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FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS WITH APPLICATIONS TO COUPLED FIXED POINT THEORY

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Abstract. In this paper, we present some fixed point theorems for generalized contractions of Hardy-Rogers and of Ćirić type in complete metric spaces endowed with a partial order relation. Some applications to coupled fixed point theory involving generalized contraction conditions are obtained. Moreover, using the concept of f-closed set, we will prove a general fixed point theorem of Ran-Reurins type. As a consequence, several general coupled fixed point theorems in the recent literature are given. Some open questions are pointed out.

1. Introduction

It is well known that Banach's contraction principle for single-valued contractions was extended in several directions. One of this research directions involved the so-called generalized contractions mappings. For example, Kannan, Reich, Rus, Ćirić, Hardy-Rogers and others (see [20] for a exhaustive synthesis) replaced the classical contraction condition (which involves $d(x, y)$ and $d(f(x), f(y))$) by different more general assumptions (involving not only $d(x, y)$ and $d(f(x), f(y))$ but also $d(x, f(x))$, $d(y, f(y))$, $d(x, f(y))$ and $d(y, f(x))$).

Another research direction regarding Banach's contraction principle was initiated by M. Turinici (see [24] for a very interesting and vast study) and further developed by the paper of Ran and Reurings [19]. The idea is to impose the contraction condition, not on the whole space, but only on the subset of comparable elements (with respect to a certain partial ordering given on the metric space). For extensions of the Ran-Reurings theorem see [12–14].

On the other hand, the concept of coupled fixed point and the study of coupled fixed point problems appeared in some papers of Amann (scientific report) and Opoitsev (see [10, 11]), but the topic was strongly developed by D. Guo and V. Lakshmikantham [5]), T. Gnana Bhaskar and V. Lakshmikantham [4] and V. Lakshmikantham and L. Ćirić [9].

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If (X, d) is a metric space and $F : X \times X \to X$ is an operator, then, by definition, a coupled fixed point for F is a pair $(x^*, y^*) \in X \times X$ satisfying the system

$$\begin{cases} x = F(x, y) \\ y = F(y, x). \end{cases} \tag{1.1}$$

In the last year, the theory of coupled fixed points in the setting of an ordered metric space and under some contraction type conditions on the operator F was reconsidered and many new works were published, see for example [1,8,15–17,21–24].

In this paper, we will continue this research by considering the generalized contraction conditions of Hardy-Rogers and Ćirić and the very interesting concepts of F-invariant and F-closed sets, see [8,22,23]. Moreover, some general fixed point theorems will be given and some consequences in coupled fixed point theory are deduced.

2. F-invariant and F-closed sets

We recall here two important concepts related to coupled fixed point theory.

Definition 2.1 (Samet-Vetro [22]). Let (X, d) be a metric space and $F : X \times X \to X$ be a mapping. A nonempty subset $\mathbb{M}$ of X^4 is said to be F-invariant if for all $x, y, u, v \in X$ the following assumptions are satisfied:

(i) $(x, y, u, v) \in \mathbb{M}$ if and only if $(v, u, y, x) \in \mathbb{M}$;
(ii) if $(x, y, u, v) \in \mathbb{M}$ then $(F(x, y), F(y, x), F(u, v), F(v, u)) \in \mathbb{M}$.

As an example of F-invariant set, the authors in [22] show that, if F is a mixed monotone operator on a metric space endowed with a partial ordering $\preceq$ (i.e., increasing in the first variable and decreasing in the second one), then the set

$$\mathbb{M} := \{(x, y, u, v) \in X^4 : \ x \succeq u, y \preceq v\}$$

is F-invariant.

An improvement of the above concept was given in [8], as follows.

Definition 2.2 (Kubti et all. [8]). Let (X, d) be a metric space and $F : X \times X \to X$ be a mapping. A nonempty subset $\mathbb{M}$ of X^4 is said to be F-closed if for all $x, y, u, v \in X$ the following implication holds:

$$(x, y, u, v) \in \mathbb{M} \ \Rightarrow \ (F(x, y), F(y, x), F(u, v), F(v, u)) \in \mathbb{M}.$$

Obviously, any F-invariant set is F-closed but not reversely, see Example 13 in [8].

Moreover, the following characterization theorem holds for a F-closed set $\mathbb{M}$.

Theorem 2.3 (see Lemma 14 in [8]). *Let X be a nonempty set, $\mathbb{M} \subset X^4$ and $F : X \times X \to X$ be an operator. Define*

$$(x, y) \preceq (u, v) \Leftrightarrow (x, y) = (u, v) \ \text{or} \ (u, v, x, y) \in \mathbb{M}$$

and

$$T_F(x, y) := (F(x, y), F(y, x)), \ \text{for all} \ (x, y) \in X \times X.$$

Then:

(a) $\mathbb{M}$ *is F-closed if and only if T_F is increasing with respect to $\preceq$;*

(b) *if* $\mathbb{M}$ *is* F*-invariant then* T_F *is increasing with respect to* $\preceq$.

(c) $\preceq$ *is a preorder (i.e., reflexive and transitive) on* $X \times X$ *if and only if* $\mathbb{M}$ *has the transitive property, i.e., for all* $x, y, u, v, z, w \in X$ *te following implication holds:*

$$(x, y, u, v) \in \mathbb{M} \text{ and } (u, v, z, w) \in \mathbb{M} \text{ imply } (x, y, z, w) \in \mathbb{M}.$$

In this setting, the following very nice result is given in [8].

Theorem 2.4 (see Theorem 16 in [8])**.** *Let* X *be a nonempty set,* $\mathbb{M} \subset X^4$ *and* $F : X \times X \to X$ *be a continuous operator. Suppose:*

(i) $\mathbb{M}$ *is* F*-closed;*

(ii) *there exists* $(x_0, y_0) \in X \times X$ *such that* $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in \mathbb{M}$*;*

(iii) *there exists* $k \in (0, 1)$ *such that, for all* $(x, y, u, v) \in \mathbb{M}$ *we have*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k\,(d(x, u) + d(y, v)).$$

Then F *has at least one coupled fixed point.*

As consequence, some coupled fixed point theorems given in [22], as well as in some other works can be obtained.

If $f : X \to X$ is a given mapping we denote by $Fix(f) := \{x \in X : x = f(x)\}$ the fixed point set of f. The following fixed point theorem, given by Ran-Reurings in [19], was the staring point of a long list of papers dealing with fixed point theorems in ordered metric spaces.

Theorem 2.5 ([19])**.** *Let* X *be a nonempty set endowed with a partial order "*$\preceq$*" and* $d : X \times X \to \mathbb{R}_+$ *be a complete metric on* X*. Let* $f : X \to X$ *be a continuous operator with respect to* d *and increasing with respect to "*$\preceq$*". Suppose that there exist a constant* $k \in (0, 1)$ *and an element* $x_0 \in X$ *such that:*

(i) $d(f(x), f(y)) \leq kd(x, y)$*, for all* $x, y \in X$ *with* $x \preceq y$.

(ii) $x_0 \preceq f(x_0)$.

Then $Fix(f) \neq \emptyset$ *and the sequence of successive approximations* $(f^n(x))_{n\in\mathbb{N}}$ *starting from any point* $x \in X$ *which is comparable to* x_0 *converges to a fixed point of* f*. If additionally, every pair* $x, y \in X$ *has a lower bound and an upper bound, then the fixed point* x^* *is unique and, for any point* $x \in X$*, the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *converges to* x^**.*

Several improvements of the condition that every pair $x, y \in X$ has a lower bound and an upper bound and of the continuity of the operator f were given later by Nieto and Rodriguez-Lopez, se [12] and [13]. See also the work of Turinici [24] for a comprehensive study.

On the other hand, in a series of papers it is shown that an existence result for the coupled fixed point problem (1.1) can be obtained as a consequence of a fixed point theorem of Ran-Reurings type for the operator $T_F(x, y) = (F(x, y), F(y, x))$.

In this paper, we will illustrate the statement that any fixed point theorem in a mixed order and metric structure implies a coupled fixed point theorem, by considering the case of generalized contractions of Hardy-Rogers and of Ćirić. A general Ran-Reurings type theorem is also given.

3. Main results

The following fixed point theorem was proved by Hardy and Rogers in [6].

Theorem 3.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping for which there exist $a, b, c \in \mathbb{R}_+$ with $a+2b+2c \in (0,1)$ such that, for all $x, y \in X$, we have*

$$d(f(x), f(y)) \leq ad(x,y) + b\left(d(x, f(x)) + d(y, f(y))\right) + c\left(d(x, f(y)) + d(y, f(x))\right).$$

Then:

(i) $Fix(f) = \{x^*\}$;
(ii) *for every $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*;*
(iii) $d(f^n(x), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x))$, *for all $x \in X$* $\left(\text{where } \alpha := \frac{a+b+c}{1-b-c}\right)$.

For our next results we need the following concepts.

Definition 3.2. Let X be a nonempty set endowed with a partial order "$\preceq$" and $d : X \times X \to \mathbb{R}_+$ be a metric on X. The triple $(X, d, \preceq)$ is said to be:

(a) i-regular if for any increasing sequence $(x_n)_{n\in\mathbb{N}}$ which is convergent to x^* as $n \to \infty$, we have that $x_n \preceq x^*$, for all $n \in \mathbb{N}$;
(b) d-regular if for any decreasing sequence $(x_n)_{n\in\mathbb{N}}$ which is convergent to x^* as $n \to \infty$, we have that $x_n \succeq x^*$, for all $n \in \mathbb{N}$;

We have now the following Ran-Reurings type theorem for the case of Hardy-Rogers operators.

Theorem 3.3. *Let X be a nonempty set, $\preceq$ be a partial order on X and d be a complete metric on X. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$, for which there exist $a, b, c \in \mathbb{R}_+$ with $a + 2b + 2c \in (0, 1)$ such that the following assertions hold:*

(i) *there is $x_0 \in X$ with $x_0 \preceq f(x_0)$;*
(ii) *f has closed graph with respect to d or the space $(X, d, \preceq)$ is i-regular;*
(iii) $d(f(x), f(y)) \leq ad(x,y) + b\left(d(x, f(x)) + d(y, f(y))\right)$
$+c\left(d(x, f(y)) + d(y, f(x))\right),$

for all $x, y \in X$ with $x \preceq y$.

Then:

(a) *$Fix(f) \neq \emptyset$ and, for every $x \in X$ with $x \preceq x_0$ or $x_0 \preceq x$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to $x^* \in Fix(f)$. Moreover, the following apriori estimation holds*

$$d(f^n(x), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x)), \ \text{for } n \in \mathbb{N} \ \left(\text{where } \alpha := \frac{a+b+c}{1-b-c}\right).$$

(b) *if additionally, for every pair $x, y \in X$ of elements which are not comparable with respect to $\preceq$, there exists $z \in X$ such that z is comparable with x and y, then $Fix(f) = \{x^*\}$ and, for every $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*.*

Proof. (a) Let $x_0 \in X$ with $x_0 \preceq f(x_0)$. Denote $x_n := f^n(x_0),\ n \in \mathbb{N}$. Then by the monotonicity of f, we get that $(x_n)_{n\in\mathbb{N}}$ is increasing. Moreover we have:

$$\begin{aligned} d(x_1,x_2) = d(f(x_0),f(x_1)) &\le ad(x_0,x_1) + b\,(d(x_0,f(x_0)) + d(x_1,f(x_1))) \\ &\quad + c\,(d(x_0,f(x_1)) + d(x_1,f(x_0))) \\ &\le (a+b+c)d(x_0,x_1) + (b+c)d(x_1,x_2). \end{aligned}$$

Thus

$$d(x_1,x_2) \le \frac{a+b+c}{1-b-c} d(x_0,f(x_0)).$$

Denote $\alpha := \frac{a+b+c}{1-b-c}$. It is easy to see that $\alpha \in (0,1)$. By mathematical induction, we immediately get that

$$d(x_n,x_{n+1}) \le \alpha^n d(x_0,f(x_0)),\ \text{for each } n \in \mathbb{N}. \tag{3.1}$$

By (3.1), we obtain that

$$d(x_n,x_{n+p}) \le \frac{\alpha^n}{1-\alpha} d(x_0,f(x_0)),\ \text{for each } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*. \tag{3.2}$$

A standard procedure implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence it is convergent in (X,d) to an element $x^* \in X$. We show now that $x^* \in Fix(f)$.

If f has closed graph, then the conclusion is obvious.

If the space $(X,d,\preceq)$ is i-regular, then we have

$$\begin{aligned} d(x^*,f(x^*)) &\le d(x^*,f(x_n)) + d(f(x_n),f(x^*)) \\ &\le d(x^*,x_{n+1}) + ad(x_n,x^*) + b\,(d(x_n,x_{n+1}) + d(x^*,f(x^*))) \\ &\quad + c\,(d(x_n,f(x^*)) + d(x^*,x_{n+1})). \end{aligned}$$

Thus

$$\begin{aligned} d(x^*,f(x^*)) &\le \frac{a+c}{1-b-c} d(x_n,x^*) + \frac{b}{1-b-c} d(x_n,x_{n+1}) \\ &\quad + \frac{1+c}{1-b-c} d(x_{n+1},x^*). \end{aligned} \tag{3.3}$$

By (3.3), letting $n \to \infty$, we get that $x^* \in Fix(f)$.

Let us consider now any $x \in X$ with $x \preceq x_0$ or $x_0 \preceq x$. Denote $y_n := f^n(x)$, for $n \in \mathbb{N}$. By our assumption, we have that $x_n \preceq y_n$ or $y_n \preceq x_n$, for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} d(y_n,x_n) &= d(f(y_{n-1}),f(x_{n-1})) \\ &\le ad(y_{n-1},x_{n-1}) + b\,(d(y_{n-1},y_n) + d(x_{n-1},x_n)) \\ &\quad + c\,(d(y_{n-1},x_n) + d(x_{n-1},y_n)) \\ &\le ad(y_{n-1},x_{n-1}) + b\,(d(y_{n-1},x_{n-1}) + d(x_{n-1},x_n) + d(x_n,y_n) + d(x_{n-1},x_n)) \\ &\quad + c\,(d(y_{n-1},x_{n-1}) + d(x_{n-1},x_n) + d(x_{n-1},x_n) + d(x_n,y_n)). \end{aligned}$$

Thus, we obtain

$$d(y_n,x_n) \le \frac{a+b+c}{1-b-c} d(y_{n-1},x_{n-1}) + \frac{2(b+c)}{1-b-c} d(x_{n-1},x_n), \forall n \in \mathbb{N}. \tag{3.4}$$

If we denote $a_n := d(y_n, x_n)$, $b_n := \frac{2(b+c)}{1-b-c} d(x_{n-1}, x_n)$, then (3.4) can be written as

$$a_n \leq \alpha a_{n-1} + b_n, \forall n \in \mathbb{N}.$$

Since $\alpha < 1$ and $\lim\limits_{n\to\infty} b_n = 0$, by Lemma 2.3 in [18], we obtain that $a_n \to 0$ as $n \to \infty$. Thus (y_n) converges to x^*. The apriori estimation follows by (3.2) by letting $p \to \infty$.

(b) We prove first the uniqueness of the fixed point. Suppose there exist $x^*, y^* \in Fix(f)$. If x^*, y^* are comparable (for example $x^* \preceq y^*$), then we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq (a + 2c)d(x^*, y^*),$$

which implies $d(x^*, y^*) = 0$ and thus $x^* = y^*$. If x^*, y^* are not comparable, then there exists $z \in X$ which is comparable to x^* and y^*. As a consequence, $f^n(x^*)$ and $f^n(z)$ and respectively $f^n(z)$ and $f^n(y^*)$ are comparable, for each $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x^*, f(z)) &= d(f(x^*), f(z)) \\ &\leq ad(x^*, z) + bd(z, f(z)) + c\left(d(x^*, f(z)) + d(z, f(x^*))\right) \\ &\leq ad(x^*, z) + bd(z, x^*) + bd(x^*, f(z)) + c\left(d(x^*, f(z)) + d(z, x^*)\right). \end{aligned}$$

Hence

$$d(x^*, f(z)) \leq \frac{a+b+c}{1-b-c} d(x^*, z).$$

By mathematical induction we can prove that

$$d(x^*, f^n(z)) \leq \left(\frac{a+b+c}{1-b-c}\right)^n d(x^*, z), \text{ for all } n \in \mathbb{N}^*. \tag{3.5}$$

Hence, the sequence $(f^n(z))$ converges to x^* as $n \to \infty$.

In a similar way, using that $f^n(z)$ and $f^n(y^*)$ are comparable for each $n \in \mathbb{N}$, we get that $(f^n(z))$ converges to y^* as $n \to \infty$. By the uniqueness of the limit we obtain that $x^* = y^*$.

Let $x \in X$ be arbitrarily chosen. If x and x_0 are comparable, then, by (a), we know that $(f^n(x))$ converges to x^* as $n \to \infty$. If x and x_0 are not comparable, then there exists $z \in X$ which is comparable with x and respectively with x_0. Since x_0 and z are comparable (with respect to $\preceq$), we obtain that $(f^n(z))$ converges to the unique fixed point x^* as $n \to \infty$. On the other hand

$$d(f^n(x), x^*) \leq d(f^n(x), f^n(z)) + d(f^n(z), x^*).$$

If $y_n := f^n(x)$ and $z_n := f^n(z)$, the we have

$$\begin{aligned} d(y_n, z_n) &= d(f(y_{n-1}), f(z_{n-1})) \\ &\leq ad(y_{n-1}, z_{n-1}) + b\left(d(y_{n-1}, y_n) + d(z_{n-1}, z_n)\right) \\ &\quad + c\left(d(y_{n-1}, z_n) + d(z_{n-1}, y_n)\right) \\ &\leq ad(y_{n-1}, z_{n-1}) + (b+c)\left(d(y_{n-1}, z_{n-1}) + 2d(z_{n-1}, z_n) + d(y_n, z_n)\right). \end{aligned}$$

Hence

$$d(y_n, z_n) \leq \frac{a+b+c}{1-b-c} d(y_{n-1}, z_{n-1}) + \frac{2(b+c)}{1-b-c} d(z_{n-1}, z_n).$$

As before, by Lemma 2.3 in [18], we obtain that $d(y_n, z_n) \to 0$ as $n \to \infty$, which proves that $(f^n(x))$ converges to x^* as $n \to \infty$. The proof is complete. □

Remark 3.4. A similar result holds if we replace the condition $x_0 \preceq f(x_0)$ with $f(x_0) \preceq x_0$ and the i-regularity of the space with its d-regularity.

From the above fixed point theorem, we get the following coupled fixed point theorem for Hardy-Rogers type operators $F : X \times X \to X$.

Theorem 3.5. *Let X be a nonempty set, $\preceq$ be a partial order on X and d be a complete metric on X. Let $F : X \times X \to X$ be an operator such that:*

(a) *F has closed graph or the triple $(X, d, \preceq)$ is i-regular;*
(b) *there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$;*
(c) *there exists $a, b, c \in \mathbb{R}_+$ with $a+2b+2c \in (0, 1)$ such that, for all $(x, y), (u, v) \in X \times X$ with $(x \preceq u, y \succeq v)$ (or reversely), we have*

$$\begin{aligned} d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) &\le a\,[d(x,u)+d(y,v)] \\ &+ b\,[d(x,F(x,y))+d(y,F(y,x))+d(u,F(u,v))+d(v,F(v,u))] \\ &+ c\,[d(x,F(u,v))+d(y,F(v,u))+d(u,F(x,y))+d(v,F(y,x))]\,. \end{aligned}$$

Then, F has a coupled fixed point $(x^*, y^*) \in X \times X$ and the sequences $(F^n(x_0, y_0))_{n\in\mathbb{N}}$ and $(F^n(y_0, x_0))_{n\in\mathbb{N}}$ converge to x^* and to y^*, respectively. Moreover, if for all $(x, y), (x_1, y_1) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(x \preceq u, y \succeq v)$ and $(u \preceq x_1, v \succeq y_1)$ (or reversely), then the coupled fixed point is unique and, for all $(x, y) \in X \times X$, the sequences $(F^n(x, y))_{n\in\mathbb{N}}$ and $(F^n(y, x))_{n\in\mathbb{N}}$ converge to x^* and to y^*, respectively.

Proof. It is easy to see that the hypotheses of this theorem imply that all the hypotheses of Theorem 3.3 applied for $f := T_F : Z \to Z$ (where $Z := X \times X$ is endowed with the l^1 type metric $d^*((x, y), (u, v)) := d(x, u) + d(y, v)$, for all $(x, y), (u, v) \in X \times X$) and $T_F(x, y) := (F(x, y), F(y, x))$ take place. □

The following result was proved in [22]. Notice that Theorem 3.5 follows by Theorem 3.6 by taking $\mathbb{M} := \{(a, b, c, d) \in X^4 : a \succeq c, b \preceq d\}$.

Theorem 3.6. *Let (X, d) be a complete metric space, $F : X \times X \to X$ be a continuous mapping and $\mathbb{M}$ be a nonempty subset of X^4. Suppose that the following assertions are satisfied:*

(a) *$\mathbb{M}$ is F-invariant;*
(b) *there exists $(x_0, y_0) \in X \times X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in \mathbb{M}$;*
(c) *there exist $\alpha, \beta, \delta, \gamma, \theta \in \mathbb{R}_+$ with $\alpha + \beta + \delta + \gamma + \theta < 1$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \le \alpha\,[d(x,u)+d(y,v)] & \\ &+ \beta\,[d(x,F(x,y))+d(y,F(y,x))] \\ &+ \beta\,[d(x,F(x,y))+d(y,F(y,x))] \\ &+ \delta\,[d(u,F(u,v))+d(v,F(v,u))] \\ &+ \gamma\,[d(x,F(u,v))+d(y,F(v,u))] \\ &+ \theta\,[d(u,F(x,y))+d(v,F(y,x))]\,, \forall (x,y,u,v) \in \mathbb{M}. \end{aligned}$$

Then, F has a coupled fixed point $(x^*, y^*) \in X \times X$. Moreover, if for all $(x, y), (x_1, y_1) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(x, y, u, v) \in \mathbb{M}$ and $(x_1, y_1, u, v) \in \mathbb{M}$, then the coupled fixed point is unique.

In the second part of our work, we will show that the above two theorems follow by applying a Ran-Reurings type fixed point theorem.

Another generalization of Theorem 3.1 is the following fixed point theorem for generalized contractions given by Lj. Ćirić in [3].

Theorem 3.7. *Let (X, d) be a metric space and $f : X \to X$ be a mapping for which there exists $q \in (0, 1)$ such that, for all $x, y \in X$, we have*

$$d(f(x), f(y)) \leq q \max\Big\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}\left(d(x, f(y)) + d(y, f(x))\right)\Big\}.$$

If X is f-orbitally complete, then:

(i) $Fix(f) = \{x^*\}$;
(ii) *for every $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*;*
(iii) $d(f^n(x), x^*) \leq \frac{q^n}{1-q} d(x, f(x))$, *for all $x \in X$.*

We have now the following Ran-Reurings type theorem for the case of Ćirić type operators.

Theorem 3.8. *Let X be a nonempty set, $\preceq$ be a partial order on X and d be a complete metric on X. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$, for which there exists $q \in (0, 1)$ such that:*

(i) *there is $x_0 \in X$ with $x_0 \preceq f(x_0)$;*
(ii) *f has closed graph with respect to d or the space $(X, d, \preceq)$ is i-regular;*
(iii) $d(f(x), f(y)) \leq q \max\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}\left(d(x, f(y)) + d(y, f(x))\right)\}$, *for all $x, y \in X$ with $x \preceq y$.*

Then:

(a) *$Fix(f) \neq \emptyset$ and the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to $x^* \in Fix(f)$ as $n \to \infty$ and the following estimation holds*

$$d(f^n(x_0), x^*) \leq \frac{q^n}{1-q} d(x_0, f(x_0)), \text{ for } n \in \mathbb{N};$$

(b) *if additionally, $q \in (0, \frac{1}{2})$, then, for every $x \in X$ with $x \preceq x_0$ or $x_0 \preceq x$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* and the following estimation holds*

$$d(f^n(x), x^*) \leq \frac{q^n}{1-q} d(x, f(x)), \text{ for } n \in \mathbb{N}.$$

Moreover, if we also suppose that for every pair $(x, y) \in X \times X$ of elements which are not comparable with respect to $\preceq$, there exists $z \in X$ such that z is comparable with x and with y, then $Fix(f) = \{x^*\}$ and, for every $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*.

Proof. (a) Let $x_0 \in X$ with $x_0 \preceq f(x_0)$. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$. Then, by the monotonicity of f, we get that $(x_n)_{n\in\mathbb{N}}$ is increasing. Moreover we have:

$$\begin{aligned} &d(x_1, x_2) = d(f(x_0), f(x_1)) \\ &\le q \max\left\{d(x_0, x_1), d(x_0, f(x_0)), d(x_1, f(x_1)), \frac{1}{2}\left(d(x_0, f(x_1)) + d(x_1, f(x_0))\right)\right\} \\ &= q \max\left\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2} d(x_0, x_2)\right\} = q \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned}$$

Since $q < 1$, we get that $d(x_1, x_2) \le q d(x_0, f(x_0))$. By mathematical induction, we immediately get that

$$d(x_n, x_{n+1}) \le q^n d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}. \tag{3.6}$$

By (3.6), we obtain that

$$d(x_n, x_{n+p}) \le \frac{q^n}{1-q} d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*. \tag{3.7}$$

A standard procedure implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence it is convergent in (X, d) to an element $x^* \in X$. We will prove that $x^* \in Fix(f)$.

If f has closed graph, then the conclusion is obvious.

If the space $(X, d, \preceq)$ is i-regular, then we have

$$\begin{aligned} &d(x^*, f(x^*)) \le d(x^*, f(x_n)) + d(f(x_n), f(x^*)) \\ &\le d(x^*, x_{n+1}) \\ &\quad + q \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)), \frac{1}{2}\left(d(x_n, f(x^*)) + d(x^*, x_{n+1})\right)\right\} \\ &= d(x^*, x_{n+1}) + q \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*))\}. \end{aligned}$$

Thus

$$d(x^*, f(x^*)) \le \frac{1}{1-q}\left[d(x^*, x_{n+1}) + q\left(d(x_n, x^*) + d(x_n, x_{n+1})\right)\right] \to 0, n \to \infty.$$

Hence we obtain $x^* = f(x^*)$. The apriori estimation follows by (3.7) by letting $p \to \infty$.

(b) Let $x \in X$ be such that x and x_0 are comparable. Denote $y_n := f^n(x)$, for $n \in \mathbb{N}$. By our assumption, we have that $x_n \preceq y_n$ or $y_n \preceq x_n$, for each $n \in \mathbb{N}$. Then we have

$$\begin{aligned} &d(y_n, x_n) = d(f(y_{n-1}), f(x_{n-1})) \\ &\le q \max\left\{d(y_{n-1}, x_{n-1}), d(y_{n-1}, y_n), d(x_{n-1}, x_n), \frac{1}{2}\left(d(y_{n-1}, x_n) + d(x_{n-1}, y_n)\right)\right\} \\ &\le q\left(d(y_{n-1}, x_{n-1}) + d(x_{n-1}, x_n) + d(x_n, y_n)\right). \end{aligned}$$

Hence

$$d(y_n, x_n) \le \frac{q}{1-q}\left(d(y_{n-1}, x_{n-1}) + d(x_{n-1}, x_n)\right), \forall n \in \mathbb{N}.$$

If we denote $a_n := d(y_n, x_n)$, $b_n := \frac{q}{1-q} d(x_{n-1}, x_n)$, then the above relation can be written as

$$a_n \le \frac{q}{1-q} a_{n-1} + b_n, \forall n \in \mathbb{N}.$$

Since $\frac{q}{1-q} < 1$ and $\lim_{n\to\infty} b_n = 0$, by Lemma 2.3 in [18], we obtain that $a_n \to 0$ as $n \to \infty$. Thus (y_n) converges to x^*.

Under additional conditions, we can also prove the uniqueness of the fixed point. Suppose there exist $x^*, y^* \in Fix(f)$. If x^*, y^* are comparable (for example $x^* \preceq y^*$), then we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq q d(x^*, y^*),$$

which implies $d(x^*, y^*) = 0$ and thus $x^* = y^*$. If x^*, y^* are not comparable, then there exists $z \in X$ which is comparable to x^* and y^*. As a consequence, $f^n(x^*)$ and $f^n(z)$ and, respectively $f^n(z)$ and $f^n(y^*)$ are comparable, for each $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x^*, f(z)) &= d(f(x^*), f(z)) \\ &\leq q \max\left\{ d(x^*, z), d(z, f(z)), \frac{1}{2}\left(d(x^*, f(z)) + d(z, x^*)\right) \right\} \\ &\leq q\left(d(x^*, z) + d(x^*, f(z))\right). \end{aligned}$$

Hence

$$d(x^*, f(z)) \leq \frac{q}{1-q} d(x^*, z).$$

By mathematical induction we can prove that

$$d(x^*, f^n(z)) \leq \left(\frac{q}{1-q}\right)^n d(x^*, z), \text{ for all } n \in \mathbb{N}^*.$$

Hence, the sequence $(f^n(z))$ converges to x^* as $n \to \infty$.
Using the fact that $f^n(z)$ and $f^n(y^*)$ are comparable for each $n \in \mathbb{N}$, we get (by a similar approach) that $(f^n(z))$ converges to y^* as $n \to \infty$. Thus $x^* = y^*$.
Let $x \in X$ be such that x and x_0 are not comparable. Then there exists $z \in X$ which is comparable with x and respectively with x_0. Since x_0 and z are comparable (with respect to $\preceq$), we obtain that $(f^n(z))$ converges to the unique fixed point x^* as $n \to \infty$. On the other hand,

$$d(f^n(x), x^*) \leq d(f^n(x), f^n(z)) + d(f^n(z), x^*).$$

If $y_n := f^n(x)$ and $z_n := f^n(z)$, the we have

$$\begin{aligned} & d(y_n, z_n) = d(f(y_{n-1}), f(z_{n-1})) \\ & \leq q \max\left\{ d(y_{n-1}, z_{n-1}), d(y_{n-1}, y_n), d(z_{n-1}, z_n), \frac{1}{2}\left(d(y_{n-1}, z_n) + d(z_{n-1}, y_n)\right) \right\} \\ & \leq q \max\left\{ d(y_{n-1}, z_{n-1}) + d(z_{n-1}, z_n) + d(z_n, y_n), \frac{1}{2}\left(d(y_{n-1}, z_n) + d(z_{n-1}, y_n)\right) \right\} \\ & = q\left(d(y_{n-1}, z_{n-1}) + d(z_{n-1}, z_n) + d(z_n, y_n)\right). \end{aligned}$$

Hence

$$d(y_n, z_n) \leq \frac{q}{1-q} d(y_{n-1}, z_{n-1}) + \frac{q}{1-q} d(z_{n-1}, z_n), \forall n \in \mathbb{N}^*.$$

As before, by Lemma 2.3 in [18], we obtain that $d(y_n, z_n) \to 0$ as $n \to \infty$, which proves that, for each $x \in X$, the sequence $(f^n(x))$ converges to x^* as $n \to \infty$. The proof is complete. □

Remark 3.9. A similar result holds if we replace the condition $x_0 \preceq f(x_0)$ with $f(x_0) \preceq x_0$ and the i-regularity of the space with its d-regularity.

From the above fixed point theorem, we get the following coupled fixed point theorem for Ćirić type operators $F : X \times X \to X$.

Theorem 3.10. *Let X be a nonempty set, $\preceq$ be a partial order on X and d be a complete metric on X. Let $F : X \times X \to X$ be an operator such that:*

(a) *F has closed graph or the triple $(X, d, \preceq)$ is i-regular;*
(b) *there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$;*
(c) *there exists $q \in (0, 1)$ such that, for all $(x, y), (u, v) \in X \times X$ satisfying $(x \preceq u, y \succeq v)$, we have*

$$d(F(x,y),F(u,v)) \leq q \max\left\{d(x,u), d(x,F(x,y)), d(u,F(u,v)), \frac{1}{2}\left[d(x,F(u,v)) + d(u,F(x,y))\right]\right\}$$

Then, F has at least one coupled fixed point $(x^*, y^*) \in X \times X$ and the sequences $(F^n(x_0, y_0))$ and $(F^n(y_0, x_0))$ converge to x^* and to y^*, respectively. If additionally, we suppose that for all $(x, y), (x_1, y_1) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(x \preceq u, y \succeq v)$ and $(u \preceq x_1, v \succeq y_1)$ (or reversely), then the coupled fixed point is unique and, for all $(x, y) \in X \times X$, the sequences $(F^n(x, y))_{n\in\mathbb{N}}$ and $(F^n(y, x))_{n\in\mathbb{N}}$ converge to x^* and to y^*, respectively.

Proof. The conclusion follows by applying Theorem 3.8 for $f := T_F : Z \to Z$ (where $Z := X \times X$ is endowed with the l^1 type metric $d^*((x, y), (u, v)) := d(x, u) + d(y, v)$, for all $(x, y), (u, v) \in X \times X$) and $T_F(x, y) := (F(x, y), F(y, x))$. □

By the above results, it is quite clear that every fixed point theorem of Ran-Reurings type generates a coupled fixed point theorem under a similar contraction type condition. In fact a more general method can be proved for these two type of problems. We will show this approach in what follows.

Definition 3.11. Let X be a nonempty set, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Then, $\mathbb{P}$ is said f-closed if the following implication holds:

$$(z, w) \in \mathbb{P} \text{ implies } (f(z), f(w)) \in \mathbb{P}.$$

Some examples of f-closed sets are presented in Remark 3.18 and Remark 3.19. Here we just notice the following characterization of f-closed sets.

Lemma 3.12. *Let X be a nonempty set, $\mathbb{P} \subset X \times X$ and $f : X \to X$ be a given operator. We define*

$$x \preceq y \iff x = y \text{ or } (x, y) \in \mathbb{P}$$

Then:

(a) *$\mathbb{P}$ has the transitive property if and only if $\preceq$ is a preorder on X.*
(b) *$\mathbb{P}$ is f-closed if and only if f is increasing with respect to $\preceq$.*

Proof. (a) Suppose $\mathbb{P}$ has the transitive property, i.e., $(x, y), (y, z) \in \mathbb{P}$ implies $(x, z) \in \mathbb{P}$. Then the binary relation $\preceq$ is transitive too, i.e., $x \preceq y$ and $y \preceq z$ implies $x \preceq z$. Moreover, by the definition, $\preceq$ is reflexive. The reverse implication is also true.
(b) Suppose that f is increasing with respect to $\preceq$. We show that $\mathbb{P}$ is f-closed. Indeed, take $x, y \in X$ with $(x, y) \in \mathbb{P}$. Then $x \preceq y$. By the monotonicity of f we

get that $f(x) \preceq f(y)$. Thus $(f(x), f(y)) \in \mathbb{P}$. Hence, $\mathbb{P}$ is f-closed. The reverse implication can be obtained in a similar manner. □

Definition 3.13. Let (X, d) be a metric space, $f : X \to X$ be an operator and $\mathbb{P} \subset X^2$. The triple $(X, d, \mathbb{P})$ is said to be:

(a) i-$\mathbb{P}$-regular if for any sequence $(x_n)_{n\in\mathbb{N}}$, with $(x_n, x_{n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$, which is convergent to x^* as $n \to \infty$, we have that $(x_n, x^*) \in \mathbb{P}$, for all $n \in \mathbb{N}$;

(b) d-$\mathbb{P}$-regular if for any sequence $(x_n)_{n\in\mathbb{N}}$, with $(x_{n+1}, x_n) \in \mathbb{P}$ for all $n \in \mathbb{N}$, which is convergent to x^* as $n \to \infty$, we have that $(x^*, x_n) \in \mathbb{P}$, for all $n \in \mathbb{N}$;

The following general Ran-Reurings type fixed point theorem (in fact, a fixed point result for almost contractions in the sense of Berinde, see [2]) is given in terms of f-closed sets.

Theorem 3.14. *Let (X, d) be a complete metric space, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Suppose:*

(i) *$\mathbb{P}$ is f-closed;*

(ii) *f has closed graph or the triple $(X, d, \mathbb{P})$ is i-$\mathbb{P}$-regular;*

(iii) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in \mathbb{P}$;*

(iv) *there exists $k \in [0, 1)$ and $l \geq 0$ such that, for all $(x, y) \in \mathbb{P}$, we have*

$$d(f(x), f(y)) \leq kd(x, y) + ld(y, f(x)).$$

Then f has at least one fixed point and the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}$ converges to an element $x^* \in Fix(f)$.

Proof. Let us denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$. By (iii) and the f-closedness property of $\mathbb{P}$ we obtain that $(x_n, x_{n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$. Then, by (iv) we obtain that

$$d(x_n, x_{n+1}) \leq k^n d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}.$$

Thus, by classical approach we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Denote by $x^* \in X$ the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. If f has closed graph, then we immediately get that $x^* \in Fix(f)$. If the triple $(X, d, \mathcal{P})$ is i-$\mathbb{P}$-regular, then, since $(x_n, x_{n+1}) \in \mathbb{P}$, we obtain that $(x_n, x^*) \in \mathbb{P}$ and so, by (iv), we get

$$d(x_{n+1}, f(x^*)) = d(f(x_n), f(x^*)) \leq kd(x_n, x^*) + ld(x^*, f(x_n)) \to 0 \text{ as } n \to \infty.$$

Hence $x^* = f(x^*)$. □

Remark 3.15. A similar result holds if we replace the condition $(x_0, f(x_0)) \in \mathbb{P}$ with the assumption $(f(x_0), x_0) \in \mathbb{P}$ and the i-$\mathbb{P}$-regularity of the space with its d-$\mathbb{P}$-regularity.

Remark 3.16. One can remark that we do not ask the symmetry of $\mathbb{P}$ nor its reflexivity.

In the particular case when $l = 0$, we obtain the following result.

Corollary 3.17. *Let (X, d) be a complete metric space, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Suppose:*

(i) *$\mathbb{P}$ is f-closed;*

(ii) *f has closed graph or the triple $(X, d, \mathbb{P})$ is i-$\mathbb{P}$-regular;*
(iii) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in \mathbb{P}$;*
(iv) *there exists $k \in [0, 1)$ such that, for all $(x, y) \in \mathbb{P}$, we have*

$$d(f(x), f(y)) \leq kd(x, y).$$

Then we have the following conclusions:

(a) *f has at least one fixed point and the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}$ converges to an element $x^* \in Fix(f)$;*
(b) *For any $x \in X$ such that (x_0, x) or (x_0, x) belongs to $\mathbb{P}$, the sequences $x_n := f^n(x_0)$ and $u_n := f^n(x)$, $n \in \mathbb{N}$ are Cauchy equivalent (i.e., $d(x_n, u_n) \to 0$ as $n \to \infty$) and hence $(f^n(x))$ converges to the same point $x^* \in Fix(f)$;*
(c) *If additionally, we suppose that for every $x \in X$ for which neither (x_0, x) nor (x_0, x) does not belong to $\mathbb{P}$ there exists $z \in X$ such that $(x_0, z), (z, x) \in \mathbb{P}$, then $Fix(f) = \{x^*\}$ and $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* as $n \to \infty$.*

Proof. (b) If (x_0, x) or (x_0, x) belongs to $\mathbb{P}$, then $(f^n(x_0), f^n(x))$ or $(f^n(x), f^n(x_0))$ are in $\mathbb{P}$, for every $n \in \mathbb{N}$. Thus, we have (for the first situation, for example), that

$$d(x_n, u_n) \leq k^n d(x_0, x) \to 0 \text{ as } n \to \infty.$$

(c) Let $x \in X$ be such that (x_0, x) and (x_0, x) do not belong to $\mathbb{P}$. Then, there exists $z \in X$ such that (x_0, z) and (z, x) are in $\mathbb{P}$. By (b) the sequences $x_n := f^n(x_0)$ and $v_n := f^n(z)$, $n \in \mathbb{N}$ are Cauchy equivalent. Thus, the sequences $v_n := f^n(z)$ and $w_n := f^n(x)$, $n \in \mathbb{N}$ are Cauchy equivalent too. Concerning the uniqueness, suppose there exists $y^* \in Fix(f)$ such that $y^* \neq x^*$. If (x_0, y^*) or (y^*, x_0) are in $\mathbb{P}$, then by

$$\begin{aligned} d(y^*, x^*) &= d(f^n(y^*), f^n(x^*)) \\ &\leq d(f^n(y^*), f^n(x_0)) + d(f^n(x_0), f^n(x^*)) \\ &\leq k^n d(y^*, x_0) + d(f^n(x_0), x^*) \to 0 \text{ as } n \to \infty, \end{aligned}$$

we obtain that $y^* = x^*$. On the other hand, if neither (x_0, y^*) nor (y^*, x_0) does not belong to $\mathbb{P}$, then there exists $z \in X$ such that (y^*, z) and (z, x_0) are in $\mathbb{P}$. By the fact that $(z, x_0) \in \mathbb{P}$, we get that $f^n(z)$ and $f^n(x_0)$ are Cauchy equivalent and, thus, $f^n(z)$ converges to $x^* \in Fix(f)$. Then we get again

$$\begin{aligned} d(y^*, x^*) &= d(f^n(y^*), f^n(x^*)) \\ &\leq d(f^n(y^*), f^n(z)) + d(f^n(z), f^n(x^*)) \\ &\leq k^n d(y^*, z) + d(f^n(z), x^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, we obtain that $y^* = x^*$. The proof is now complete. □

Remark 3.18. 1) In particular, in the above theorem, if the metric space (X, d) is endowed with a partial order $\preceq$ and we consider $\mathbb{P} := \{(x, y) \in X \times X : x \preceq y\}$ or $\mathbb{P} := \{(x, y) \in X \times X : y \preceq x\}$ then, under the additional hypothesis that f is increasing with respect to $\preceq$, we obtain Ran-Reurings Theorem, see Theorem 2.5.
2) In the above framework, if we choose $\mathbb{P} := \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$ and we suppose that f is increasing with respect to $\preceq$, then $\mathbb{P}$ is f-closed and some generalizations of the results given in [14] can be obtained.

Remark 3.19. Let (X,d) be a metric space and $f : X \to X$. Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. If we define $\mathbb{P} := \{(x,y) \in X \times X : (x,y) \in E(G)\}$ and suppose that f preserves edges (see (2.2) in [7]), then $\mathbb{P}$ is f-closed and, by Theorem 3.14, we get some extensions of the results given in [7].

Remark 3.20. The result can be extended, under additional conditions, to the case of nonlinear contractions (also called φ-contractions). Recall that $f : X \to X$ is said to be a φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (i.e., it is increasing and satisfies the condition $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$) and

$$d(f(x), f(y)) \le \varphi(d(x,y)), \text{ for all } x, y \in X.$$

More precisely, we have the following result.

Theorem 3.21. *Let (X,d) be a complete metric space, $\mathbb{P} \subset X^2$ and $f : X \to X$ be an operator. Suppose:*

(i) *$\mathbb{P}$ is f-closed;*
(ii) *f has closed graph or the triple $(X,d,\mathbb{P})$ is i-$\mathbb{P}$-regular;*
(iii) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in \mathbb{P}$;*
(iv) *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$d(f(x), f(y)) \le \varphi(d(x,y)), \text{ for all } (x,y) \in \mathbb{P};$$

(v) *for every $x, y \in X$ for which neither (x,y) nor (y,x) does not belong to $\mathbb{P}$ there exists $z \in X$ such that $(x,z), (z,y) \in \mathbb{P}$;*

Then, $Fix(f) = \{x^*\}$ and $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* as $n \to \infty$.

Proof. Let us denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$. By (iii) and the f-closedness property of $\mathbb{P}$ we obtain that $(x_n, x_{n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$. Then, by (iv) we obtain that

$$d(x_n, x_{n+1}) \le \varphi^n(d(x_0, f(x_0))), \text{ for all } n \in \mathbb{N}. \tag{3.8}$$

We can prove now that the sequence (x_n) is Cauchy.

Let $\epsilon > 0$ be arbitrary. Since $\varphi^n(\epsilon) \to 0$ as $n \to \infty$, there exists $n(\epsilon) > 0$ such that $\varphi^n(\epsilon) < \frac{\epsilon}{4}$, for each $n \ge n(\epsilon)$. Let $g := f^{n(\epsilon)}$ and $y_m := g^m(x_0)$, $m \in \mathbb{N}$. Then we have

$$d(y_m, y_{m+1}) = d(f^{n(\epsilon)m}(x_0), f^{n(\epsilon)m}(g(x_0))) \le \varphi^{n(\epsilon)m}(d(x_0, g(x_0))) \to 0, \ n \to \infty.$$

Hence, for $\epsilon > 0$ there exists $m(\epsilon) > 0$ such that $d(y_m, y_{m+1}) < \frac{\epsilon}{2}$, for each $m \ge m(\epsilon)$. Let

$$\tilde{B}(y_{m(\epsilon)}; \epsilon) := \{y \in X \mid d(y, y_{m(\epsilon)}) \le \epsilon\}.$$

We will show that $g : \tilde{B}(y_{m(\epsilon)}; \epsilon) \to \tilde{B}(y_{m(\epsilon)}; \epsilon)$. Indeed, let $u \in \tilde{B}(y_{m(\epsilon)}; \epsilon)$. Then

$$\begin{aligned} d(g(u), y_{m(\epsilon)}) &\le d(g(u), g(y_{m(\epsilon)})) + d(g(y_{m(\epsilon)}), y_{m(\epsilon)}) \\ &= d(g(u), g(y_{m(\epsilon)})) + d(y_{m(\epsilon)+1}, y_{m(\epsilon)}). \end{aligned}$$

If $(u, y_{m(\epsilon)})$ or $(u, y_{m(\epsilon)})$ are in $\mathbb{P}$, then we can write directly

$$d(g(u), g(y_{m(\epsilon)})) \le \varphi^{n(\epsilon)}(d(u, y_{m(\epsilon)})) \le \varphi^{n(\epsilon)}(\epsilon) < \frac{\epsilon}{2}.$$

If neither $(u, y_{m(\epsilon)})$ nor $(u, y_{m(\epsilon)})$ does not belong to $\mathbb{P}$, then there exists $z \in X$ such that $(u, z)$and $(z, y_{m(\epsilon)})$ are in $\mathbb{P}$. Then

$$\begin{aligned} d(g(u), g(y_{m(\epsilon)})) &\leq d(g(u), g(z)) + d(g(z), g(y_{m(\epsilon)})) \\ &\leq \varphi^{n(\epsilon)}(d(u, z)) + \varphi^{n(\epsilon)}(d(z, y_{m(\epsilon)})). \end{aligned}$$

Hence

$$d(g(u), y_{m(\epsilon)}) \leq \Big(\varphi^{n(\epsilon)}(d(u, z)) + \varphi^{n(\epsilon)}(d(z, y_{m(\epsilon)}))\Big) + d(y_{m(\epsilon)+1}, y_{m(\epsilon)}) < \epsilon.$$

As a consequence, for every $i, j \in \mathbb{N}$ with $i, j \geq m(\epsilon)$, we get

$$d(y_i, y_j) \leq d(y_i, y_{m(\epsilon)}) + d(y_j, y_{m(\epsilon)}) \leq 2\epsilon,$$

which proves that the sequence (y_m) is Cauchy and hence (x_m) is Cauchy. The rest of the proof is similar to the previous one. □

Remark 3.22. It is an open problem to obtain an existence result for the fixed points result of f without the hypothesis (v) of the above theorem. for example, a solution for this question is the case when we suppose, instead of the assumption

$$\lim_{n\to\infty} \varphi^n(t) = 0, \text{ for each } t > 0,$$

that

$$\sum_{n\geq 0} \varphi^n(t) < \infty, \text{ for each } t > 0. \tag{3.9}$$

Recall that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which is increasing and has the above property (3.9) is called a strong comparison function. Then, by (3.8), we can prove that the sequence (x_n) is Cauchy and, by (ii), we get that the limit of the sequence (x_n) is a fixed point for f.

Using the above results we can derive now some general coupled fixed point theorems. For example, Theorem 2.4 given in [8] can be immediately derived.

Theorem 3.23. *Theorem 2.4 follows by Corollary 3.17.*

Proof. Let $Z := X \times X$ endowed with the metric

$$d^*((x, y), (u, v)) := d(x, u) + d(y, v), \text{ for all } (x, y), (u, v) \in X \times X,$$

generated by d. We consider the operator $T : Z \to Z$ given by

$$T_F(x, y) := (F(x, y), F(y, x)), \text{ for all } (x, y) \in X \times X$$

and define

$$\mathbb{P} := \{(z, w) \in Z \times Z : z = (x, y), w = (u, v), (x, y, u, v) \in \mathbb{M}\}.$$

Since $\mathbb{M}$ is F-closed, we get that $\mathbb{P}$ is T_F-closed. Moreover, by the hypotheses of Theorem 2.4, T_F satisfies the following assumptions:

(i) T_F has closed graph with respect to d^*;
(ii) there exists $z_0 := (x_0, y_0) \in Z$ such that $(z_0, T_F(z_0)) \in \mathbb{P}$;
(iii) there exists $k \in [0, 1)$ such that, for all $(z, w) \in \mathbb{P}$, we have

$$d^*(T_F(z), T_F(w)) \leq kd^*(z, w).$$

Since the fixed points of T_F are coupled fixed points for F, the conclusion of Theorem 2.4 follows by Corollary 3.17. □

Moreover, by Corollary 3.17 or, more generally, by Theorem 3.14, we can obtain several known coupled fixed point theorems, as well as some new ones. For example, we can prove the following result.

Theorem 3.24. *Let X be a nonempty set, $\mathbb{M} \subset X^4$ and $F : X \times X \to X$ be an operator with closed graph. Suppose:*

(i) *$\mathbb{M}$ is F-closed;*
(ii) *there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0, y_0), F(y_0, x_0)) \in \mathbb{M}$;*
(iii) *there exists $k \in [0, 1)$ and $l \geq 0$ such that, for all $(x, y, u, v) \in \mathbb{M}$ we have*

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leq$$
$$k\,(d(x,u) + d(y,v)) + l\,(d(u, F(x,y)) + d(v, F(y,x)))\,.$$

Then F has at least one coupled fixed point.

In a similar way, by Theorem 3.21, we obtain the following general existence and uniqueness result for the coupled fixed point problem.

Theorem 3.25. *Let X be a nonempty set, $\mathbb{M} \subset X^4$ and $F : X \times X \to X$ be an operator with closed graph. Suppose:*

(i) *$\mathbb{M}$ is F-closed;*
(ii) *there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0, y_0), F(y_0, x_0)) \in \mathbb{M}$;*
(iii) *there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $(x, y, u, v) \in \mathbb{M}$ we have*

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leq \varphi((d(x,u) + d(y,v))).$$

Then F has a unique coupled fixed point $(x^*, y^*) \in X \times X$ and, for all $(x, y) \in X \times X$, the sequences $(F^n(x,y))_{n\in\mathbb{N}}$ and $(F^n(y,x))_{n\in\mathbb{N}}$ converge to x^* and y^*, respectively.

Remark 3.26. It is also very clear that, if instead of the l^1 *type metric d^*, we consider on $X \times X$ the l^∞ type metric $\hat{d}((x,y),(u,v)) := \max\{d(x,u), d(y,v)\}$, we can obtain another chain of consequences of Theorem 3.14 and Theorem 3.21.*

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