k} \to \hat{q}$. By the same as the proof above, we have $\hat{q} \in Fix(T) \cap \Omega$. Moreover, it follows from (3.14) that

$$\langle(\mu F - \gamma V)q, \hat{q} - q\rangle \geq 0. \tag{3.15}$$

Interchanging q and $\hat{q}$, we obtain

$$\langle(\mu F - \gamma V)\hat{q}, q - \hat{q}\rangle \geq 0. \tag{3.16}$$

Lemma 2.6 and adding these two inequalities (3.15) and (3.16) yields

$$(\mu\eta - \gamma l)\|q - \hat{q}\|^2 \leq \langle(\mu F - \gamma V)q - (\mu F - \gamma V)\hat{q}, q - \hat{q}\rangle \leq 0.$$

Hence $q = \hat{q}$. Therefore we conclude that $x_t \to q$ as $t \to 0$.

The variational inequality (3.2) can be rewritten as

$$\langle(\mu F-\gamma V)q, p-q\rangle = \langle(I+\gamma V-\mu F)q-q, q-p\rangle \geq 0, \quad \forall p \in Fix(T)\cap\Omega.$$

By (2.1), this is equivalent to the fixed point equation

$$q = P_{Fix(T)\cap\Omega}(I+\gamma V-\mu F)q.$$

□

From Theorem 3.1, we can deduce the following result.

Corollary 3.2. *Let $\{x_t\}$ be a net generated by*

$$x_t = T_t P_C[(1-t)K_r(x_t)], \quad \forall t \in (0,1). \tag{3.17}$$

Then $\{x_t\}$ converges strongly, as $t \to 0$, to a point $q \in Fix(T)\cap\Omega$, which solves the following minimum norm problem: find $x^* \in Fix(T)\cap\Omega$ such that

$$\|x^*\| = \min_{x\in Fix(T)\cap\Omega}\|x\|. \tag{3.18}$$

Proof. In (3.8) with $F\equiv I$, $\mu=1$, $\tau=1$, $V\equiv 0$, and $l=0$, letting $t\to 0$ yields

$$\|q-p\|^2 \leq \langle p, p-q\rangle, \quad \forall p \in Fix(T)\cap\Omega.$$

Equivalently,

$$\langle q, p-q\rangle \geq 0, \quad \forall p \in Fix(T)\cap\Omega.$$

This obviously implies that

$$\|q\|^2 \leq \langle p, q\rangle \leq \|p\|\|q\|, \quad \forall p \in Fix(T)\cap\Omega.$$

It turns out that $\|q\| \leq \|p\|$ for all $p \in Fix(T)\cap\Omega$. Therefore, q is the minimum-norm point of $Fix(T)\cap\Omega$. □

Next, we show strong convergence of the explicit algorithm (3.3) to a point q of $Fix(T)\cap\Omega$, which is also a solution of the variational inequality (3.2).

Theorem 3.3. *Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_n P_C[\alpha_n\gamma V x_n + (I-\alpha_n\mu F)K_r(x_n)], \quad \forall n \geq 0, \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then $\{x_n\}$ converges strongly to a point $q \in Fix(T)\cap\Omega$, which solves the variational inequality (3.2).

Proof. First, from $\alpha_n \to 0$ in condition (C1), without loss of generality, we assume that $2(1-\beta_n)(\tau-\gamma l)\alpha_n < 1$. From now, let $p \in Fix(T)\cap\Omega$ and set $u_n = K_r(x_n)$ for all $n \geq 0$.

We divides the proof several steps:

Step 1. We show that $\|x_n - p\| \leq \max\left\{\|x_0-p\|, \frac{\gamma\|Vp\|+\mu\|\mu Fp\|}{\tau-\gamma l}\right\}$ for all $n \geq 0$. Indeed, from Lemma 2.1,

$$\|u_n - p\| = \|K_r(x_n) - K_r(p)\| \leq \|x_n - p\|.$$

Then, from Lemma 2.7 and $p = T_n p = T_n P_C p$, we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\beta_n x_n + (1-\beta_n)T_n P_C[\alpha_n \gamma V(x_n) + (I - \alpha_n \mu F)u_n - T_n P_C p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)\|\alpha_n \gamma V x_n + (I - \alpha_n \mu F)u_n - p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)[\alpha_n \gamma \|V x_n - Vp\| \\
&\qquad + \|(I - \alpha_n \mu F)u_n - (I - \alpha_n \mu F)p\| + \alpha_n \|(\gamma V - \mu F)p\|] \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)[\alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau)\|u_n - p\|] \\
&\quad + (1-\beta_n)\alpha_n(\|\gamma V p\| + \|\mu F p\|) \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)[\alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau)\|u_n - p\|] \\
&\quad + (1-\beta_n)\alpha_n(\|\gamma V p\| + \|\mu F p\|) \\
&\le \beta_n \|x_n - p\| + (1-\beta_n)[\alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau)\|x_n - p\|] \\
&\quad + (1-\beta_n)\alpha_n(\|\gamma V p\| + \|\mu F p\|) \\
&= [1 - (1-\beta_n)\alpha_n(\tau - \gamma l)]\|x_n - p\| + \alpha_n(1-\beta_n)(\|\gamma V p\| + \|\mu F p\|) \\
&= [1 - (\tau - \gamma l)\alpha_n(1-\beta_n)]\|x_n - p\| + (\tau - \gamma l)\alpha_n(1-\beta_n)\frac{\gamma\|Vp\| + \mu\|Fp\|}{\tau - \gamma l}.
\end{aligned}$$

From induction, we have

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\gamma\|Vp\| + \mu\|Fp\|}{\tau - \gamma l}\right\}, \quad n \ge 0.$$

Hence $\{x_n\}$ is bounded. From (3.3), $\{u_n\}$, $\{Vx_n\}$, and $\{Fu_n\}$ are also bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. To show this, let $z_n = T_n P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)u_n]$ for all $n \ge 0$, Then, we write (3.3) as follows:

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \quad \forall n \ge 0.$$

It follows from the definition of z_n that

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&= \|T_n P_C[\alpha_{n+1}\gamma V x_{n+1} + (I - \alpha_{n+1}\mu F)u_{n+1}] \\
&\qquad - T_n P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)u_n]\| \\
&\le \|\alpha_{n+1}\gamma V x_{n+1} + (I - \alpha_{n+1}\mu F)u_{n+1} - \alpha_n \gamma V x_n - (I - \alpha_n \mu F)u_n\| \\
&\le \|u_{n+1} - u_n\| + \alpha_{n+1}(\gamma\|V x_{n+1}\| + \|\mu F u_{n+1}\|) + \alpha_n(\gamma\|V x_n\| + \mu\|F x_n\|).
\end{aligned} \tag{3.19}$$

From Lemma 2.1, we also have

$$\|u_{n+1} - u_n\| = \|K_r(x_{n+1}) - K_r(x_n)\| \le \|x_{n+1} - x_n\|. \tag{3.20}$$

By (3.19) and (3.20), we derive

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le (\alpha_{n+1} + \alpha_n)M_1,$$

where $M_1 = \sup\{\gamma\|V x_n\| + \mu\|F u_n\|\}$. Therefore,

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, by Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Thus, from condition (C2), it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Indeed, In fact, from (3.3) and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\beta_n(x_n - p) + (1-\beta_n)(z_n - p)\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\alpha_n\gamma V x_n + (I - \alpha_n\mu F)u_n - p\|^2 \\
&= \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n)\|\alpha_n\gamma V x_n + (I - \alpha_n\mu F)u_n - (I - \alpha_n\mu F)p - \alpha_n\mu F p\|^2 \qquad (3.21)\\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)[(1-\tau\alpha_n)\|u_n - p\| + \alpha_n(\gamma\|V x_n\| + \mu\|Fp\|)]^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|u_n - p\| + \alpha_n(\gamma\|V x_n\| + \mu\|Fp\|)]^2 \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|u_n - p\|^2 + \alpha_n^2(\gamma\|V x_n\| + \mu\|Fp\|)^2 \\
&\qquad + 2\alpha_n\|u_n - p\|(\gamma\|V x_n\| + \mu\|\mu F p\|) \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 + \alpha_n M_2,
\end{aligned}$$

where $M_2 = \sup\{(\gamma\|V x_n\| + \mu\|Fp\|)^2 + 2\|u_n - p\|(\gamma\|V x_n\| + \mu\|Fp\|)\}$. Since K_r is firmly nonexpansive in Lemma 2.1, we obtain

$$\begin{aligned}
\|u_n - p\|^2 &= \|K_r(x_n) - K_r(p)\|^2 \\
&\le \langle K_r x_n - K_r p, x_n - p\rangle \\
&= \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\}.
\end{aligned}$$

Thus, we deduce

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \qquad (3.22)$$

From (3.21) and (3.22), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\|x_n - p\|^2 - \|x_n - u_n\|^2) + \alpha_n M_2 \\
&\le \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2 + \alpha_n M_2.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
(1-\beta_n)\|x_n - u_n\|^2 &\le \|x_n - u_n\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_2 \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n M_2.
\end{aligned}$$

Since $\alpha_n \to 0$ in condition (C1) and $\|x_{n+1} - x_n\|$ by Step 2, we deduce

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|u_n - T_n u_n\| = 0$. To this end, first, from $\alpha_n \to 0$ in condition (C1), we note that

$$\begin{aligned}\|z_n - T_n u_n\| &= \|T_n P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)u_n] - T_n P_C u_n\| \\ &\le \alpha_n(\gamma\|Vx_n\| + \mu\|Fu_n\|) \to 0.\end{aligned}$$

Then we obtain

$$\|x_n - T_n u_n\| \le \|x_n - z_n\| + \|z_n - T_n u_n\| \to 0.$$

Hence, from Step 3 with this fact, we have

$$\|T_n u_n - u_n\| \le \|T_n u_n - x_n\| + \|x_n - u_n\| \to 0.$$

Step 5. We show that

$$\limsup_{n\to\infty} \langle(\mu F - \gamma V)q, q - u_n\rangle \le 0,$$

where $q = P_{Fix(T)\bigcap\Omega}(I + \gamma V - \mu F)q$ is a unique solution of the variational inequality (3.2). To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty} \langle(\mu F - \gamma V)q, q - u_{n_i}\rangle = \limsup_{n\to\infty} \langle(\mu F - \gamma V)q, q - u_n\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ which converges weakly to w. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From Step 3 and Step 4, we obtain $x_{n_i} \rightharpoonup w$ and $T_{n_i} u_{n_i} \rightharpoonup w$. We show that $w \in Fix(T)$. To this end, define $S : C \to C$ by $Sx = \lambda x + (1 - \lambda)Tx$, $\forall x \in C$. Then S is nonexpansive with $Fix(S) = Fix(T)$ by Lemma 2.2 (iii). Notice that

$$\begin{aligned}\|Su_{n_i} - u_{n_i}\| &\le \|Su_{n_i} - T_{n_i} u_{n_i}\| + \|T_{n_i} u_{n_i} - u_{n_i}\| \\ &= (\lambda - \lambda_{t_{n_i}})\|u_{n_i} - Tu_{n_i}\| + \|T_{n_i} u_{n_i} - u_{n_i}\| \\ &= \frac{\lambda - \lambda_{t_{n_i}}}{1 - \lambda_{t_{n_i}}}\|u_{n_i} - T_{n_i} u_{n_i}\| + \|T_{n_i} u_{n_i} - u_{n_i}\| \\ &= \frac{1 + \lambda - 2\lambda_{n_i}}{1 - \lambda_{n_i}}\|T_{n_i} u_{n_i} - u_{n_i}\|.\end{aligned}$$

By Step 4 and $\lambda_{t_{n_i}} \to \lambda$, we have $\|Su_{n_i} - u_{n_i}\| \to 0$. So, we can use Lemma 2.5 to get $w \in Fix(S)$. By Lemma 2.2 (iii), $w \in Fix(T)$.

By the same argument as in the proof of Theorem 3.1, we also have $w \in \Omega$, and hence $w \in Fix(T) \cap \Omega$. Since $q = P_{Fix(T)\cap\Omega}(I + \gamma V - \mu F)q$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} \langle(\mu F - \gamma V)q, q - u_n\rangle &= \lim_{i\to\infty} \langle(\mu F - \gamma V)q, q - u_{n_i}\rangle \\ &= \langle(\mu F - \gamma V)q, q - w\rangle \le 0.\end{aligned}$$

Step 6. We show that $\lim_{n\to\infty} \|x_n - q\| = 0$ where $q = P_{Fix(T)\cap\Omega}(I + \gamma V - \mu F)q$ is a unique solution of the variational inequality (3.2). From (3.3), we know that

$$x_{n+1} - q = \beta_n(x_n - q) + (1 - \beta_n)(z_n - q),$$

where $z_n = T_n P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)u_n]$ for all $n \geq 0$. Applying the convexity of $\|\cdot\|^2$ and Lemma 2.7, we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)\|z_n - q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)\|\alpha_n\gamma V x_n + (I - \alpha_n\mu F)u_n - q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)\|\alpha_n\gamma(Vx_n - Vq) + (I - \alpha_n\mu F)u_n - (I - \alpha_n\mu F)q\|^2 \\
&- (1-\beta_n)2\alpha_n\langle(\mu F - \gamma V)q, u_n - q\rangle + (1-\beta_n)2\alpha_n^2\gamma l\|x_n - q\|\|(\mu F - \gamma V)q\| \\
&+ (1-\beta_n)2\alpha_n^2(\|\mu F u_n\| + \|\mu F q\|)\|(\mu F - \gamma V)q\| \\
&+ (1-\beta_n)\alpha_n^2\|(\mu F - \gamma V)q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)[\alpha_n\gamma l\|x_n - q\| + (1 - \alpha_n\tau)\|u_n - q\|]^2 \\
&+ (1-\beta_n)2\alpha_n\langle(\mu F - \gamma V)q, q - u_n\rangle + 2\alpha_n^2\gamma l\|x_n - q\|\|(\mu F - \gamma V)q\| \\
&+ 2\alpha_n^2(\|\mu F u_n\| + \|\mu F q\|)\|(\mu F - \gamma V)q\| + \alpha_n^2\|(\mu F - \gamma V)q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)[\alpha_n\gamma l\|x_n - q\| + (1 - \alpha_n\tau)\|x_n - q\|]^2 \\
&+ (1-\beta_n)2\alpha_n\langle(\mu F - \gamma V)q, q - u_n\rangle + 2\alpha_n^2\gamma l\|x_n - q\|\|(\mu F - \gamma V)q\| \\
&+ 2\alpha_n^2(\|\mu F u_n\| + \|\mu F q\|)\|(\mu F - \gamma V)q\| + \alpha_n^2\|(\mu F - \gamma V)q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)(1 - (\tau - \gamma l)\alpha_n)^2\|x_n - q\|^2 \\
&+ (1-\beta_n)2\alpha_n\langle(\mu F - \gamma V)q, q - u_n\rangle + 2\alpha_n^2\gamma l\|x_n - q\|\|(\mu F - \gamma V)q\| \\
&+ 2\alpha_n^2(\|\mu F u_n\| + \|\mu F q\|)\|(\mu F - \gamma V)q\| + \alpha_n^2\|(\mu F - \gamma V)q\|^2 \\
=\ & (1 - 2(1-\beta_n)(\tau - \gamma l)\alpha_n)\|x_n - q\|^2 \\
&+ (1-\beta_n)2\alpha_n\langle(\mu F - \gamma V)q, q - u_n\rangle + \alpha_n^2 M_3 \\
=\ & (1 - \xi_n)\|x_n - q\|^2 + \xi_n\delta_n,
\end{aligned}$$

where

$$\begin{aligned}
M_3 = \sup\{&(\tau - \gamma l)^2\|x_n - p\|^2 + 2\gamma l\|x_n - q\|\|(\mu F - \gamma V)q\| \\
&+ 2\mu(\|Fu_n\| + \|Fq\|)\|(\mu F - \gamma V)q\| + \|(\mu F - \gamma V)q\|^2\},
\end{aligned}$$

$\xi_n = 2(1-\beta_n)(\tau - \gamma l)\alpha_n$ and

$$\delta_n = \frac{1}{\tau - \gamma l}\langle(\mu F - \gamma V)q, q - u_n\rangle + \frac{\alpha_n M_3}{2(1-\beta_n)(\tau - \gamma l)}.$$

From conditions (C1) and (C2) and Step 5, it is easy to see that $\xi_n \to 0$, $\sum_{n=0}^{\infty} \xi_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence, by Lemma 2.3, we conclude $x_n \to q$ as $n \to \infty$. This completes the proof. □

From Theorem 3.3, we deduce immediately the following result.

Corollary 3.4. *Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_nP_C[(1-\alpha_n)K_r(x_n)], \quad \forall n \geq 0. \tag{3.23}$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying conditions (C1) *and* (C2) *in Theorem 3.3. Then $\{x_n\}$ converges strongly to a point $q \in Fix(T) \cap \Omega$, which solves the minimum norm problem* (3.18).

Proof. Take $F \equiv I$, $\mu = 1$ $\tau = 1$, $V \equiv 0$, and $l = 0$ in Theorem 3.3. Then the variational inequality (3.2) is reduced to the inequality

$$\langle q, p-q\rangle \geq 0, \quad \forall p \in Fix(T) \cap \Omega.$$

This is equivalent to $\|q\|^2 \leq \langle p, q\rangle \leq \|p\|\|q\|$ for all $p \in Fix(T) \cap \Omega$. It turns out that $\|q\| \leq \|p\|$ for all $p \in Fix(T) \cap \Omega$ and q is the minimum-norm point of $Fix(T) \cap \Omega$. □

Remark 3.5. 1) We point out that our implicit algorithms (3.1) and (3.17) are different from those considered by many authors. The explicit algorithms (3.3) and (3.23) are also different from those introduced by many authors in this direction.

2) As special cases of Theorem 3.1 and Theorem 3.3, we can also provide the corresponding iterative methods for finding a common element of the set of solutions of the GEP(1.2), MEP(1.3), EP(1.4) and VIP(1.5), respectively, and the set of fixed points of a strictly pseudocontractive mapping.

3) As in Corollary 3.2 and Corollary 3.4, we can obtain the minimum-norm point of $Fix(T) \cap GEP(\Theta, A)$, $Fix(T) \cap MEP(\Theta, \varphi)$, $Fix(T) \cap EP(\Theta)$, and $Fix(T) \cap VI(C, A)$, respectively.

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