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THE LERAY-SCHAUDER CONDITION IN THE COINCIDENCE PROBLEM FOR TWO MAPPINGS

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Dedicated to Prof. Wataru Takahashi on the occasion of his 70th birthday

Abstract. The purpose of this paper is to study the existence of a coincidence point for two nonlinear mappings defined on a Banach space and taking values on another one using the Leray-Schauder condition. Later on, we apply these results to obtain the existence of solution to some classes of differential equations.

1. Introduction

From a mathematical point of view, many problems arising from diverse areas of natural science involve the existence of solutions of nonlinear equations with the form

$$t(u) = s(u), \quad u \in M, \tag{1.1}$$

where M is a nonempty subset of a Banach space X, and $s, t : M \to Y$ are nonlinear mappings taking values on another Banach space Y. The problem of finding a solution for Equation (1.1) is known as a *coincidence problem*. Coincidence theory is a very powerful technique especially in problems about of existence of solutions in nonlinear equations. For instance, in [7, 8, 15, 12, 13, 14, 17] several of such results are applied to solve boundary value problems.

The coincidence problem can be considered as a generalization of the fixed point problem since if $s : M \subseteq X \to X$ is a mapping, to study the existence of a fixed point for s is the same that to find a solution of the coincidence problem where t is the identity mapping on M. In this sense, R. Machuca [16] proved a coincidence theorem which is a generalization of the well known Banach contraction principle. Generalizations of this result can be found, for instance in [12, 13, 18]. On the other hand, Gaines and Mawhin introduced coincidence degree theory in the 70s in analyzing functional and differential equations [10]. The main goal in the coincidence degree theory is to search for the existence of solutions of Equation (1.1) in some bounded and open set M in some Banach space X for t being a linear operator and s a nonlinear operator using Leray-Schauder degree theory (see [22] to find some sharpening results).

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In this paper, we intend to obtain several versions, without invoking degree theory, of the coincidence problem where s and t can both be nonlinear. In Section 3.1, we are concerned with the solvability of the following differential equation:

$$\begin{cases} u'(t) - g(t, u(t), u'(t)) = f(t), & t \in (0,1) \text{ a.e.} \\ u(0) = \xi, \end{cases} \tag{1.2}$$

where $f \in L^1([0,1]; \mathbb{R}^n)$ is a fixed function and $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function.

Finally, we will use this kind of coincidence results to obtain the existence of solutions for a Dirichlet problem of the form:

$$\begin{cases} \Delta\rho(u(x)) = f(x, u(x)) & \text{in } \Omega \\ \rho(u(x)) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, Δ is the standard Laplace operator, $\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

2. Preliminaries

In this section, we shall introduce notations, definitions and preliminary facts which are used throughout this article.

Let X be a real normed space. As usual, given a nonempty subset $A \subseteq X$, denote the closure of A by $\overline{A}$, the boundary of A by ∂A, the convex hull of A by $co(A)$, the diameter of A by $\operatorname{diam}(A)$, and the family of bounded subsets of A by $\mathcal{B}(A)$.

Definition 2.1. *Let X be a normed space. A measure of non-compactness is a function $\mu : \mathcal{B}(X) \to \mathbb{R}^+$ which satisfies:*

(1) $\mu(A) = 0 \Leftrightarrow \overline{A}$ *is compact.*
(2) $\mu(A) = \mu(\overline{A})$.
(3) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$
(4) $\mu(co(A)) = \mu(A)$.

To avoid confusion when dealing with different spaces, we will in some cases add the name of a subspace as a subscript.

Point 3 in the above definition implies that $\mu(A) \leq \mu(B)$, whenever $A \subset B$.

The most important examples of measures of noncompactness are the *Kuratowski* measure of noncompactness (or set measure of noncompactness):

Given a bounded subset A of X,

$$\alpha(A) = \inf\left\{r > 0 : A \subset \cup_{i=1}^n D_i,\ \operatorname{diam}(D_i) \leq r\right\}.$$

And the *Hausdorff* measure of noncompactness (or ball measure of noncompactness):

Given a bounded subset A of X,

$$\chi(A) = \inf\left\{r > 0 : A \subset \cup_{i=1}^n B(x_i, r), x_i \in X\right\}.$$

A detailed account of theory and applications of measures of noncompactness may be found in the monographs [1, 4]

Definition 2.2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces endowed with the measures of noncompactness μ_X and μ_Y respectively. If C is a nonempty subset of X and $T : C \to Y$ is a mapping,*

(a) *Given $k > 0$, the mapping T is called (μ_X, μ_Y)-k-set contractive if $\mu_Y(T(A)) \leq k\mu_X(A)$ for all $A \in \mathcal{B}(C)$.*
(b) *The mapping T is called (μ_X, μ_Y)-condensing if $\mu_Y(T(A)) < \mu_X(A)$ for all bounded subset A of C with $\mu_X(A) > 0$.*
(c) *The mapping T is called expansive if the inequality $\|T(x) - T(y)\|_Y \geq \|x - y\|_X$ holds for every $x, y \in C$.*
(d) *The mapping T is called nonexpansive if the inequality $\|T(x) - T(y)\|_Y \leq \|x - y\|_X$ holds for every $x, y \in C$.*
(e) *The mapping T is said to be bounded if there exists $k > 0$ such that $\|T(x)\|_Y \leq k$ for all $x \in C$.*

The following well known theorem was proved in 1967 by Sadovskii [21], it is a generalization of Darbo's fixed point theorem [9]. We refer to [3] where the reader will find many applications of these theorems.

Theorem 2.3. *Suppose that C is a closed convex bounded subset of a Banach space X and $T : C \to C$ a continuous and condensing mapping, then T has a fixed point.*

When the domain C, in Sadovskii's theorem, is unbounded the following result is also well known.

Theorem 2.4. *Suppose that C is a closed convex and unbounded subset of a Banach space X and $T : C \to C$ a continuous and condensing mapping. If there exist $R > 0$ and $z \in C$ such that for all $u \in C \cap S_R(z)$*

$$T(u) - z \neq \lambda(u - z), \qquad \forall \lambda > 1,$$

then T has a fixed point.

We recall the following theorem proved by Petryshyn in [19] (also see [20]).

Theorem 2.5. *Suppose that U is an open bounded subset of a Banach space X and $T : \overline{U} \to X$ a continuous and condensing mapping. If there exists $z \in U$ such that for all $u \in \partial U$*

$$u \neq \lambda T(u) + (1 - \lambda)z, \qquad \forall \lambda \in (0, 1),$$

then T has a fixed point.

Let $T : X \to Y$ be a mapping which transforms bounded subsets of X into bounded subsets of Y. For a such mapping, we define

$$l(T) := \sup\{r > 0 : r\mu_X(A) \leq \mu_Y(T(A)), A \in \mathcal{B}(X)\}.$$

In the following we are going to use the Kuratowski measure of noncompactness. Assuming that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and if a mapping $T : C \to Y$ is (α_X, α_Y)-condensing we will say simply that T is α-condensing.

If $T : X \to Y$ is an invertible and bounded linear map and $N : \overline{\Omega} \to Y$ is a k-set contractive map with $k < l(T)$ such that for all $x \in \partial\Omega$ we have $T(x) \neq N(x)$, one can associate with the pair (T,N) a topological degree, the so-called Sadovskii-Nussbaum degree (see [10] we also refer to the reader to [22]). This degree allows us to show the following.

Theorem 2.6. *Let X and Y be two Banach spaces and let $T : X \to Y$ be an invertible, bounded linear map and $\Omega \subseteq X$ bounded open and symetric about $0 \in \Omega$. Let $N : \overline{\Omega} \to Y$ be a k-set contractive map with $k < l(T)$. Then given $y \in Y$ such that $T(x) \neq \lambda N(x) + \lambda y$, for all $x \in \partial\Omega$ and $\lambda \in (0,1)$, there exists $x \in \overline{\Omega}$ such that $T(x) = N(x) + y$.*

Let X and Y be two normed spaces and $A : X \to 2^Y$ a multivalued mapping. Recall some useful notation, namely: the (effective) domain of A is $D(A) := \{x \in X : A(x) \neq \emptyset\}$, the range of A is $R(A) := \{u \in A(x) : x \in D(A)\}$ and its graph is the subset of $X \times Y$ defined as $G(A) := \{(x,u) \in X \times Y : x \in D(A),\ u \in A(x)\}$.

We note that one may identify each subset $G \subseteq X \times Y$ with a multivalued mapping $A : X \to 2^Y$ by defining $A(x) := \{u \in Y : (x,u) \in G\}$. The inverse $A^{-1} : R(A) \to 2^X$ is then defined by $A^{-1}(u) := \{x \in X : u \in A(x)\}$ and it is clear that $G(A^{-1}) = \{(u,x) \in X \times Y : (x,u) \in G(A)\}$.

Given an operator $A : D(A) \to 2^X$, we define

$J_\lambda := (I + \lambda A)^{-1} : R(I + \lambda A) \to 2^{D(A)}$, here I means the identity operator. As usual, we call to J_λ a resolvent of A.

Definition 2.7. *An operator $A \subset X \times X$ is said to be accretive if and only if $\|x - y\| \leq \|x - y + \lambda(u - v)\|$ for all $x, y \in D(A)$, for each $u \in Ax$ and $v \in Ay$, and for all $\lambda > 0$. If moreover, $R(A + I) = X$, we say that A is m-accretive.*

Proposition 2.8. *The operator $A \subset X \times X$ is accretive if and only if for each $\lambda > 0$, the resolvent $J_\lambda : R(I + \lambda A) \to D(A)$ is a single-valued nonexpansive mapping.*

A detailed account of theory and applications of accretive operators may be found, for instance, in the monograph [5].

3. A Leray-Schauder condition to the coincidence problem

3.1. **Single-valued case.** The first purpose here is to establish several coincidence results by using Theorem 2.4.

Theorem 3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Consider C a convex closed subset of X such that the mappings $t : C \to Y$ and $s : C \to Y$ satisfy:*

1. *$\overline{t(C)}$ is a convex subset of Y and $t^{-1} : t(C) \to C$ is uniformly continuous on bounded subsets of $t(C)$,*
2. *s is a continuous k-set contractive mapping,*
3. *$s(C) \subset t(C)$,*
4. *$k < l(t)$,*
5. *There are $R > 0$ and $x_0 \in C$ such that for every $x \in C$ with $\|x - x_0\| \geq R$ we have*

$$s(x) - t(x_0) \neq \lambda(t(x) - t(x_0)) \qquad \forall \lambda > 1. \tag{3.1}$$

Then there is $z \in C$ such that $s(z) = t(z)$.

Proof. It is not difficult to check that $t^{-1} : t(C) \to C$ is $\frac{1}{l(t)}$-set contractive. However, if we do not assume the continuity of t, then $t(C)$ may be unclosed (for instance see [12]). Thus, we have to consider $\overline{t^{-1}} : \overline{t(C)} \to C$ given by

$$\overline{t^{-1}}(y) = \begin{cases} t^{-1}(y) & , \text{ si } y \in t(C) \\ \lim_n t^{-1}(x_n) & , \ x_n \in t(C), \ x_n \to y \in \partial(t(C)). \end{cases}$$

We claim that $\overline{t^{-1}}$ is also $\frac{1}{l(t)}$-set contractive. Let A be a bounded subset of $\overline{t(C)}$. Since $A = (A \cap t(C)) \cup (A \cap \partial(t(C)))$, and $t^{-1} : t(C) \to C$ is $\frac{1}{l(t)}$-set contractive, it is enough to assume that $A \subset \partial(t(C))$.

Take $r = \alpha_Y(A)$ and $\varepsilon > 0$, there exist $A_1, A_2, \dots, A_n$ such that $A \subset \cup_{i=1}^n A_i \subset \partial(t(C))$ and $\operatorname{diam}_{\|\cdot\|_Y}(A_i) \leq r + \frac{\varepsilon}{3}$.

$$\alpha_X(\overline{t^{-1}}(A)) \leq \max\{\alpha_X(\overline{t^{-1}}(A_i)) : i = 1, \dots, n\}. \tag{3.2}$$

Let

$$B_i := \bigcup_{y \in A_i} \left\{ x \in t(C) : \|x - y\|_Y \leq \frac{\varepsilon}{3} \right\}.$$

Note that $A_i \subset \overline{B_i}$. Then

$$\begin{aligned} \alpha_X(\overline{t^{-1}}(A_i)) &\leq \alpha_X(\overline{t^{-1}}(\overline{B_i})) = \alpha_X(\overline{t^{-1}(B_i)}) \\ &= \alpha_X(t^{-1}(B_i)) \leq \frac{1}{l(t)} \alpha_Y(B_i). \end{aligned} \tag{3.3}$$

For $u, v \in B_i$ there exist $x_u, x_v \in A_i$ such that $\max\{\|u - x_u\|_Y, \|v - x_v\|_Y\} \leq \frac{\varepsilon}{3}$, then

$$\begin{aligned} \|u - v\|_Y &\leq \|u - x_u\|_Y + \|x_u - x_v\|_Y + \|x_v - v\|_Y \\ &\leq \frac{\varepsilon}{3} + \operatorname{diam}_{\|\cdot\|_Y}(A_i) + \frac{\varepsilon}{3} \\ &\leq \frac{2\varepsilon}{3} + r + \frac{\varepsilon}{3} \\ &= r + \varepsilon. \end{aligned}$$

So $\operatorname{diam}(B_i) \leq r + \varepsilon$, therefore $\alpha_Y(B_i) \leq r + \varepsilon$.

Using (3.2), (3.3) and since ε is an arbitrary positive number we have

$$\alpha_X(\overline{t^{-1}}(A)) \leq \frac{1}{l(t)} \alpha_Y(A)$$

as we claimed.

Since by assumption (3), $s(C) \subseteq t(C)$, we may consider the mapping $h : \overline{t(C)} \to t(C)$ given by $h(x) = s(\overline{t^{-1}}(x))$, clearly h is a continuous mapping and for every bounded subset B of $\overline{t(C)}$ we have

$$\alpha_Y(h(B)) = \alpha_Y(s \circ \overline{t^{-1}}(B)) \leq k \alpha_X(\overline{t^{-1}}(B)) \leq \frac{k}{l(t)} \alpha_Y(B),$$

in consequence h is a condensing mapping because $\frac{k}{l(t)} < 1$.

Next step will be to check that h satisfies a Leray-Schauder condition with $y_0 := t(x_0)$. Otherwise, without loss of generality, we may assume that for every $n \in \mathbb{N}$, there exists $y_n \in t(C)$ (i.e., there exits $x_n \in C$ such that $y_n = t(x_n)$) and $\lambda_n > 1$ such that

$$\|y_n - y_0\|_Y \geq n \tag{3.4}$$

and

$$h(t(x_n)) - y_0 = \lambda_n(t(x_n) - y_0). \tag{3.5}$$

From (3.1) and (3.5) we deduce that

$$\|x_n - x_0\|_X < R,$$

which means that $\{x_n\}$ is a bounded sequence.

Since s is α-condensing then $\{s(x_n)\}$ is a bounded sequence. From (3.5) then

$$t(x_n) = \frac{1}{\lambda_n}(s(x_n) - y_0) + y_0,$$

this implies that $\{t(x_n)\}$ is also bounded which is in contradiction with (3.4).

Finally, applying Theorem 2.4, there exists $z_0 \in \overline{t(C)}$ such that $h(z_0) = z_0$. Since $s(C) \subseteq t(C)$, there is $z \in C$ such that $z_0 = t(z)$, which means that $t(z) = s(z)$. □

Corollary 3.2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $t : X \to Y$ be a continuous invertible linear map and C a convex closed subset of X.Let $s : C \to Y$ be a continuous k-set contraction with $k < l(t)$ and satisfying that $s(C) \subset t(C)$. If there are $R > 0$ and $x_0 \in C$ such that*

$$x \in C, \|x - x_0\|_X \geq R \Rightarrow s(x) - t(x_0) \neq \lambda(t(x) - t(x_0)) \qquad \forall \lambda > 1,$$

then there is $z \in C$ such that $s(z) = t(z)$.

Proof. Notice that $t(C)$ is a convex subset of Y since C is convex and t is a linear mapping. Moreover, the open mapping theorem says that $t^{-1} : Y \to X$ is a continuous linear mapping and then uniformly continuous. Now, in order to obtain the conclusion we may apply Theorem 3.1. □

Next result may be considered as a sharpening of Theorem 2.5.

Theorem 3.3. *Let X be a normed space and let Y be a Banach space. Assume that U is a bounded open subset of X, $t : \overline{U} \to Y$ an expansive mapping such that $t(U)$ is an open bounded subset of Y with $\partial(t(U)) \subset t(\partial U)$ and $s : \overline{U} \to Y$ is a continuous condensing mapping. If there exists $x_0 \in U$ such that for all $x \in \partial U$*

$$t(x) \neq \lambda s(x) + (1 - \lambda)t(x_0) \qquad \forall \lambda \in (0, 1) \tag{3.6}$$

then there exists $z_0 \in Y$ such that $t(z_0) = s(z_0)$.

Proof. Since t is expansive then it is injective. Since t is an injection, we deduce that $t(\partial U) \subset \partial(t(U))$. That means, with our assumptions, $t(\partial U) = \partial(t(U))$. Then

$$t(\overline{U}) = t(U) \cup t(\partial U) = t(U) \cup \partial(t(U)) = \overline{t(U)}.$$

Define $h = s \circ t^{-1} : \overline{t(U)} \to Y$. Since t^{-1} and s are continuous then h is a continuous mapping.

We claim that h is α-condensing. Let A be a bounded subset of Y. Take $B = t^{-1}(A)$, since t^{-1} is nonexpansive then

$$\mathrm{diam}_{\|\cdot\|_X}(B) = \mathrm{diam}_{\|\cdot\|_X}(t^{-1}(A)) \leq \mathrm{diam}_{\|\cdot\|_Y}(A).$$

This means that B is bounded.

Notice that if $\alpha_Y(A) = 0$, then $\alpha_X(B) = 0$. Hence if we assume that $\alpha_Y(A) > 0$:

$$\alpha_Y(h(A)) = \alpha_Y(s(B)) < \alpha_X(B) = \alpha_X(t^{-1}(A)) \leq \alpha_Y(A),$$

otherwise $0 = \alpha_Y(h(A)) < \alpha_Y(A)$. Therefore h is α-condensing as we claimed.

Now, we are going to show that

$$y \neq \lambda h(y) + (1-\lambda)t(x_0), \qquad \forall \lambda \in (0,1),$$

whenever $y \in \partial(t(U))$.

Otherwise, there would be $y \in \partial(t(U)) = t(\partial U)$ and a number $\lambda \in (0,1)$ such that

$$y = \lambda h(y) + (1-\lambda)t(x_0).$$

Then there exists $x \in \partial U$ such that $t(x) = y$, so

$$t(x) = \lambda(s \circ t^{-1})(t(x)) + (1-\lambda)t(x_0)$$

this implies

$$t(x) = \lambda s(x) + (1-\lambda)t(x_0),$$

which contradicts (3.6).

Finally, since $t(U)$ is open, Theorem 2.5 guarantees the existence of a fixed point $y_0 \in t(\overline{U})$ for h, then there exists $z_0 \in \overline{U}$ such that $y_0 = t(z_0)$ and so $s(z_0) = t(z_0)$. □

Remark 3.4. *If in the above theorem we add that X is a Banach space and that t is continuous, then $t(\overline{U})$ is a closed subset of Y (for instance see* [12]*). Therefore the assumption $\partial(t(U)) \subset t(\partial U)$ is directly satisfied.*

The following corollary allows us to give a completion of Theorem 2.6.

Corollary 3.5. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $t : X \to Y$ be an expansive continuous invertible affine map and U a bounded open subset of X.Let $s : \overline{U} \to Y$ be a continuous condensing mapping. If there is $x_0 \in U$ such that for all $x \in \partial U$,*

$$t(x) \neq \lambda s(x) + (1-\lambda)t(x_0) \qquad \forall \lambda \in (0,1),$$

then there is $z \in \overline{U}$ such that $s(z) = t(z)$.

Proof. The open mapping theorem guarantees that $t(U)$ is an open subset of Y and that $t^{-1} : Y \to X$ is a continuous affine mapping and then uniformly continuous. Now, in order to obtain the conclusion we may apply Theorem 3.3. □

Example 3.6. We would like to know if the the system of equations given by

$$\left\{ \begin{array}{ll} x^2 & = \sqrt{x+y} \\ y^2 & = 3\sin(2x+y) \end{array} \right. \tag{3.7}$$

has at least a non trivial solution.

Consider $X = (\mathbb{R}^2, \frac{1}{3}\|\cdot\|_\infty)$, $Y = (\mathbb{R}^2, \|\cdot\|_\infty)$, $U = (1/2, 2) \times (1/2, 2) \subset X$, the mapping $t : \overline{U} \to Y$ given by

$$t(x, y) = (x^2, \frac{1}{3}y^2)$$

and the mapping $s : X \to Y$ given by

$$s(x, y) = (\sqrt{x+y}, \sin(2x + y)).$$

Note that s is continuous and it is compact since we are working in finite dimensional spaces. It can be shown that

$$\|t(x_1, y_1) - t(x_2, y_2)\|_\infty \geq \frac{1}{3}\|(x_1, y_1) - (x_2, y_2)\|_\infty,$$

that is, t is expansive. Moreover, $t(\partial U) = \partial(t(U))$ and $t(U)$ is an open set.

We need to check that (3.6) is satisfied in Theorem 3.3.

Observe that

$$\partial U = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4,$$

where

$$\begin{aligned} \sigma_1 &= \{(1/2, t) : t \in [1/2, 2]\}, \\ \sigma_2 &= \{(t, 2) : t \in [1/2, 2]\}, \\ \sigma_3 &= \{(2, t) : t \in [1/2, 2]\}, \\ \sigma_4 &= \{(t, 1/2) : t \in [1/2, 2]\}. \end{aligned}$$

We will check that (3.6) is satisfied with the point $(1, 1)$ for every $(x, y) \in \partial U$, i.e., for all $(x, y) \in \partial U$ we will see that

$$(x^2, \frac{1}{3}y^2) \neq \left(\lambda\sqrt{x+y} + (1-\lambda), \lambda \sin(2x + y) + \frac{1}{3}(1-\lambda)\right) \tag{3.8}$$

for all $\lambda \in (0, 1)$.

For $t \in [1/2, 2]$ and for $\lambda \in (0, 1)$ we observe that

$$\lambda\sqrt{\frac{1}{2} + t} + (1-\lambda) \geq \lambda + (1-\lambda) = 1 > \frac{1}{4}, \tag{3.9}$$

$$\begin{aligned} \lambda \sin(2t + 2) + \frac{1}{3}(1-\lambda) &\leq \lambda + \frac{1}{3}(1-\lambda) = \frac{2}{3}\lambda + \frac{1}{3} \\ &< \frac{2}{3} + \frac{1}{3} = 1 < \frac{4}{3}, \end{aligned} \tag{3.10}$$

$$\lambda\sqrt{2 + t} + (1-\lambda) < 2\lambda + (1-\lambda) = \lambda + 1 < 2 < 4. \tag{3.11}$$

then (3.9) shows that (3.8) holds on σ_1, (3.11) shows that (3.8) holds on σ_2 and (3.11) shows that (3.8) holds on σ_3.

For $t \in [12/10, 2]$ and for $\lambda \in (0, 1)$

$$\begin{aligned} \frac{\lambda}{t^2}\sqrt{t + \frac{1}{2}} + \frac{1-\lambda}{t^2} &\leq \lambda\sqrt{\left(\frac{10}{12}\right)^3 + \left(\frac{1}{2}\right)\left(\frac{10}{12}\right)^4} + \frac{1-\lambda}{t^2} \\ &< \lambda + (1-\lambda) < 1. \end{aligned} \tag{3.12}$$

For $t \in [1/2, 12/10]$ and for $\lambda \in (0,1)$

$$\begin{aligned}\lambda \sin\left(2t+\frac{1}{2}\right)+\frac{1-\lambda}{3} &\geq \lambda \sin\left(2\left(\frac{12}{10}\right)+\frac{1}{2}\right)+\frac{1-\lambda}{3}\\ &= \lambda \sin\left(\frac{29}{10}\right)+\frac{1-\lambda}{3} > \frac{\lambda}{5}+\frac{1-\lambda}{3}\\ &= \frac{1}{3}-\lambda\left(\frac{2}{15}\right) > \frac{1}{3}-\frac{2}{15} = \frac{3}{15} > \frac{1}{12}. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13) we deduce that (3.8) holds on σ_4.

Therefore, by Theorem 3.3 there exists $(x,y) \in \overline{U}$ such that $t(x,y) = s(x,y)$ and this point is a solution of the system.

3.1.1. *Existence of strong solution to a differential equation.* Consider the Banach space $(\mathbb{R}^n, \|\cdot\|_n)$ and let $L^1(0,1;\mathbb{R}^n)$ be the Banach space of Bochner integrable functions $x : [0,1] \to \mathbb{R}^n$ endowed with the norm

$$\|x\|_1 = \int_0^1 \|x(t)\|_n dt.$$

It is well known that if $x : [0,1] \to \mathbb{R}^n$ is absolutely continuous, then it is almost everywhere differentiable on $[0,1]$, its derivative $x' \in L^1(0,1;\mathbb{R}^n)$ and

$$x(t) = x(0) + \int_0^t x'(s)ds.$$

In this section we are concerned to find an absolutely continuous function $u : [0,1] \to \mathbb{R}^n$ such that its derivative $u' \in L^1(0,1;\mathbb{R}^n)$ satisfies almost for every point in $(0,1)$ the following differential equation

$$\begin{cases} u'(t) - g(t,u(t),u'(t)) = f(t), & t \in (0,1) \text{ a.e.} \\ u(0) = \xi \in \mathbb{R}^n, \end{cases} \tag{3.14}$$

where $f \in L^1(0,1;\mathbb{R}^n)$ is a fixed function and $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function. A such function u is called strong solution of Eq.(3.14).

First, let us notice that (3.14) is equivalent to the differential equation

$$\begin{cases} u'(t) - g(t,u(t)+\xi,u'(t)) = f(t), & t \in (0,1) \text{ a.e.} \\ u(0) = 0, \end{cases} \tag{3.15}$$

Thus, our goal will be to study the existence of a strong solution of (3.15).

Let us introduce the Sobolev space $W^{1,1}(0,1;\mathbb{R}^n)$ as the space of all absolutely continuous functions. Then we can write this space as:

$$W^{1,1}(0,1;\mathbb{R}^n) := \left\{u \in L^1(0,1;\mathbb{R}^n) : u' \in L^1(0,1;\mathbb{R}^n)\right\},$$

The space $W^{1,1}(0,1;\mathbb{R}^n)$ can be endowed with the norm

$$\|u\|_{1,1} := \max\{\|u\|_1, \|u'\|_1\},$$

where $\|\cdot\|_1$ is the usual norm in $L^1(0,1;\mathbb{R}^n)$. $(W^{1,1}(0,1;\mathbb{R}^n), \|\cdot\|_{1,1})$ is a Banach space.

Now we can consider the following subspace $X := \{u \in W^{1,1}(0,1;\mathbb{R}^n) : u(0) = 0\}$. This is a closed subspace of $(W^{1,1}(0,1;\mathbb{R}^n), \|\cdot\|_{1,1})$ and thus it is also a Banach space.

Lemma 3.7. *Let u be an element in X. Then $\|u\|_{1,1} = \|u'\|_1$.*

Proof. It is well know that if $u \in X$ then $u(t) = u(t) - u(0) = \int_0^t u'(\tau)d\tau$ in $[0,1]$. Therefore

$$\|u(t)\|_n \leq \int_0^t \|u'(\tau)\|_n d\tau,$$

this means that $\|u\|_1 \leq \|u'\|_1$ and consequently $\|u\|_{1,1} = \|u'\|_1$. □

Lemma 3.8. *Let f be a fixed element of $L^1(0,1;\mathbb{R}^n)$. The mapping $T : X \to L^1(0,1;\mathbb{R}^n)$ defined by $T(u)(t) = u'(t) - f(t)$ is an expansive bijection.*

Proof. T is an expansive mapping. Indeed, by Lemma 3.7, we know that if $u, v \in X$, then $\|u - v\|_{1,1} = \|u' - v'\|_1$, then,

$$\|Tu - Tv\|_1 = \|u' - v'\|_1 = \|u - v\|_{1,1}.$$

Now, let us see that T is onto. Indeed, given $u \in L^1(0,1;\mathbb{R}^n)$ it is enough to consider

$$w(t) := \int_0^t (u(\tau) + f(\tau))d\tau,$$

since in this case, $w \in X$ and $T(w) = u$. □

Let $\mathcal{M}(0,1;\mathbb{R}^n)$ be the set of all measurable functions $\varphi : [0,1] \to \mathbb{R}^n$. If $f : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, then f defines a mapping $N_f : \mathcal{M}(0,1;\mathbb{R}^n) \to \mathcal{M}(0,1;\mathbb{R}^n)$ by $N_f(\varphi)(t) := f(t,\varphi(t))$. This mapping is called the superposition (or Nemytskii) operator generated by f. The next three lemmas are of foremost importance for our subsequent analysis.

Lemma 3.9. *Let $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function, if there exist a constant $b \geq 0$ and a function $a(\cdot) \in L^1_+(0,1;\mathbb{R})$ such that*

$$\|f(t,x)\|_n \leq a(t) + b\|x\|_n,$$

then N_f maps continuously $L^1(0,1;\mathbb{R}^n)$ into itself.

In order to do a proof of the above lemma we can follow a similar argument as in [2, Theorems 3.1 and 3.7]).

If we argue as in [2, Lemma 9.5] we obtain:

Lemma 3.10. *Let $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function, if there exist a constant $b \geq 0$ and a function $a(\cdot) \in L^1_+(0,1;\mathbb{R})$ such that*

$$\|g(t,x,y)\|_n \leq a(t) + b(\|x\|_n + \|y\|_n),$$

then the map $N_g : W^{1,1}(0,1;\mathbb{R}^n) \to L^1(0,1;\mathbb{R}^n)$ defined by

$$N_g(\varphi)(t) = g(t,\varphi(t),\varphi'(t))$$

is continuous.

Lemma 3.11. *Let $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function such that there exist $a \in L^1_+(0,1,\mathbb{R})$, $b, k > 0$ satisfying that*

(1) $\|g(t,x,0)\|_n \le a(t) + b\|x\|_n$,
(2) $\|g(t,x,y_1) - g(t,x,y_2)\|_n \le k\|y_1 - y_2\|_n$.

Then, the operator $N_g : X \to L^1(0,1;\mathbb{R}^n)$ *is* $2k$*-set contractive.*

Proof. From assumptions (1) and (2) we obtain that

$$\|g(t,x,y)\|_n \le a(t) + b\|x\|_n + k\|y\|_n,$$

for $(t,x,y) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$. Hence, using Lemma 3.10, we infer that $N_g : X \to L^1(0,1;\mathbb{R}^n)$ is a continuous mapping.

Let A be a bounded subset of X, and let $r = \alpha_X(A)$. Then for every $\varepsilon > 0$ there exist subsets $A_1,\dots,A_n$ of X such that $A = \cup_{i=1}^n A_i$ and $\operatorname{diam}_{\|\cdot\|_{1,1}}(A_i) < r + \varepsilon$. Since by [23, Theorem 1], we know that the injection of $W^{1,1}(0,1;\mathbb{R}^n)$ in $L^1(0,1,\mathbb{R}^n)$ is compact then $A, A_1,\dots,A_n$ are relatively compact in $L_1(0,1;\mathbb{R}^n)$.

Let $u_i \in A_i$ for $i = 1,\dots,n$. Define the mapping $g_{u_i} : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ given by $g_{u_i}(t,x) = g(t,x,u_i'(t))$. Since g is a Carathéodory function then the g_{u_i}'s are also. Moreover from assumption (1) and Lemma 3.9, the mapping $N_{u_i} : L^1(0,1;\mathbb{R}^n) \to L^1(0,1;\mathbb{R}^n)$ defined as $N_{u_i}v(t) = g_{u_i}(t,v(t))$ is well defined and it is also continuous. Then N_{u_i} are uniformly continuous on $\cup_{i=1}^n A_i$. So, there exists $\delta > 0$ such that for $\|v - w\|_1 < \delta$ with $v,w \in \cup_{i=1}^n A_i$ we have

$$\|N_{u_i}(v) - N_{u_i}(w)\|_1 := \int_0^1 \|g(t,v(t),u_i'(t)) - g(t,w(t),u_i'(t))\|_n dt < \varepsilon.$$

For each A_i there is a finite family of subsets $A_{i,j}$ such that $A_i = \cup_j A_{i,j}$ and $\operatorname{diam}_{\|\cdot\|_1}(A_{i,j}) < \delta$.

Therefore for any $v,w \in A_{i,j}$ we have

$$\begin{aligned}
\|N_g(v) - N_g(w)\|_1 &= \int_0^1 \|g(t,v(t),v'(t)) - g(t,w(t),w'(t))\|_n dt \\
&\le \int_0^1 \|g(t,v(t),v'(t)) - g(t,v(t),u_i'(t))\|_n dt \\
&\quad + \int_0^1 \|g(t,v(t),u_i'(t)) - g(t,w(t),u_i'(t))\|_n dt \\
&\quad + \int_0^1 \|g(t,w(t),u_i'(t)) - g(t,w(t),w'(t))\|_n dt \\
&\le k\int_0^1 \|v'(t) - u_i'(t)\|_n dt + \varepsilon + k\int_0^1 \|u_i'(t) - w'(t)\|_n dt \\
&\le k\|v - u_i\|_{1,1} + \varepsilon + k\|w - u_i\|_{1,1} \\
&\le 2kr + \varepsilon.
\end{aligned}$$

That is

$$\alpha_{L^1}(N_g(A)) \le 2k\alpha_X(A).$$

□

Now, for studying the existence of a strong solution to (3.15), we define

$$T : X \to L^1(0,1;\mathbb{R}^n) \quad \text{by} \quad T(u) = u' - f$$

and

$$S : X \to L^1(0,1;\mathbb{R}^n) \ \text{ by } \ S(u) = N_{\tilde{g}}(u),$$

where $\tilde{g}(t,x,y) = g(t, x+\xi, y)$.

Thus, to show that (3.15) has a solution is to see that the coincidence problem, $T(u) = S(u)$ admits a solution.

Theorem 3.12. *If* $\max\{b+k, 2k\} < 1$, *(3.15) has at least a solution in the Sobolev space* $W^{1,1}(0,1;\mathbb{R}^n)$.

Proof. In order to show that T, S fulfill the conditions of Corollary 3.5. First we shall show that there exists $r > 0$ such that if $\|u\|_{1,1} \geq r$ then $T(u) \neq \mu N_{\tilde{g}}(u)$ for all $\mu \in (0,1)$, since the rest of conditions are consequences of the above lemmas. Thus, let us take $u \in X$ satisfying that $T(u) = \lambda N_{\tilde{g}}(u)$ for some $\lambda \in (0,1)$. Hence

$$u'(t) - f(t) = \lambda g(t, u(t)+\xi, u'(t)), \quad t \in (0,1) \text{ a.e.}$$

from this equality we infer that

$$\|u'(t)\|_n \leq \lambda \left(a(t) + b(\|u(t)\|_n + \|\xi\|_n) + k\|u'(t)\|_n\right) + \|f(t)\|_n \ \text{ a.e.}$$

Therefore,

$$\|u'\|_1 \leq \lambda\|a||_1 + \lambda b\|u\|_1 + \lambda b|\xi| + \lambda k\|u'\|_1 + \|f\|_1.$$

Applying Lemma 3.7, we obtain that

$$(1 - \lambda(b+k))\|u\|_{1,1} \leq \lambda(\|a\|_1 + b|\xi|) + \|f\|_1.$$

Since by hypothesis $b + k < 1$ if we call $r := \frac{\|a\|_1 + |\xi| + \|f\|_1}{1-(b+k)}$, it is easy to see that $\|u\|_{1,1} < r$. This inequality allows us to conclude that if $\|u\|_{1,1} \geq r$, then $Tu \neq \lambda N_{\tilde{g}}(u)$ for all $\lambda \in (0,1)$.

Now, let $u_0(t) = \int_0^t f(\tau)d\tau$. We choose $x_0 = Tu_0 = 0$ and

$$U = \{u \in X : \|u - u_0\|_{1,1} < r + \|u_0\|\}.$$

If $u \in \partial(U)$ we have $\|u - u_0\|_{1,1} = r + \|u_0\|$ which implies $\|u\|_{1,1} \geq r$ so for all $\lambda \in (0,1)$

$$Tu \neq \lambda N_{\tilde{g}}(u) + (1-\lambda)x_0.$$

Applying Corollary 3.5 there exists $z_0 \in \overline{U}$ such that $Tz_0 = N_{\tilde{g}}z_0$, as we want to show. □

A trivial consequence of Theorem 3.12 is the following one:

The equation

$$\text{(3.16)} \qquad \begin{cases} u'(t) - g(t, u(t)) = f(t), & t \in (0,1) \text{ a.e.} \\ u(0) = \xi, \end{cases}$$

where $f \in L^1(0,1;\mathbb{R}^n)$ is a fixed function and $g : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that there exist $a \in L^1_+(0,1)$, $0 \leq b < 1$ satisfying that $\|g(t,x)\|_n \leq a(t) + b\|x\|_n$, has a strong solution.

Example 3.13.

$$\text{(3.17)} \qquad \begin{cases} u'(t) - \frac{\cos(u(t))}{\sqrt{t}} - \frac{u(t)+\sin(u'(t))}{2\sqrt{t+2}} = f(t), & t \in (0,1) \\ u(0) = \xi, \end{cases}$$

has a strong solution since in this example we have that $g(t,x,y) = \frac{\cos(x)}{\sqrt{t}} + \frac{x+\sin(y)}{2\sqrt{t+2}}$ and therefore $|g(t,x,0)| \leq \frac{1}{\sqrt{t}} + \frac{1}{2\sqrt{2}}|x|$ and $|g(t,x,y_1) - g(t,x,y_2)| \leq \frac{1}{2\sqrt{2}}|y_1 - y_2|$, which implies that g fulfills the conditions of Theorem 3.12.

3.2. **Multivalued case.**

Theorem 3.14. *Let X be a normed space and let Y be Banach space. Consider a nonempty subset D of X. Suppose that $t : D \to 2^Y$ is a multivalued mapping and $s : D \to Y$ is a mapping which satisfy:*

(1) *$R(t) = Y$ and $t^{-1} : Y \to D$ is a univalued continuous and compact mapping,*
(2) *s is continuous and it maps bounded subsets into bounded subsets,*
(3) *There exists $R > 0$ such that*

$$\|x\|_X \geq R,\ x \in D \quad \Rightarrow \quad \lambda s(x) \notin t(x) \quad \forall \lambda \in (0,1). \tag{3.18}$$

Then there exists $x_0 \in D$ with $s(x_0) \in t(x_0)$.

Proof. The mapping $h : Y \to Y$ given by $h(y) = s \circ t^{-1}(y)$ is continuous and compact. Therefore h is a continuous condensing mapping.

We will see that h satisfies a Leray-Schauder condition with 0_Y. Otherwise, we can assume that for each $n \in \mathbb{N}$ there are $y_n \in Y$ with $\|y_n\| \geq n$ and $\lambda_n > 1$ such that

$$h(y_n) = \lambda_n y_n. \tag{3.19}$$

Since $R(t) = Y$ for each $n \in \mathbb{N}$ there exists $x_n \in D$ with $y_n \in t(x_n)$, from this and (3.19)

$$\frac{1}{\lambda_n} s(x_n) = y_n \in t(x_n).$$

Therefore $\|x_n\| \leq R$, from (3.18). Using assumption (2) we have that $(s(x_n))$ is a bounded sequence, then (y_n) is a bounded sequence which is a contradiction.

Hence there exists $M > 0$ such that

$$\|y\|_Y \geq M \Rightarrow h(y) \neq \lambda y, \quad \forall \lambda > 1,$$

and from Theorem 2.4 we have that there exist $y_0 \in Y$ such that $h(y_0) = y_0$, but since $R(t) = Y$, there exists $x_0 \in D$ such that $t^{-1}(y_0) = x_0$. Then we obtain that $s(x_0) = y_0 \in t(x_0)$ as we want to prove. □

Corollary 3.15. *Let X be a Banach space and $A : D(A) \to 2^X$ an m-accretive operator such that $0 \in A(0)$ and $s : D(A) \to X$ a continuous mapping. Suppose that the following conditions are fulfilled:*

(1) *J_λ^A is compact,*
(2) *there exists $R > 0$ such that $\|s(x)\| \leq a + b\|x\|$ whenever $x \in D(A)$ with $\|x\| \geq R$.*

Then given $\rho > b$ there exists $x_0 \in D(A)$ such that $s(x_0) \in \rho x_0 + A(x_0)$.

Proof. Let $\lambda := \frac{1}{\rho}$. To show the result we are going to apply Theorem 3.14 to the mappings $t := I + \lambda A$ and λs. Thus we have to see that the coincidence problem $\lambda s(x) \in t(x)$ has a solution.

Since A is m-accretive, $t^{-1} = J_\lambda^A : X \to D(A)$ is single-valued and nonexpansive and by assumption (1) it is also compact. Moreover, it is not difficult to see that $\|y\| \geq \|x\|$ whenever $y \in x + \lambda A(x)$. Indeed, we know that there is $z \in A(x)$ such that $y = x + \lambda z$, hence since A is accretive and $0 \in A(0)$, we obtain

$$\|x - 0\| \leq \|x - 0 + \lambda(z - 0)\| = \|x + \lambda z\| = \|y\|. \tag{3.20}$$

Assumption (2) guarantees that λs maps bounded set into bounded sets.

Finally, let us see that there exists $\beta > 0$ such that $\mu\lambda s(x) \notin (I+\lambda A)(x)$ whenever $\|x\| \geq \beta$ and $x \in D(A)$. Indeed, if there exists $\mu \in (0, 1)$ such that

$$\mu\lambda s(x) \in (I + \lambda A)(x), \tag{3.21}$$

then by (3.20)and (3.21) we have that $\mu\lambda\|s(x)\| \geq \|x\|$. In this case, assumption (2) yields

$$\|x\| \leq \mu\lambda(a + b\|x\|),$$

which is a contradiction when we take $\beta > R$ larger enough and $\rho > b$. □

Next result works with mappings which are condensing but not necessarily k-set contractive, examples of such mappings can be found for instance in [3, 4].

Theorem 3.16. *Let $(X, \|\cdot\|_X)$ be a normed space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Assume that $t : X \to 2^Y$ is a multivalued mappings with $R(t) = Y$ such that $t^{-1} : Y \to X$ is single-valued nonexpansive and $s : D(t) \to Y$ a continuous α-condensing mapping satisfying that there exists $R > 0$ and $y_0 \in Y$ such that*

$$\|x - t^{-1}y_0\|_X \geq R \Rightarrow \mu s(x) + (1 - \mu)y_0 \notin t(x) \qquad \forall\mu \in (0, 1). \tag{3.22}$$

Then there exists $x_0 \in X$ such that $s(x_0) \in t(x_0)$.

Proof. Since $t^{-1} : Y \to X$ is nonexpansive, then t^{-1} is continuous. Consider the mapping $h := s \circ t^{-1} : Y \to Y$, it is continuous because it is composition of continuous functions. Reasoning as in the proof of Theorem 3.3 we can prove that h is condensing.

Finally, we show that h satisfies a Leray-Schauder condition with y_0. If this was false, we could find $y_n \in Y$ and $\lambda_n > 1$, for each $n \in \mathbb{N}$, satisfying

$$\|y_n - y_0\| \geq n \qquad \text{and} \qquad h(y_n) - y_0 = \lambda_n(y_n - y_0).$$

Taking $x_n = t^{-1}(y_n)$, the previous assumption along with the definition of h yields

$$s(x_n) - y_0 = \lambda_n(y_n - y_0),$$

so

$$\frac{1}{\lambda_n}s(x_n) + \left(1 - \frac{1}{\lambda_n}\right)y_0 = y_n \in t(x_n).$$

Using (3.22) we conclude that $\|x_n - t^{-1}y_0\|_X < R$, which means that the sequence (x_n) is bounded. Then the sequence $(s(x_n))$ is bounded, because s is α-condensing. Therefore y_n is bounded, but this is a contradiction.

By Theorem 2.4 there exists $y \in Y$ such that $h(y) = y$. Choosing $x_0 = t^{-1}(y)$ we have that $s(x_0) = y \in t(x_0)$. □

Example 3.17. Let $(H, \langle\cdot,\cdot\rangle)$ be a finite dimensional real Hilbert space and let D be a nonempty closed convex subset of H.

Given a mapping $f : D \to H$, the *variational inequality* defined by f and D is

$$VI(f,D) : \begin{cases} \text{find } x_0 \in D \text{ such that} \\ \langle f(x_0), y - x_0\rangle \geq 0, \text{ for all } y \in D. \end{cases} \tag{3.23}$$

As an application of Theorem 3.16, we shall see that $VI(f,D)$ admits a solution whenever f is a continuous mapping which satisfies that there exists $R \geq 0$ such that

$$\|x\|_H \geq R,\ x \in D \Rightarrow \langle f(x), x\rangle > 0, \tag{3.24}$$

and $0 \in D$.

Let us introduce the indicator function of D: $I_D : H \to [0,+\infty]$ defined by

$$I_D(x) := \begin{cases} 0, \text{ if } x \in D, \\ +\infty, \text{ if } x \in H \setminus D. \end{cases}$$

It is well known (for instance see [5]) that I_D is a proper convex lower semi continuous function and its subdifferential $\partial I_D : H \to 2^H$ given by

$$\partial I_D(x) = \{\xi \in H :\ \langle \xi, y - x\rangle \leq I_D(y) - I_D(x),\ \text{for all } y \in H\},$$

is clearly an m-accretive operator on H where its effective domain is $D(\partial I_D) = D$. Moreover, it is easy to see that

$$\partial I_D(x) = \{y \in H : \langle y, z - x\rangle \leq 0 \ \text{ for every } z \in D\}.$$

Thus, a solution of $VI(f,D)$ will be a point $x_0 \in D$ such that $-f(x_0) \in \partial I_D(x_0)$.

In order to study the existence of solution for this problem , we call $t := I + \partial I_D : D \to 2^H$ and $s := -f + I : D \to H$.

Since ∂I_D is m-accretive then t^{-1} is a single-valued nonexpansive mapping and $R(t) = H$. The mapping s is α-condensing because s is compact, since it is continuous and H is finite dimensional.

Note that $0 \in D$ implies that $0 \in \partial I_D(0)$. So $t^{-1}(0) = 0$. We will choose $y_0 = 0$ and R given by (3.24) in Theorem 3.16.

If condition (3.22) does not hold the exist $x \in D$ with $\|x\|_H \geq R$ and $\mu \in (0,1)$ such that $\mu(-f(x) + x) \in x + \partial I_D(x)$, i.e. $-\mu f(x) + (\mu - 1)x \in \partial I_D(x)$. Which means that for all $v \in D$

$$\langle -\mu f(x) + (\mu - 1)x, v - x\rangle \leq 0.$$

The convexity of D along with that 0 and x are elements of D implies $(1-\mu)x \in D$. Then

$$\langle -\mu f(x) + (\mu - 1)x, (1-\mu)x - x\rangle \leq 0,$$

$$\mu^2\langle f(x), x\rangle - (\mu - 1)\mu\|x\|_H^2 \leq 0,$$

$$\langle f(x), x\rangle \leq \frac{\mu - 1}{\mu}\|x\|_H^2 \leq 0,$$

which contradicts assumption (3.24).

The above facts allow us to say that t and s are under the hypotheses of Theorem 3.16, so we conclude that there exists $x_0 \in D$ such that $s(x_0) \in x_0 + \partial I_D(x_0)$ and this means that $-f(x_0) \in \partial I_D(x_0)$.

Next result shows that if s, t are under the conditions of Theorem 3.16 and we add that s is a bounded mapping then Leray-Schauder's condition is directly fulfilled.

Corollary 3.18. *Let $(X, \|\cdot\|_X)$ be a normed space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Assume that $t : X \to Y$ is an expansive surjection and $s : X \to Y$ a continuous, bounded and α-condensing mapping. Then there exists $x_0 \in X$ such that $s(x_0) = t(x_0)$.*

Proof. In order to obtain the result, we only have to see that $t, s : X \to Y$, defined as in Theorem 3.16, satisfy condition (3.22) with $y_0 := 0_Y$.

To do this, we argue as follows:

Since $t : X \to Y$ is expansive, we have that

$$\|t(x) - t(0_X)\|_Y \geq \|x\|_X \text{ for all } x \in X,$$

thus, since t is onto, we infer

$$\|t(x)\|_Y \geq \|x - t^{-1}(0_Y)\|_X - \|t^{-1}(0_Y)\|_X - \|t(0_X)\|_Y.$$

On the other hand, since s is a bounded mapping, there exists $M > 0$ such that $\|s(x)\|_Y \leq M$ for every $x \in X$.

If now we take $R := M + \|t^{-1}(0_Y)\|_X + \|t(0_X)\|_Y$. We may conclude that if $\|x - t^{-1}(0_Y)\|_X \geq R$, then

$$\|t(x)\|_Y \geq \|x - t^{-1}(0_Y)\|_X - \|t^{-1}(0_Y)\|_X - \|t(0_X)\|_Y \geq M,$$

which means that $s(x) - 0_Y \neq \lambda(t(x) - 0_Y)$ whenever $\lambda > 1$. □

Corollary 3.19. *Let $(X, \|\cdot\|_X)$ be a normed space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Assume that $t : X \to Y$ is an expansive surjection and $s : X \to Y$ a continuous, α-condensing mapping satisfying that there exists $R > 0$ and $y_0 \in Y$ such that*

$$\|x - t^{-1}y_0\|_X \geq R \Rightarrow \|s(x) - y_0\|_Y \leq \|x - t^{-1}(y_0)\|_X. \tag{3.25}$$

Then there exists $x_0 \in X$ such that $s(x_0) = t(x_0)$.

Proof. We are going to prove that $t, s : X \to Y$, defined as in Theorem 3.16, satisfies condition (3.22). Indeed, since t^{-1} is a nonexpansive mapping, we have

$$\|x - t^{-1}(y_0)\|_X = \|t^{-1}(t(x)) - t^{-1}(y_0)\|_X \leq \|t(x) - y_0\|_Y.$$

The above inequality along with (3.25) implies that if $\|x - t^{-1}y_0\| \geq R$, then

$$\|s(x) - y_0\|_Y \leq \|x - t^{-1}(y_0)\|_X \leq \|t(x) - y_0\|_Y.$$

Consequently

$$s(x) - y_0 \neq \lambda(t(x) - y_0) \text{ whenever } \lambda > 1.$$

□

3.2.1. *A nonlinear Dirichlet problem.* Let Ω be a measurable subset on $\mathbb{R}^n$ which for simplicity will be assumed to be bounded.

The Sobolev space $W^{m,p}(\Omega)$ is the Banach space of all functions in $L^p(\Omega)$ all of whose weak derivatives up to order m also belong to $L^p(\Omega)$. The norm in this space is given by

$$\|u\|_{m,p} = \|u\|_p + \sum_{1\le|\alpha|\le m} \|D^\alpha u\|_p,$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $D^\alpha u = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}} u$.

$W_0^{m,p}(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

In this section, We shall study the existence of solutions in $L^1(\Omega)$ for the equation

$$\left\{ \begin{array}{ll} \Delta\rho(u(x)) = f(x, u(x)) & x \in \Omega \\ \rho(u(x)) = 0 & x \in \partial\Omega \end{array} \right. . \tag{3.26}$$

Let us now specify the conditions assuring the existence of a solution for Equation (3.26):

(1) Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.
(2) $\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}),\ \rho(0) = 0$.
(3) There exists $C > 0$ and $\gamma \in \mathbb{R}^+$ with $\gamma > 1$ such that

$$\rho'(r) \ge C|r|^{\gamma-1} \text{ for each } r \in \mathbb{R} \setminus \{0\}.$$

(4) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $|f(s, x)| \le a(s) + b|x|$, where $a \in L^1(\Omega)$ and $b \ge 0$. This condition guarantees that the superposition operator associated to f,

$$N_f(u)(s) = f(s, u(s)),$$

acts form $L^1(\Omega)$ into $L^1(\Omega)$ and is continuous. We refer to [2] for background material on superposition operators.

H. Brezis and W. Strauss in [6] showed that under the above conditions (1) and (2), the operator

$$\left\{ \begin{array}{l} D(P) = \{u \in L^1(\Omega) :\ \rho(u) \in W_0^{1,1}(\Omega),\ \Delta\rho(u) \in L^1(\Omega)\} \\ P(u) = \Delta\rho(u),\ u \in D(P) \end{array} \right. \tag{3.27}$$

is m-dissipative, which means that $-P$ is m-accretive.

Definition 3.20. *We say that $v \in L^1(\Omega)$ is a solution of Problem (3.26) whenever $v \in L^1(\Omega),\ \rho(v) \in W_0^{1,1}(\Omega),\ \Delta\rho(v) \in L^1(\Omega)$ and $\Delta\rho(v(x)) = f(x, v(x))$ a.e. $x \in \Omega$. That is, whenever $v \in D(P)$ is a solution of the coincidence problem $P(v) = N_f(v)$, where $D(P)$ and P are defined in (3.27).*

Theorem 3.21. *If Conditions (1–4) are fulfilled, then Problem (3.26) has a solution.*

Proof. Let us consider the following operator

$$D(Q) = \{u \in W_0^{1,1}(\Omega),\ \Delta u \in L^1(\Omega)\}$$
$$Q(u) = \Delta u,\ u \in D(Q),$$

where Δu is understood in the sense of distributions. From [6, Theorem 8] we know that there exists $D > 0$ such that

$$D\|u\|_{1,1} \leq \|Qu\|_1 \tag{3.28}$$

for each $u \in D(Q)$. Moreover, [11, Remark 4.12] shows that there exists

$$Q^{-1} : L^1(\Omega) \to D(Q)$$

and it is continuous.

In [11, Theorem 4.11] was proved that the superposition operator

$$S : L^1(\Omega) \to L^1(\Omega) \text{ such that } S(u)(x) := \rho^{-1}(u(x)).$$

is well defined and it is continuous.

As a consequence of the above facts we may introduce the operator:

$$T : L^1(\Omega) \to L^1(\Omega) \text{ defined by } T(u) = S(Q^{-1}(u)).$$

Now, we will see that $T(u) \in D(P)$ for every $u \in L^1(\Omega)$.

Indeed, we know that $T(u) \in L^1(\Omega)$. Moreover $\rho(T(u)) = Q^{-1}(u) \in D(Q)$.

Consequently

$$\rho(T(u)) \in W_0^{1,1}(\Omega) \text{ and } \Delta\rho(T(u)) \in L^1(\Omega)\},$$

i.e. $T(u) \in D(P)$.

The above argument says that T is the inverse operator, in $L^1(\Omega)$, of P.

Next, let us see that T is a compact mapping.

Indeed, let A be a bounded subset of $L^1(\Omega)$. From (3.28), we have that $(Q^{-1}(A))$ is a bounded subset of $W^{1,1}(\Omega)$ and since the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, we have that $Q^{-1}(A)$ relatively compact in $L^1(\Omega)$, and thus, since $S : L^1(\Omega) \to L^1(\Omega)$ is a continuous mapping, $T(A)$ must be a relatively compact subset of $L^1(\Omega)$.

On the other hand, Condition (4) implies that the superposition operator $N_f : L^1(\Omega) \to L^1(\Omega)$ is continuous and maps bounded subsets into bounded subsets and since $P(D(P)) = L^1(\Omega)$, we also have that $N_f(D(P)) \subseteq P(D(P))$.

In order to find a solution of Problem (3.26) it will be enough to apply Theorem 3.14. To this end, we shall show that there exists $R > 0$ such that if $u \in D(P)$ and there exists $\mu \in (0, 1)$ with

$$P(u) = \mu N_f(u), \tag{3.29}$$

then $\|u\|_1 \leq R$.

Suppose that u satisfies (3.29). Since $|f(s, x)| \leq a(s) + b|x|$, we have that

$$\mu\|N_f(u)\|_1 \leq \|a\|_1 + b\|u\|_1 \tag{3.30}$$

On the other hand, we know that

$$\rho'(r) \geq C|r|^{\gamma-1} \text{ for each } r \in \mathbb{R} \setminus \{0\},$$

which implies that

$$|\rho(r)| \geq \frac{C}{\gamma}|r|^{\gamma}.$$

The above inequality means that $\frac{C}{\gamma}|u(s)|^{\gamma} \leq |\rho(u(s))|$ a.e. $s \in \Omega$. Thus, we infer that $u \in L^{\gamma}(\Omega)$. Since Ω is a bounded set, Hölder's inequality yields

$$\int_{\Omega} |u(s)|ds \leq \Big(\int_{\Omega} |u(s)|^{\gamma} ds\Big)^{\frac{1}{\gamma}} (\lambda(\Omega))^{\frac{\gamma-1}{\gamma}},$$

where $\lambda(\Omega)$ is Lebesgue measure of Ω. Hence, if we call $K := \frac{C}{\gamma}\lambda(\Omega)^{1-\gamma}$, we obtain that

$$K\|u\|_1^{\gamma} \leq \frac{C}{\gamma}\int_{\Omega} |u(s)|^{\gamma} ds \leq \|\rho(u)\|_1.$$

Moreover, by (3.28), $D\|\rho(u)\|_1 \leq \|\Delta\rho(u)\|_1$. Therefore,

$$DK\|u\|_1^{\gamma} \leq \|\Delta\rho(u)\|_1. \tag{3.31}$$

(3.30) along with (3.31) implies

$$DK\|u\|_1^{\gamma} \leq \|a\|_1 + b\|u\|_1.$$

Consequently,

$$DK \leq \frac{\|a\|_1 + b\|u\|_1}{\|u\|_1^{\gamma}}.$$

However, since $\gamma > 1$ it is clear that

$$\lim_{\|u\|_1 \to \infty} \frac{\|a\|_1 + b\|u\|_1}{\|u\|_1^{\gamma}} = 0.$$

Hence there exists $R > 0$ such that if $\|u\|_1 > R$ then

$$\frac{\|a\|_1 + b\|u\|_1}{\|u\|_1^{\gamma}} < \frac{DK}{2}.$$

The above inequality allows us to conclude that $\|u\|_1 \leq R$ whenever u is a solution of (3.29). $\square$

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