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METRIC FIXED POINT RESULTS FOR GENERALIZED CONTRACTIVE MAPPINGS AND APPLICATIONS

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Abstract. In this paper, we establish some new results on the existence of fixed points for general mutivalued contractive mappings with respect to w-distance. In support of our results, example and applications are also presented. Our results either improve or generalize results of metric fixed point theory, including the corresponding fixed point results of Nadler [23], Mizoguchi and Takahashi [22], Suzuki and Takahashi [27], Latif [17], Du and Hung [7] and many others.

1. Introduction and preliminaries

It is well-known that the classical Banach contraction principle, (which asserts that each single-valued contraction self mapping on a complete metric space has a unique fixed point) plays a central role in nonlinear functional analysis and is widely considered as the source of metric fixed point theory. Due to its significance and application in different scientific fields, a lot of generalizations of this result have been done in different directions by several authors, see; [1–28] and references therein.

Let us recall some basic notions, definitions, facts and well-known results needed in this paper.

Let (X, d) be a metric space. We use 2^X to denote the collection of all nonempty subsets of X, $Cl(X)$ for the collection of all nonempty closed subsets of X and $CB(X)$ for the collection of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, \quad A,B \in CB(X),$$

where $d(x,B) = \inf_{y\in B} d(x,y)$. The function $H : CB(X) \times CB(X) \to [0,\infty)$ is called the Pompeiu- Hausdorff metric induced by the metric d.

An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to 2^X$ if $x \in T(x)$. The set of fixed points of T is denoted by $F(T)$. A sequence $\{x_n\}$ in X is called an *orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A function $f : X \to \mathbb{R}$ is called *lower semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $f(x) \leq \liminf_{n\to\infty} f(x_n)$. A function $\varphi : [0,\infty) \to [0,1)$ is said to be a MT-function if $\limsup_{r\to t^+} k(r) < 1$ for all $t \in [0,\infty)$. In [5], Du characterizes MT-function as follows: A function φ is MT-function iff for any strictly decreasing

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sequence $\{x_n\}$ in $[0, \infty)$, we have $0 \le \sup_n \varphi(x_n) < 1$. For more characterizations of MT-function, see; [5, 6]. A single valued self mapping on a metric space (X, d) is said to be Caristi mapping if there exists a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that for each $x \in X$,

$$d(x, f(x)) \le \varphi(x) - \varphi(f(x)).$$

One of the most important extension of the Banach contraction principle was given by Caristi [2], known as Caristi's fixed point theorem.

Theorem 1.1 ([2]). *Let (X, d) be a complete metric space. Then each Caristi mapping $f : X \to X$ has a fixed point.*

Many authors have studied and generalized Caristi's fixed point theorem to various directions. On the other hand investigations on the existence of fixed points for multivalued mappings in the frame work of metric spaces were initiated by Nadler [23]. Using the concept of Hausdorff metric and an iterative method, he established the following multivalued version of the Banach contraction principle.

Theorem 1.2 ([23]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a map such that for a fixed constant $k \in (0, 1)$ and for each $x, y \in X$,*

$$H(T(x), T(y)) \le k\ d(x, y).$$

Then $Fix(T) \ne \emptyset$.

It is worth to mention that without using an iterative method a beautiful proof of Theorem 1.2 was given by Jachymski [12] by using Theorem 1.1. In fact he proved that Caristi's fixed point result (Theorem 1.1) yields Nadler's fixed point result (Theorem 1.2).

Nadler's fixed point result has been generalized in many different directions. Mizoguchi and Takahashi [22] have obtained the following real generalization of Theorem 1.2 (which is also a partial answer to the problem proposed by Reich [25]).

Theorem 1.3 ([22]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$. Assume that there exists a MT-function φ such that for each $x, y \in X$,*

$$H(T(x), T(y)) \le \varphi(d(x, y))d(x, y).$$

Then $Fix(T) \ne \emptyset$.

In [22], a multivalued version of the Theorem 1.1 is given as under.

Theorem 1.4 ([22]). *Let (X, d) be a complete metric space and let $T : X \to 2^X$ be a mapping such that for each $x \in X$, there exists $y \in Tx$ satisfying*

$$\varphi(y) + d(x, y) \le \varphi(x),$$

where φ is a lower semicontinuous function of X into $\mathbb{R}^+$. Then $Fix(T) \ne \emptyset$.

Using Theorem 1.4, Daffer and Kaneko [4] have obtained the following fixed point result which also contains the corresponding results given in [3, 13, 26].

Theorem 1.5 ([4]). *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued mapping such that for a fixed constant $k \in [0, 1)$ and for each $x, y \in X$*

$$H(Tx, Ty) \leq k\, M(x, y),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$. If $x \to d(x, Tx)$ is lower semicontinuous on X, then $Fix(T) \neq \emptyset$.

In [14], Kada et al. introduced a notion of w-distance on metric spaces and then improved several classical results in metric fixed point theory including the Caristi fixed point theorem. While, Suzuki and Takahashi [27] have introduced notions of single-valued and multivalued weakly contractive mappings with respect to w-distance and proved several fixed point results for such mappings.

Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a *w-distance* on X if it satisfies the following for any $x,\ y,\ z \in X$:

- (i) $p(x, z) \leq p(x, y) + p(y, z)$;
- (ii) a function $p(x, .) : X \to [0, \infty)$ is lower semicontinuous;
- (iii) for any $\epsilon > 0$, then exists $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $d\,(x, y) \leq \epsilon$.

Note that, in general for $x, y \in X$, $p(x, y) \neq p(y, x)$ and not either of the implications $p(x, y) = 0 \Leftrightarrow x = y$ necessarily hold. Clearly, the metric d is a w-distance on X. Let $(Y, \|.\|)$ be a normed space. Then the functions $p_1, p_2 : Y \times Y \to [0, \infty)$ defined by $p_1(x, y) = \|y\|$ and $p_2(x, y) = \|x\| + \|y\|$ for all $x, y \in Y$ are w-distances [14]. Many other examples and properties of the w-distance can be found in [14, 27, 28]. We denote by $W(X)$ the set of all w-distance on X. For $p \in W(X)$ and $A \subseteq X$, define $p(x, A) = \inf\{p(x, y) : y \in A\}$ and $p(x, A) = 0$ provided $x \in A$.

The following two lemmas concerning w-distance are fundamental.

Lemma 1.6 ([14]). *Let (X, d) be a metric space and let $p \in W(X)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. Then, the following statements hold for any $x, y, z \in X$:*

- (a) *if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$; in particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;*
- (b) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*
- (c) *if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
- (d) *if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.7 ([21]). *Let A be a closed subset of a metric space (X, d) and $p \in W(X)$. Suppose that there exists $u \in X$ such that $p(u, u) = 0$. Then $p(u, A) = 0 \Leftrightarrow u \in A$, where $p(x, A) = \inf\{p(x, y) : y \in A\}$.*

Using the concept of w-distance, Kada et al. [14] improved Theorem 1.1 as follows:

Theorem 1.8 ([14]). *Let (X,d) be a complete metric space and $p \in W(X)$. Then each Caristi mapping f on X with respect to p has a fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$.*

Among others, a multivalued version of this result obtained by Latif [16], which contains Theorem 1.4. In particular, an improved version of Theorem 1.4 is as follows; also see [14, 20, 21, 28].

Theorem 1.9 ([14, 16]). *Let (X,d) be a complete metric space and $p \in W(X)$. Let $T : X \to 2^X$ be a mapping such that for each $x \in X$, there exists $y \in Tx$ satisfying*

$$\varphi(y) + p(x,y) \leq \varphi(x),$$

where φ is a lower semicontinuous function of X into $\mathbb{R}^+$. Then T has a fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$.

Let (X,d) be a metric space. A multivalued mapping $\mathrm{T} : X \to 2^X$ is said to be p-contractive [27] if there exist $p \in W(X)$ and $k \in [0,1)$ such that for any $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ with

$$p(u,v) \leq kp(x,y).$$

In particular, a singlevalued mapping $f : X \to X$ is said to be p-contractive if there exist $p \in W(X)$ and $k \in [0,1)$ such that $p(f(x), f(y)) \leq kp(x,y)$ for all $x, y \in X$.

In [27], Suzuki and Takahashi proved that each p-contractive closed valued map on a complete metric space X has fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$. Which is an improved version of the corresponding results in [10, 23]. Further, they deduced that each single-valued p-contractive self map on X has a unique fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$. Which is an improved version of the Banach contraction principle. Using w-distance, Latif [17] improved Theorem 1.3, which also improves and generalizes [15, Theorem 1].

In the sequel, we consider X as a metric space with metric d. Let $T : X \to 2^X$. We say
(a) T is generalized p-contractive if there exist $p \in W(X)$ and a fixed constant $0 \leq k < 1$ such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ with

$$p(u,v) \leq kM_p(x,y), \tag{1.1}$$

where $M_p(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty), \alpha[p(x,Ty) + p(y,Tx)]\}$, and $0 < \alpha < \frac{1}{2k+\delta}$ for some $\delta \geq 1$.

In particular, if $p = d$, then we say T is generalized d-contractive.
Note that a class of generalized p-contractive mappings contains the class of generalized d-contractive mappings and a class of p-contractive mappings.
(b) T is weak p-contractive if there exist $p \in W(X)$ and a MT-function φ such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ with

$$\min\{p(u,v), p(y,Ty)\} \leq \varphi(p(x,y))p(x,y). \tag{1.2}$$

In particular, if $p = d$, then we say T is weak d-contractive.

The purpose of this paper is to establish some new results on the existence of fixed points for general mutivalued contractive mappings with respect to w-distance. In support of our results, some examples and applications are also presented. Our results either generalize or improve many fixed point results including the corresponding result of Nadler [23], Mizoguchi and Takahashi [22], Suzuki and Takahashi [27], Latif [17], Du and Hung [7] and many others.

2. Main Results

Applying Theorem 1.9, first we prove a result on the existence of fixed points for generalized p-contractive mappings.

Theorem 2.1. *Let (X, d) be a complete metric space. Then, for each multivalued generalized p-contractive mapping $T : X \to Cl(X)$ there exists a point $x_0 \in X$ such that $x_0 \in T(x_0)$ and $p(x_0, x_0) = 0$, provided a real valued function $x \to p(x, Tx)$ is lower semicontinuous on X.*

Proof. Let $x \in X$ and $\epsilon > 0$. Here we choose ϵ so small that $\frac{1}{1+\epsilon} > k \geq 0$. Let $y \in Tx$ so that $p(x, y) \leq (1 + \epsilon)p(x, Tx)$. Since $y \in Tx$ we get $p(y, Tx) = 0$ and there exists $y' \in Ty$, such that

$$p(y, Ty) \leq p(y, y') \leq k \ \max\{p(x, y), p(x, Tx), p(y, Ty), \alpha p(x, Ty)\},$$

and thus finally we get

$$p(y, Ty) \leq k \ \max\{p(x, y), p(x, Tx), \alpha p(x, Ty)\}.$$

We need to examine the following three cases:
Case 1. Suppose that

$$p(x, y) = \max\{p(x, y), p(x, Tx), \alpha p(x, Ty)\}.$$

Then, we obtain

$$\begin{aligned} p(x, Tx) - p(y, Ty) &\geq p(x, Tx) - kp(x, y) \\ &\geq \frac{1}{1+\epsilon}p(x, y) - kp(x, y) \\ &= (\frac{1}{1+\epsilon} - k)p(x, y) \end{aligned}$$

Case 2. Suppose that

$$p(x, Tx) = \max\{p(x, y), p(x, Tx), \alpha p(x, Ty)\}.$$

Then, we have

$$\begin{aligned} p(x, Tx) - p(y, Ty) &\geq p(x, Tx) - kp(x, Tx) \\ &= (1 - k)p(x, Tx) \\ &\geq \frac{1-k}{1+\epsilon}p(x, y) \\ &\geq (\frac{1}{1+\epsilon} - k)p(x, y) \end{aligned}$$

Case 3. Suppose that

$$\alpha\, p(x, Ty) = \max\{p(x, y), p(x, Tx), \alpha\, p(x, Ty)\}$$

Then, note that

$$p(y,Ty) \le k\alpha\, p(x,Ty) \le k\alpha[p(x,y)+p(y,Ty)],$$

that is,

$$(1-k\alpha)p(y,Ty) \quad \le \quad k\alpha\, p(x,y).$$

Now, since $\alpha < \frac{1}{2k+\delta}$, we have

$$\big(1-\frac{k}{2k+\delta}\big)p(y,Ty) \le (1-k\alpha)p(y,Ty) \le k\alpha\, p(x,y) \le \frac{k}{2k+\delta}p(x,y),$$

and thus,

$$p(y,Ty) \le \frac{k}{k+\delta}\, p(x,y).$$

Since $k+\delta > 1$, we get $k > \frac{k}{k+\delta}$ and thus, we have

$$p(y,Ty) \le \frac{k}{k+\delta}\, p(x,y) \le k\, p(x,y).$$

Then,

$$\begin{aligned} p(x,Tx)-p(y,Ty) &\ge p(x,Tx)-kp(x,y) \\ &\ge \Big(\frac{1}{1+\epsilon}-k\Big)p(x,y). \end{aligned}$$

Hence for all the three cases, we have

$$p(x,Tx)-p(y,Ty) \ge \Big(\frac{1}{1+\epsilon}-k\Big)p(x,y).$$

Now, we can define a lower semicontinuous function $\varphi(x) = (\frac{1}{1+\epsilon}-k)^{-1}p(x,Tx)$ for all $x \in X$. Note that

$$p(x,y) \le \varphi(x)-\varphi(y).$$

Using Theorem 1.9, there exists $z \in X$ such that $z \in Tz$ and $p(z,z)=0$. □

Remark 2.2. Theorem 2.1 is a generalization of Theorem 1.5. Consequently it contains the corresponding fixed point results given in [3, 13, 26].

The following result is a direct consequence of Theorem 2.1.

Theorem 2.3. *Let (X,d) be a complete metric space. Then each multivalued generalized d-contractive mapping $T : X \to Cl(X)$ has a fixed point, provided a real valued function $x \to d(x,Tx)$ is lower semicontinuous on X.*

The following lemma for weak p-contractive mapping is crucial for the next fixed point result.

Lemma 2.4. *Let (X,d) be a metric space. Then for each weak p-contractive mapping $T : X \to Cl(X)$ then there exists an orbit $\{x_n\}$ of T at $x_0 \in X$ such that the sequence of nonnegative numbers $\{p(x_n,x_{n+1})\}$ is decreasing to zero and the sequence $\{x_n\}$ is Cauchy.*

Proof. Let φ be a MT-function on X. Define a function $\mu : [0,\infty) \to [0,1)$ by

$$\mu(t) = \frac{1+\varphi(t)}{2} \quad \text{for all} \quad t \in [0,\infty). \tag{2.1}$$

Clearly, $0 \le \varphi(t) < \mu(t) < 1$ for all $t \in [0,\infty)$. Let $x_0 \in X$ be an arbitrary but fixed element and let $x_1 \in Tx_0$. If $x_0 \ne x_1$, then there exists $y_1 \in Tx_1$ such that

$$\min\{p(x_1,y_1), p(x_1,Tx_1)\} \le \varphi(p(x_0,x_1))p(x_0,x_1) < \mu(p(x_0,x_1))p(x_0,x_1).$$

Since $y_1 \in Tx_1$, thus we have

$$\min\{p(x_1,y_1), p(x_1,Tx_1)\} = p(x_1,Tx_1) < \mu(p(x_0,x_1))p(x_0,x_1).$$

It follows that for some $x_2 \in Tx_1$, we get

$$p(x_1,x_2) < \mu(p(x_0,x_1))p(x_0,x_1).$$

If $x_1 \ne x_2$, then similarly there exists $y_2 \in Tx_2$ such that

$$p(x_2,Tx_2) = min\{p(x_2,y_2), p(x_2,Tx_2)\} < \mu(p(x_1,x_2))p(x_1,x_2).$$

and for some $x_3 \in Tx_2$ we get

$$p(x_2,x_3) \le \mu(p(x_1,x_2))p(x_1,x_2).$$

By induction, we obtain an orbit $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of T at x_0 satisfying

$$p(x_n,x_{n+1}) \le \mu(p(x_{n-1},x_n))p(x_{n-1},x_n) < p(x_{n-1},x_n).$$

It follows that the sequence $\{p(x_{n-1},x_n)\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0,\infty)$. Since φ is a MT-function, we have

$$0 \le \sup_n \varphi(p(x_{n-1},x_n)) < 1.$$

Using the definition of μ, we have

$$0 \le \sup_n \varphi(p(x_{n-1},x_n)) = \frac{1}{2}[1 + \sup_n \varphi(p(x_{n-1},x_n))] < 1.$$

Let $\gamma = \sup_n \varphi(p(x_{n-1},x_n))$. Note that, $\gamma \in [0,1)$, and then $\{\gamma^n\}$ converges to 0 and

$$p(x_n,x_{n+1}) \le \mu(p(x_{n-1},x_n))p(x_{n-1},x_n) < \gamma p(x_{n-1},x_n). \tag{2.2}$$

Thus, we get

$$p(x_n,x_{n+1}) < \gamma^n p(x_0,x_1), \tag{2.3}$$

which implies that the decreasing sequence $\{p(x_n,x_{n+1})\}$ converges to 0. Now, for any $n,m \in \mathbb{N}$ with $m > n$, we have

$$p(x_n,x_m) \le \sum_{j=n}^{m-1} p(x_j,x_{j+1}) < \frac{\gamma^n}{1-\gamma} p(x_0,x_1).$$

Thus, by Lemma 1.6 $\{x_n\}$ is a Cauchy sequence in X. □

Applying Lemma 2.4, we prove a fixed point result for weak p-contractive mappings.

Theorem 2.5. *Let (X, d) be a complete metric space. Then for each weak p-contractive mapping $T : X \to Cl(X)$, there exists $z \in X$ such that either $z \in Tz$ or $p(z, Tz) = 0$.*

Proof. If follows from Lemma 2.4 that there exists an orbit $\{x_n\}$ in X such that the sequence $\{p(x_n, x_{n+1})\}$ decreasing to 0 and $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since $p(x_n, .)$ is lower semicontinuous, then it follows from of the proof Lemma 2.4 that

$$p(x_n, z) \leq \liminf_{m\to\infty} p(x_n, x_m) < \frac{\gamma^n}{1-\gamma} p(x_0, x_1).$$

Note that the sequence $\{p(x_n, z)\}$ converges to 0. Now, since $x_n \in Tx_{n-1}$, for each $n \geq 1$, then it follows from the definition of T that there exists $w_n \in Tz$ such that

$$\min\{p(x_n, w_n), p(z, Tz)\} < \varphi(p(x_{n-1}, z))p(x_{n-1}, z). \tag{2.4}$$

Here, we consider the following two possible cases:
Case 1 Assume that for all $n \in \mathbb{N}$,

$$\min\{p(x_n, w_n), p(z, Tz)\} = p(x_n, w_n).$$

Since $0 \leq \varphi(t) < \mu(t) < 1$ for all $t \in [0, \infty)$, then for all $n \in \mathbb{N}$, we have

$$p(x_n, w_n) \leq \varphi(p(x_{n-1}, z))p(x_{n-1}, z) < \mu(p(x_{n-1}, z))p(x_{n-1}, z)$$

and thus $p(x_n, w_n) < p(x_{n-1}, z)$ for all $n \in \mathbb{N}$. Thus it follows from Lemma 1.6 that $\{w_n\}$ converges to z. Since Tz is closed, we get $z \in Tz$.
Case 2 Assume that for all $n \in \mathbb{N}$

$$\min\{p(x_n, w_n), p(z, Tz)\} = p(z, Tz), \tag{2.5}$$

Then, for all $n \in \mathbb{N}$ we get

$$p(z, Tz) \leq \varphi(p(x_{n-1}, z))p(x_{n-1}, z) < p(x_{n-1}, z).$$

Thus, we deduces that $p(z, Tz) = 0$. □

Remark 2.6. If $p(z, Tz) = 0$ and further assume that $p(z, z) = 0$. Then by Lemma 1.7 it follows that $z \in Tz$.

Example 2.7. Let ℓ^∞ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be the canonical basis of ℓ^∞. Let $\{\tau_n\}$ be a sequence of positive real numbers satisfying $\tau_1 = \tau_2$ and $\tau_{n+1} < \tau_n$ for all $n \geq 2$. Then, note that $\{\tau_n\}$ is a convergent sequence. Put $\upsilon_n = \tau_n e_n$ for all $n \in \mathbb{N}$ and let $X = \{\upsilon_n\}_{n\in\mathbb{N}} \subseteq \ell^\infty$. Note that (X, d_∞) is a complete metric space with $d_\infty(\upsilon_n, \upsilon_m) = \tau_n$ if $m > n$. Define a w-distance p on X by

$$p(\upsilon_n, \upsilon_m) = \begin{cases} \tau_n & \text{if } m > n; \\ 0 & \text{if } n = m. \end{cases}$$

Define $T : X \to CB(X)$ by

$$T\upsilon_n = \begin{cases} \{\upsilon_1, \upsilon_2\} & \text{if } n \in \{1, 2\}; \\ X\backslash\{\upsilon_1, \upsilon_2, ..., \upsilon_n, \upsilon_{n+1}\} & \text{if } n \geq 3. \end{cases}$$

and define $\varphi : [0, \infty) \to [0, 1)$ by

$$\varphi(t) = \begin{cases} \frac{\tau_{n+1}}{\tau_n} & \text{if } t = \tau_n \text{ for some } n \in \mathbb{N}, n \geq 2; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, φ is an MT-function because $\limsup_{s \to t^+} \varphi(t) = 0 < 1$ for all $t \in [0, \infty)$. Also, note that $F(T) = \{\upsilon_1, \upsilon_2\}$. Now, we show that T is weak p-contractive, that is; for each $x, y \in X$ with $x \neq y, u \in Tx$ and there is $w \in Ty$ such that

$$\min\{p(u, w), p(y, Ty)\} \leq \varphi(p(x, y))p(x, y).$$

For this, we consider the following four cases
Case 1. For any $u_1 \in T\upsilon_1$, there is $w_2 \in T\upsilon_2$ satisfying

$$\min\{p(u_1, w_2), p(\upsilon_2, T\upsilon_2)\} = 0 \leq \tau_2 = \varphi(p(\upsilon_1, \upsilon_2))p(\upsilon_1, \upsilon_2).$$

Case 2. For any $u_1 \in T\upsilon_1$, there is $w_m \in T\upsilon_m,\ m \geq 3$ such that

$$\min\{p(u_1, w_m), p(\upsilon_m, T\upsilon_m)\} = \tau_m \leq \tau_2 = \varphi(p(\upsilon_1, \upsilon_m))p(\upsilon_1, \upsilon_m).$$

Case 3. For any $u_2 \in T\upsilon_2$ there is $w_m \in T\upsilon_m,\ m \geq 3$ such that

$$\min\{p(u_2, w_m), p(\upsilon_m, T\upsilon_m)\} = \tau_m \leq \tau_3 = \varphi(p(\upsilon_2, \upsilon_m))p(\upsilon_2, \upsilon_m).$$

Case 4. For any $u_n \in T\upsilon_n$ there is $w_m \in T\upsilon_m,\ n \geq 3$ and $m > n$ such that

$$\min\{p(u_n, w_m), p(\upsilon_m, T\upsilon_m)\} = \tau_m \leq \tau_{n+1} = \varphi(p(\upsilon_n, \upsilon_m))p(\upsilon_n, \upsilon_m).$$

Thus, T is weak p-contractive mapping. Therefore, Theorem 2.5 is applicable.

The following results are direct consequences of Theorem 2.5.

Theorem 2.8. *Let (X, d) be a complete metric space. Then each weak d-contractive mapping $T : X \to Cl(X)$ has a fixed point.*

Theorem 2.9. [17] *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a multivalued mapping. Suppose that there exist $p \in W(X)$ and MT-function φ such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ satisfying*

$$p(u, v) \leq \varphi(p(x, y))p(x, y).$$

Then $F(T) \neq \emptyset$.

Theorem 2.10. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a multivalued mapping. Suppose that there exist $p \in W(X)$ and MT-function φ such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ satisfying*

$$p(y, Ty) \leq \varphi(p(x, y))p(x, y).$$

Then there exists $z \in X$ such that either $z \in Tz$ or $p(z, Tz) = 0$.

Remark 2.11. (1) Theorem 2.5 is a generalization of [7, Theorem 2.1], which contains Theorem 1.3 [22, Theorem 5] as a special case. Moreover, Theorem 2.5 generalizes [17, Theorem 2.2] which contains [27, Theorem 1] as a special case.
(2) Theorem 2.8 generalizes [10, Theorem 2.3].

3. Applications

Applying Theorem 2.5, we obtain the following new fixed point results.

Theorem 3.1. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a multivalued mapping. Suppose that there exist $p \in W(X)$ and MT-function φ such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ satisfying*

$$p(u, v) + p(y, Ty) \leq 2\varphi(p(x, y))p(x, y), \tag{3.1}$$

then there exists $z \in X$ such that either $z \in Tz$ or $p(z, Tz) = 0$.

Theorem 3.2. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a multivalued mapping. Suppose that there exist $p \in W(X)$ and MT-function φ such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ satisfying*

$$\sqrt{p(u, v)p(y, Ty)} \leq \varphi(p(x, y))p(x, y), \tag{3.2}$$

then there exists $z \in X$ such that either $z \in Tz$ or $p(z, Tz) = 0$.

Remark 3.3. Clearly condition (3.1) implies condition (3.2). Thus Theorem 3.1 can be proved by applying Theorem 3.2.

In fact, we can establish a wide generalization of Theorem 3.1 as follows.

Theorem 3.4. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a multivalued mapping. Suppose that there exist $p \in W(X)$ and MT-function φ such that for each $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ satisfying*

$$\frac{\alpha p(u, v) + \beta p(y, Ty)}{\alpha + \beta} \leq \varphi(p(x, y))p(x, y), \tag{3.3}$$

where α and β are nonnegative real numbers with $\alpha + \beta > 0$. Then there exists $z \in X$ such that either $z \in Tz$ or $p(z, Tz) = 0$.

Proof. Note that

$$\alpha(\min\{p(u, v), p(y, Ty)\}) \leq \alpha p(u, v)[\text{or } \alpha p(y, Ty)],$$

and

$$\beta(\min\{p(u, v), p(y, Ty)\}) \leq \beta p(u, v)[\text{or } \beta p(y, Ty)].$$

Thus

$$(\alpha + \beta) \min\{p(u, v), p(y, Ty)\} \leq \alpha p(u, v) + \beta p(y, Ty).$$

Since $\alpha + \beta > 0$, we get

$$\min\{p(u, v), p(y, Ty)\} \leq \frac{\alpha p(u, v) + \beta p(y, Ty)}{\alpha + \beta} \leq \varphi(p(x, y))p(x, y).$$

That is;

$$\min\{p(u, v), p(y, Ty)\} \leq \varphi(p(x, y))p(x, y).$$

Hence, the conclusion follows by Theorem 2.5. □

Remark 3.5.

(a) If we take $\alpha = 1$ and $\beta = 0$ in Theorem 3.3, then we obtain Theorem 2.9.
(b) If we take $\alpha = 0$ and $\beta = 1$ in Theorem 3.3, then we obtain Theorem 2.10.
(c) For $\alpha = 1$ and $\beta = 1$, Theorem 3.3, reduces to Theorem 3.1.

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