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THE CONTINUITY OF $\mathbb{Q}_+$-HOMOGENEOUS SUPERADDITIVE CORRESPONDENCES

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Dedicated to Prof. Wataru Takahashi on the occasion of his 70th birth day

Abstract. In this paper we first show that the images of compact and convex sets are bounded under lower semicontinuous $\mathbb{Q}_+$-homogeneous superadditive correspondences and then investigate the continuity of $\mathbb{Q}_+$-homogeneous superadditive correspondences and the existence of their linear selections. It is also shown that every superadditive correspondence from a cone with finite basis into a finite dimensional space admits a family of continuous linear selections.

1. Introduction

Let X be a real vector space. A subset C of X is called a cone if $tx \in C$ for each $x \in C$ and $t > 0$. Moreover, a cone C of X is called a convex cone if it is convex. If E is a linearly independent (finite) subset of X such that

$$C = \Big\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0, x_i \in E, n \in \mathbb{N} \Big\},$$

then E is said to be a (finite) cone-basis (briefly, basis) of C.

By a correspondence φ from a set X into set Y, denoted $\varphi : X \twoheadrightarrow Y$, we mean a set-valued function $\varphi : X \to 2^Y \setminus \{\emptyset\}$.

Superadditive correspondences defined on semigroups were investigated in [11] and, in particular, the Banach-Steinhaus theorem of uniform boundedness was extended to the class of lower semicontinuous and $\mathbb{Q}_+$-homogeneous correspondences in cones. In [9], it is shown that every superadditive correspondence from a cone with a basis of a real topological vector space into the family of all convex, compact subsets of a locally convex space admits an additive selection. Also, some results on the existence of selections and the continuity of linear correspondences have given in [2]. For more information the reader is also referred to [1, 7, 10].

In the present note we first investigate boundedness of the images of compact and convex sets under lower semicontinuous $\mathbb{Q}_+$-homogeneous superadditive correspondences and then give a more general form of Lemma 1 in [9], when the range of a superadditive correspondence is a finite dimensional space. We also show that every lower semicontinuous and $\mathbb{Q}_+$-homogeneous superadditive correspondence from a cone with a basis of a real topological vector space into the family of all convex, compact subsets of a locally convex space admits a linear selection. Finally, for every superadditive correspondence defined between two cones with finite basis we find a family of linear continuous selections.

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We commence some notations and basic concepts. Throughout the paper, we assume that X and Y are real Hausdorff topological vector spaces and C is a convex cone in X, unless otherwise stated. By notations $\phi : C \to b(Y)$, $\varphi : C \to c(Y)$ and $\psi : C \to cc(Y)$ we mean the correspondences $\phi : C \twoheadrightarrow Y$, $\varphi : C \twoheadrightarrow Y$ and $\psi : C \twoheadrightarrow Y$ with bounded, compact and convex, compact values, respectively.

Definition 1.1. A correspondence $\varphi : C \twoheadrightarrow Y$ is additive if $\varphi(x+y) = \varphi(x)+\varphi(y)$, for every $x, y \in C$ and is superadditive if $\varphi(x+y) \supseteq \varphi(x)+\varphi(y)$, for every $x, y \in C$.

Definition 1.2. A correspondence $\varphi : C \twoheadrightarrow Y$ is $\mathbb{Q}_+$-homogeneous if $\varphi(rx) = r\varphi(x)$ for each $x \in C$ and $r \in \mathbb{Q}_+$. If $\varphi(tx) = t\varphi(x)$ for each $x \in C$ and $t > 0$, it is called positively homogeneous.

Every additive and positively homogeneous correspondence $\varphi : C \twoheadrightarrow Y$ is called linear. Also, every correspondence $\varphi : C \twoheadrightarrow Y$ with

$$\varphi\Big(\frac{x+y}{2}\Big) \supseteq \frac{\varphi(x)+\varphi(y)}{2}$$

for each $x, y \in C$ is called midpoint convex.

Recall that a correspondence $\varphi : C \to c(Y)$ is continuous at x if it is upper and lower semicontinuous at x. If it is continuous at every point x then it is called continuous on C (see e.g. [3]).

Let (X, d) be a metric space and $c(X)$ be the set of all nonempty compact subsets of X. Then the formula

$$\mathfrak{h}(A, B) = \max\Big\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(A, b)\Big\} \qquad (A, B \in c(X)),$$

defines a metric, called Hausdorff metric, on $c(X)$.

2. Main results

We start with the following theorem:

Theorem 2.1. *Let $\{\varphi_\alpha\}_{\alpha\in I}$ be a family of lower semicontinuous and $\mathbb{Q}_+$-homogeneous superadditive correspondences $\varphi_\alpha : C \twoheadrightarrow Y$. If K is a convex and compact subset of C and $\bigcup_{\alpha\in I} \varphi_\alpha(x)$ is bounded for every $x \in K$, then $\bigcup_{\alpha\in I} \varphi_\alpha(K)$ is bounded.*

Proof. Let W be an open neighborhood of zero in Y. There exists an open balanced neighborhood U of zero such that $\overline{U} + \overline{U} \subseteq W$. If

$${\varphi_\alpha}^{-1}(\overline{U}) := \{x : \varphi_\alpha(x) \subseteq \overline{U}\}$$

and

$$E := \cap_\alpha {\varphi_\alpha}^{-1}(\overline{U}),$$

then for every $x \in K$ there exists $n \in \mathbb{N}$ such that

$$\varphi_\alpha(n^{-1}x) \subseteq U \subseteq \overline{U}, \qquad (\alpha \in I).$$

Therefore $n^{-1}x \in E$, that is, $K \subseteq \cup_{n=1}^\infty nE$. Since $K = \cup_{n=1}^\infty K \cap nE$ is of the second category, there exist $n \in \mathbb{N}$, $x_0 \in int_K(K \cap nE)$ and a balanced open neighborhood V of zero such that

$$(x_0 + V) \cap K \subseteq K \cap nE \subseteq nE.$$

That is, $(x_0 + V) \cap K \subseteq nE$. On the other hand for each $k \in K$ there exists $\lambda > 0$ such that $k \in x_0 + \lambda V$. Since K is compact, there exist positive numbers $\lambda_1, \ldots, \lambda_n$ such that

$$K \subseteq (x_0 + \lambda_1 V) \cup \cdots \cup (x_0 + \lambda_n V).$$

If γ is a rational number such that $\gamma > \max\{\lambda_1, \ldots, \lambda_n, 1\}$, we have $K \subseteq x_0 + \gamma V$. For any $x \in K$,

$$z = \frac{1}{\gamma}x + \Big(1 - \frac{1}{\gamma}\Big)x_0 \in K,$$

and so

$$z \in (x_0 + V) \cap K \subseteq nE.$$

Hence $z, x_0 \in nE$ and consequently $\varphi_\alpha(n^{-1}z) \subseteq \overline{U}$ and $\varphi_\alpha(n^{-1}x_0) \subseteq \overline{U}$ for each α. Since $x + (\gamma - 1)x_0 = \gamma z$,

$$\begin{aligned}
\varphi_\alpha(x) &\subseteq \varphi_\alpha(x) + (\gamma - 1)\varphi_\alpha(x_0) - (\gamma - 1)\varphi_\alpha(x_0) \\
&\subseteq \gamma\varphi_\alpha(z) - (\gamma - 1)\varphi_\alpha(x_0) \\
&\subseteq \gamma n\overline{U} + (\gamma - 1)n\overline{U} \\
&\subseteq \gamma n\overline{U} + \gamma n\overline{U} \\
&\subseteq \gamma nW,
\end{aligned}$$

for each α. Thus $\bigcup_\alpha \varphi_\alpha(K) \subseteq \gamma nW$, that is $\bigcup_\alpha \varphi_\alpha(K)$ is bounded. □

The image of every compact set under an upper semicontinuous correspondence $\varphi : C \to c(Y)$ is compact (see [3]). Theorem 2.1 says that the image of a convex and compact set under a lower semicontinuous and $\mathbb{Q}_+$-homogeneous superadditive correspondence $\varphi : C \twoheadrightarrow Y$ is bounded if $\varphi(x)$ is bounded for each x in that convex and compact set.

Let D be an open convex subset of X and K be a convex cone with zero in Y. A correspondence $\varphi : D \twoheadrightarrow Y$ is called K-midpoint convex if

$$\frac{\varphi(x) + \varphi(y)}{2} \subseteq \varphi\Big(\frac{x + y}{2}\Big) + K$$

for each $x, y \in D$ and K-midpoint concave if

$$\varphi\Big(\frac{x + y}{2}\Big) \subseteq \frac{\varphi(x) + \varphi(y)}{2} + K$$

for each $x, y \in D$. In the following, we bring two lemmas from [5].

Lemma 2.2. *If a correspondence $\varphi : D \to b(Y)$ is midpoint K-convex and K-lower semicontinuous at some point of D, then it is K-lower semicontinuous at every point of D.*

Lemma 2.3. *If a correspondence $\varphi : D \to b(Y)$ is midpoint K-convex and K-lower semicontinuous at some point of D, then it is K-upper semicontinuous at this point.*

Hereafter we assume that C is also with a finite cone-basis, unless otherwise stated. We shall denote by L the subspace of X spanned by the finite cone-basis of C.

Theorem 2.4 ([3]). *Let C be a topological space, Y be a metric space and that $\varphi : C \to c(Y)$. Let $c(Y)$ be endowed with Hausdorff metric topology. Then the function $f : C \to c(Y)$ defined by $f(x) = \varphi(x)$ for each $x \in C$, is continuous in Hausdorff metric topology if and only if φ is continuous.*

The next lemma gives a general form of Lemma 1 presented in [9].

Lemma 2.5. *Let Y be a real finite dimensional topological vector space and $E = \{e_1, \dots, e_n\}$ be a finite basis of the convex cone C. If $\varphi : C \to cc(Y)$ is a $\mathbb{Q}_+$-homogeneous superadditive correspondence that is lower semicontinuous at some point, then it is positively homogeneous and continuous on $int_L C$.*

Proof. Since every $\mathbb{Q}_+$-homogeneous superadditive correspondence is midpoint convex, Lemmas 2.2 and 2.3, imply the continuity of φ on $int_L C$.

Now we show that φ is positively homogeneous. Since φ is compact-valued so $\varphi(tx) = t\varphi(x)$ for $x = 0$ and $t > 0$. Let $0 \neq x \in int_L C$, $t > 0$ and $(t_n)_n$ be a sequence in $\mathbb{Q}_+$ such that $t_n \to t$. By the continuity of φ we have $\varphi(t_n x) \to \varphi(tx)$, in the Hausdorff metric topology on $cc(Y)$. On the other hand $\varphi(t_n x) = t_n \varphi(x) \to t\varphi(x)$. Therefore $t\varphi(x) = \varphi(tx)$ for each $0 \neq x \in int_L C$ and $t > 0$. If x is an arbitrary nonzero element of C and $t > 0$, then the correspondence $\varphi_0 : \{\lambda x : \lambda \geq 0\} = \langle x \rangle \to cc(Y)$ is positively homogeneous on $int_M \langle x \rangle$, where $M = \langle x \rangle - \langle x \rangle$. Therefore φ is positively homogeneous on C. □

The next example shows that Lemma 2.5 does not guarantee the continuity of φ on the whole C.

Example 2.6. Define $\varphi : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2$ by

$$\varphi(x, y) = \begin{cases} \{(0,0)\} & x \geq 0, y = 0; \\ \{(t,0) : 0 \leq t \leq x\} & x \geq 0, y > 0. \end{cases}$$

Obviously φ is a lower semicontinuous and positively homogeneous superadditive correspondence. But it is not continuous on the $C = [0, \infty) \times [0, \infty)$.

In the next result we show that the additive selections can even be linear.

According to Theorem 4 in [9], superadditive correspondence $\varphi : C \to cc(Y)$ has an additive selection a if all assumptions of Theorem 2.7 are fulfilled. By Theorem 4.3 in [6], a is continuous on $int_L C$ and thus it is linear continuous on $int_L C$. Now Theorem 2.7, gives the existence of a selection that is linear on the whole of C.

Theorem 2.7. *Let C be a convex cone with basis and Y be a real Hausdorff locally convex topological vector space. Then a $\mathbb{Q}_+$-homogeneous superadditive correspondence $\varphi : C \to cc(Y)$ admits a linear selection provided that φ is lower semicontinuous on $int_L C$.*

Proof. First we show that if C is a convex cone with finite basis, then the lower semicontinuous superadditive correspondence $\varphi : C \to cc(Y)$ admits a linear selection. By Theorem 3 in [9], φ admits an additive selection $a : C \to Y$. By Theorem 4.3 in [6], a is continuous on $int_L C$, and hence it is linear on $int_L C$. If $x \in C$, then by Lemma 5.28 in [3], $\frac{1}{2}(x + y) \in int_L C$ for each $y \in int_L C$ and so

$$\alpha \frac{a(x)}{2} + \alpha \frac{a(y)}{2} = a\Big(\alpha \frac{x + y}{2}\Big) = \frac{1}{2}(a(\alpha x) + a(\alpha y)).$$

Since $a(\alpha y) = \alpha a(y)$ for each $y \in int_L C$, so a is a bounded linear selection of φ. Now, let C be a convex cone with arbitrary basis. By Theorem 4 in [9], φ admits an additive selection $a : C \to Y$. Let $x \in C$ be fixed. There are vectors $e_1, \dots, e_n$ and non-negative scalars $\lambda_1, \dots, \lambda_n$ such that $x = \sum_{i=1}^n \lambda_i e_i$. Therefore superadditive correspondence $\varphi_0 : C_0 \to cc(Y)$ with $\varphi_0(z) = \varphi(z)$ for $z \in C_0$, is a lower semicontinuous superadditive correspondence on $int_{L_0} C_0$, where C_0 and L_0 are the convex cone and the linear space generated by $\{e_1, \dots, e_n\}$, respectively. Similar to the previous case $a_0 : C_0 \to Y$ defined by $a_0(z) = a(z)$ for $z \in C_0$ is linear so $a(\alpha x) = \alpha a(x)$ for each $\alpha > 0$. That is, a is a linear selection. □

Theorem 2.8 ([2]). *Let Y be a real normed space. If $\varphi : C \to cc(Y)$ is linear, then φ is automatically continuous.*

Theorem 2.9. *Let Y be a finite dimensional space and $\varphi : C \to cc(Y)$ be a superadditive correspondence. Then there exists a family of continuous linear functions contained in φ.*

Proof. Let $E = \{e_1, e_2, \dots, e_n\}$ and $\acute{E} = \{\acute{e}_1, \acute{e}_2, \dots, \acute{e}_m\}$ be basis for C and Y, respectively. From Theorem 1 in [9], there exists a minimal $\mathbb{Q}_+$-homogeneous superadditive correspondence $\phi : C \to cc(Y)$ contained in φ. For each $i = 1, \dots, m$ consider the correspondence $\phi_i : C \to cc(\mathbb{R})$ as

$$\phi_i(x) = \{\pi_i((\lambda_1, \dots, \lambda_m)) : \lambda_1 \acute{e}_1 + \dots + \lambda_m \acute{e}_m \in \phi(x)\}$$

where π_i is the i^{th} projection mapping. For each $i = 1, \dots, m$, ϕ_i is positively homogeneous and continuous on $int_L C$ according to Lemma 2.5. By Theorem 17.28 in [3], ϕ is continuous on $int_L C$. By Theorem 2.8, ϕ contains a continuous linear correspondence $\psi : C \to cc(Y)$ with $\psi(x) = \sum_{i=1}^n \lambda_i \phi(e_i)$ for every $x = \sum_{i=1}^n \lambda_i e_i$ in C. Define $l : C - C \to \mathbb{R}^n$ and $\acute{l} : \acute{C} - \acute{C} \to \mathbb{R}^m$ by

$$l\Big(\sum_{i=1}^n \lambda_i e_i\Big) = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$$

and

$$\acute{l}\Big(\sum_{i=1}^m \gamma_i \acute{e}_i\Big) = (\gamma_1, \gamma_2, \dots, \gamma_m)^T,$$

respectively. The functions l and $\acute{l}$ are linear isomorphisms on $C - C$ and $\acute{C} - \acute{C}$, respectively. Set $M_i = \acute{l}(\phi(e_i))$ for $i = 1, \dots, n$ and $M_\phi = M_1 \times M_2 \times \dots \times M_n$. Then, M_ϕ is a convex and compact valued multimatrix. Let $x = \sum_{i=1}^n \lambda_i e_i$ and

$$z \in \psi(x) = \sum_{i=1}^n \lambda_i \phi(e_i).$$

There are y_i's in $\phi(e_i)$ such that $z = \sum_{i=1}^n \lambda_i y_i$. If $y_i = \sum_{j=1}^m \gamma_{ji} \acute{e}_j$, then $z = \sum_{i=1}^n \lambda_i \sum_{j=1}^m \gamma_{ji} \acute{e}_j$. Putting $A = [\gamma_{ji}]_{m \times n}$ we have $A \in M_\phi$ and

$$\psi(x) \subseteq \Big\{ \sum_{j=1}^m \sum_{i=1}^n \lambda_i \gamma_{ji} \acute{e}_j : A = [\gamma_{ji}]_{m \times n} \in M_\phi \Big\}.$$

On other hand,

$$\psi(x) \supseteq \Big\{ \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \gamma_{ji} \acute{e}_j : \ A = [\gamma_{ji}]_{m\times n} \in M_\phi \Big\}.$$

Therefore we get

$$\psi(x) = \Big\{ \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i \gamma_{ji} \acute{e}_j ; A = [\gamma_{ji}]_{m\times n} \in M_\phi \Big\} = \{\acute{l} A l(x)\}_{A \in M_\phi}.$$

Since every matrix A can be considered as a continuous linear mapping, and $\varphi(x) \supseteq \psi(x)$, the proof is complete. □

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