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ON THE FIXED POINT PROPERTY FOR NONEXPANSIVE MAPPINGS IN HYPERBOLIC GEODESIC SPACES

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Abstract. Motivated by the well-known results concerning the complex Hilbert ball with the hyperbolic metric and metric trees, we give a characterization of the convex and closed subsets of Busemann and hyperbolic geodesic spaces in terms of the fixed point property for nonexpansive and firmly nonexpansive mappings. Furthermore, motivated by Goebel and Reich's results concerning the complex Hilbert ball with the hyperbolic metric, we describe how the fixed point free mappings behave in a much more general class of spaces.

1. Introduction

Let us suppose that C is a closed and convex subset of a Hilbert space X. Then we say that C has the fixed point property (fpp) if each nonexpansive mapping $T\colon C \to C$ (i.e., a mapping for which $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$) has at least one fixed point, i.e., a point $x \in C$ such that $Tx = x$. The very well-known result due to Browder (see e. g. [12, Introduction to Chapter 4]) shows that such a bounded C must have the fpp. In 1980 William O. Ray [19], as the first, proposed the solution of the converse problem. More precisely, he proved that the boundedness is a necessary condition for a closed and convex subset of a Hilbert space to have the fpp. The reader may find some similar results in the different classes of Banach spaces in [7, 22, 23] and [27].

Simultaneously, from Theorem 32.2 in [13] it follows that the situation for a convex and closed subset of the real Hilbert ball with the hyperbolic metric is completely different, namely, to obtain the same result it suffices to assume that C is geodesically bounded. The same property holds for geodesically bounded subsets of so–called metric trees (compare with [9, Theorem 4.3]). Both of these spaces, the real Hilbert ball with the hyperbolic metric and metric trees are special subclasses of more general CAT(κ) spaces (where κ may take all real values, but in these considerations we will restrict our attention to the case $\kappa \leq 0$. This will be explained more precisely in Section 2). The notion of CAT(κ) spaces was introduced in the 80's and it describes the geodesic spaces with a general condition of the curvature bounded above by κ. This guarantees that CAT(κ) spaces play a role of the natural generalization of Riemann manifolds with a bounded sectional curvature. The reader may find comprehensive expositions of this type of spaces in [3] or [4].

Very recently, the author in [21] proved that the same property of the existence of fixed points for nonexpansive mappings defined on geodesically bounded subsets

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holds true for each CAT(κ) space with a negative κ. However, Example 3.3 from [10] shows that, unlike in Hilbert spaces, also the unbounded subsets of CAT(0) spaces can have the fpp. The key-condition seems to be here the Gromov hyperbolicity of X introduced by Mikhail Gromov (see [14] or [5]). The main goal of this paper is to show for which hyperbolic spaces one may obtain a similar result that geodesic boundedness of a convex and closed subset of X guarantees the fpp. Moreover, we will consider a fixed point free nonexpansive mapping and we will show how such a mapping behaves in connection with the results due to Kazimierz Goebel and Simeon Reich on approximating curves (compare with [13, Section 25]). In this work we present a synthetic approach to this problem based mainly on the idea of δ-hyperbolicity and Gromov product.

The paper is organized as follows. Section 2 provides the main facts, definitions and elementary properties of geometry of geodesic spaces which are relevant to our discussion. In Section 3 and Section 4 we formulate our main results – in Theorem 3.1 we establish the fixed point property for nonexpansive mappings defined on the unbounded subsets of some special subclasses of geodesic spaces and in Theorem 4.1 we study the behaviour of approximating curves for the fixed point free nonexpansive mappings in the same types of spaces.

2. Preliminaries

Let X be a metric space. A *geodesic* joining x to y is a mapping $\gamma\colon [0,l] \to X$, where $l = d(x,y)$ and such that $\gamma(0) = x$, $\gamma(l) = y$, and $d(\gamma(t),\gamma(\tau)) = |t-\tau|$ for all $t, \tau \in [0,l]$. In particular, γ is an isometry. The image of γ is called a *geodesic (or metric) segment* joining x and y and it will be denoted by $[x,y]$. The space X is said to be a *geodesic space* if every two points are joined by a geodesic, and *a uniquely geodesic space* if there is exactly one geodesic joining x to y for each $x, y \in X$. In this case one may define a point $(1-t)x+ty$, $t \in (0,1)$ as a unique element of $[x,y]$ such that $d(x,(1-t)x+ty) = td(x,y)$. A subset C of a uniquely geodesic space X is called *a convex subset* if for each pair of points $x, y \in C$ we have $[x,y] \subset C$ and *a geodesically bounded subset* if there is no geodesic ray, i.e., the image of an isometric mapping $\gamma\colon \mathbb{R}_+ \to X$. In the sequel, we will assume that X is a geodesic space unless otherwise will be noted.

A mapping $T\colon C \to X$, where C is a nonempty subset of X, is called *a firmly nonexpansive mapping*, if for each $\lambda \in [0,1]$ the following inequality holds:

$$d(Tx,Ty) \leq d\left((1-\lambda)x + \lambda Tx, (1-\lambda)y + \lambda Ty\right)$$

(compare with [2, Section 3], [13, p. 123], [16] and [24]).

Let us choose a triple of points $o, x, y \in X$. Then one may find so-called equiradial points u, v, w belonging to $[o,x]$, $[o,y]$, $[x,y]$, respectively, and such that

$$d(o,u) = d(o,v), \quad d(x,u) = d(x,w), \quad d(y,v) = d(y,w).$$

The number $d(o,u)$ is called the *Gromov product* and is denoted by $(x|y)_o$, in that case o is called a base point (see [5, Section 1.2.1]). Let us note that in this definition the space X does not have to be geodesic and in the general case the

Gromov product can be defined by

$$(x|y)_o = \frac{1}{2}\left(d(x,o) + d(y,o) - d(x,y)\right),$$

but the first version will be more useful in the sequel.

Definition 2.1. A geodesic space X is called *δ-hyperbolic* for $\delta \geq 0$, if for all triples of points $x, y, z \in X$ the following implication holds: if $y' \in [x,y]$ and $z' \in [x,z]$ are such points that $d(x,y') = d(x,z') \leq (y|z)_x$, then $d(y',z') \leq \delta$.

In [4] one may find another definition of δ-hyperbolic spaces introduced by so-called slim triangles. Let $x_1, x_2, x_3 \in X$. A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in X consists of three points x_1, x_2, x_3, being its *vertices*, and a selection of three geodesics $[x_1,x_2]$, $[x_1,x_3]$, $[x_2,x_3]$ joining them, being its sides. If a point $x \in X$ lies in the union of $[x_1,x_2]$, $[x_1,x_3]$ and $[x_2,x_3]$, then we write $x \in \Delta(x_1,x_2,x_3)$. Then the triangle $\Delta(x_1,x_2,x_3)$ is *δ-slim* if each of its sides is contained in δ-neighbourhood of the union of the other two sides. X is said to be δ-hyperbolic if every triangle in X is δ-slim. Clearly, this definition, proposed first by Eliyahu Rips is equivalent to the previous one with a different value of δ (see [5, Lemma 1.2.3 and Exercise 1.2.4]).

The definition of hyperbolic spaces can be also reformulated in the following way. Let X be a metric, but not necessarily geodesic space, then X is *δ-hyperbolic* if the inequality

$$(x|z)_o \geq \min\{(x|y)_o, (y|z)_o\} - \delta$$

holds true for each base point $o \in X$ and all $x, y, z \in X$ (see [5, Section 2.1, p. 11]).

Next we focus on the special subclasses of uniquely geodesic spaces. First, for each pair of the geodesic segments $[x,y]$ and $[u,v]$ with $x, y, u, v \in X$, one may consider the following inequality:

$$d(tx + (1-t)y, tu + (1-t)v) \leq td(x,u) + (1-t)d(y,v), \qquad t \in (0,1). \tag{2.1}$$

Definition 2.2. If for each pair of metric segments formula (2.1) holds true, then X is said to satisfy the *Busemann convexity* or just to be *the Busemann space*.

Let us notice that in some papers the space satisfying the above condition is called the hyperbolic space (compare with e. g. [25] and [26]). The reader may find a generous exposition of this topic in [20, Section 8].

Now we introduce the concept of asymptotic center first proposed by Michael Edelstein in the case of uniformly convex Banach spaces. If we consider a bounded sequence (x_n) in X, we are able to define a function $r(\cdot, x_n)$, called the type, such that for each x

$$r(x, x_n) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic center of a bounded sequence with respect to a subset C of X is then defined as

$$A_C((x_n)) = \{x \in C : r(x,x_n) \leq r(y,x_n) \text{ for any } y \in C\}.$$

We will say that X has the *unique asymptotic center property* if for each closed and convex subset $C \subset X$ and for each bounded sequence (x_n) of elements from C there exists a unique asymptotic center $A_C((x_n))$ belonging to C.

The natural examples of spaces with the unique asymptotic center property are given by the real and complex Hilbert balls with the hyperbolic metric (see [13, Section 21] and [18]) and so-called uniformly convex geodesic spaces (see [8, Corollary 3.9]) with the decreasing modulus of convexity or the lower semicontinuous from the right.

As a natural example of spaces defined above one may consider the following remark.

Remark 2.3 ([20, Proposition 8.1.6] and [12, Lemma 15.2 and Theorem 15.3])**.** A Banach space X is the Busemann convex space if and only if X is a strictly convex Banach space. If X is a uniformly convex Banach space then X is Busemann convex and uniformly convex as a geodesic space.

Now we introduce some basic concepts concerning CAT(κ). For $\kappa \leq 0$, let (M^2_κ, d_κ) denote a complete, simply connected, Riemannian 2-manifold of the constant Gaussian curvature κ, which we shall call as a *model space* [4, I.2]. Let us recall that a *comparison triangle* in M^2_κ for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in X, is a triangle $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M^2_κ such that $d(x_i, x_j) = d_\kappa(\bar{x}_i, \bar{x}_j)$ for all $i, j \in \{1, 2, 3\}$. Such a triangle $\bar{\Delta} \subset M^2_\kappa$ always exists and is unique up to isometries.

For the given triangle $\Delta(x_1, x_2, x_3)$ in X and its comparison triangle $\bar{\Delta}$ in M^2_κ, the *comparison point* for a point $z \in [x_i, x_j]$ is a point $\bar{z}$ in $[\bar{x}_i, \bar{x}_j]$ such that $d(x_i, z) = d_\kappa(\bar{x}_i, \bar{z})$.

Definition 2.4. Space X is called *a CAT(κ) space*, if all the triangles of X satisfy the inequality

$$d(x, y) \leq d_\kappa(\bar{x}, \bar{y}).$$

This idea can be generalized for all real κ. However, in the case of positive κ the considered triangles have to satisfy an additional assumption on their perimeters and therefore they are not useful in the study of unbounded spaces.

It is worth mentioning that each CAT(κ) space with $\kappa \leq 0$ is uniquely geodesic and, moreover, it is uniformly convex and Busemann convex, thus these spaces can be considered another special case of our subclasses of geodesic spaces. Furthermore, each CAT(κ) space with negative κ is also a CAT(0) space. Next we will show how the concept of CAT(κ) spaces can be connected with hyperbolic spaces.

Proposition 2.5 (see [4, Proposition III.1.2] and [5, Section 1.4.3])**.** *Each CAT(κ) space, where $\kappa < 0$, is δ-hyperbolic and δ depends only on κ.*

Metric trees represent a particular class of CAT(0) spaces with many applications in various fields. They are also referred to as spaces of "$-\infty$" constant curvature (see [4, p. 67] for more details). In addition, these spaces can be also considered a special case of δ-hyperbolic spaces with $\delta = 0$ (see [5, Example 1.2.5] and [14, Remark 1.1.D]).

Definition 2.6. *A metric tree* is a uniquely geodesic metric space M for which the following implication holds: if x, y and $z \in M$ are such that $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Let X be a Busemann space and $o \in X$. One may introduce a projection of X onto a closed ball $B = \overline{B}(o, R)$ defined in the following way: $P_R(x) = x$ for $d(o, x) \leq R$, and $P_R(x) \in [o, x]$ with $d(o, P_R(x)) = R$, otherwise. It is easy to see that the Busemann convexity implies the fact that this mapping is 2-lipschitz. In the same way as it was done in [4, p. 263] we consider the inverse limit $\varprojlim \overline{B}(o, R)$. The inverse limit topology coincides with the topology of convergence on compact subsets of $\mathbb{R}_+$ and it will be called the cone topology. The generous exposition on this topology may be found in [20, Section 10.2].

There is a natural bijection $\varphi\colon X \to \varprojlim \overline{B}(o, R)$, associating to $x \in X$ a mapping $c_x\colon [0, \infty) \to X$ whose restriction to $[0, d(o, x)]$ is a geodesic segment $[o, x]$ and the restriction to $[d(o, x), \infty)$ is a constant mapping at x (see [4, p. 263]). Furthermore, the boundary $\partial_o X$ is the set of all geodesic rays issuing from o. The cone topology on X coincides with the natural one and for $\xi \in \partial_o X$ the subbasis of the cone topology is the set $\{U(c, R, \varepsilon) \mid R, \varepsilon > 0\}$, where

$$U(c, R, \varepsilon) = \{X \cup \partial_o X \mid d(o, x) > R \wedge d(P_R(x), c(R)) < \varepsilon\}$$

(compare with [4, II.8.6]).

Let us assume that for two geodesic rays c and c' there exists a positive number M such that $d(c(t), c'(t)) \leq M$ for all $t \geq 0$. Then we say that c and c' are *the asymptotic rays* (see [4, Definition II.8.1]).

Definition 2.7. *A geodesic boundary* of X (denoted by $\partial^g X$) is a set of the equivalence classes of geodesic rays, where two rays are equivalent if and only if they are asymptotic.

Although the above definition works in each geodesic space, if additionally X is a Busemann space, then the following property holds:

Proposition 2.8 ([20, Proposition 10.1.6])**.** *For every pair c and c' of the asymptotic rays issuing from the same point it must be that $c = c'$.*

Moreover, for each couple of points x and y of the complete Busemann space X their boundaries $\partial_x X$ and $\partial_y X$ are homeomorphic, therefore $\partial^g X$ equipped with the cone topology is well defined (see [6]; the similar results for more special cases may be also found in [15]).

It is worth mentioning that if X is also the δ-hyperbolic space, then this boundary coincides with the Gromov boundary – the proof of this fact and necessary definitions may be found in [11, Chapter 7].

3. The fixed point property of nonexpansive mappings

Our first result is a very natural generalization of the main theorem presented in [21] and simultaneously it is a counterpart of Theorem 3.1 in [27].

Theorem 3.1. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us suppose that X is δ-hyperbolic for some nonnegative δ. If C is a convex and closed subset of X, then the following facts are equivalent:*

a) *C is geodesically bounded.*

b) *Each nonexpansive mapping* $T: C \to C$ *has at least one fixed point.*
c) *Each firmly nonexpansive mapping* $T: C \to C$ *has at least one fixed point.*

Proof. For better understanding we will divide the proof into four steps.

Step 1.

First we assume that T is a fixed point free nonexpansive mapping.

Let us fix x_0. Then for $t \in (0,1)$ from the Busemann convexity it follows that the mapping $T_t(x) = (1-t)Tx + tx_0$ is the strict contraction. Hence for each $t \in (0,1)$ there is a unique fixed point z_t and

$$\gamma := \{z_t = (1-t)Tz_t + tx_0 : \ t \in (0,1)\} \tag{3.1}$$

forms an approximating curve (compare with [13, (24.1), p. 122] and [17]). Clearly, this curve is continuous in each point and $z_t \to x_0$ if $t \to 1$. Indeed, let us notice that

$$d(x_0, z_t) = (1-t)d(x_0, Tz_t) \leq (1-t)d(x_0, Tx_0) + (1-t)d(Tx_0, Tz_t)$$

and the nonexpansivity of T implies that

$$d(x_0, z_t) \leq \frac{1-t}{t} d(x_0, Tx_0).$$

In this step we will consider how the approximating curve behaves if $t \to 0^+$. First let us notice that the existence of a sequence (z_{t_n}) of $\{z_t : t \in (0,1)\}$ where $t_n \to 0^+$ and a positive M such that $d(x_0, z_{t_n}) \leq M$ implies that $d(z_{t_n}, Tz_{t_n}) \to 0$ and the asymptotic center $A((z_{t_n}))$ (considered with respect to C) is unique in view of our assumption and therefore this must be the fixed point of T. So we get that $d(x_0, z_t) \to \infty$ when $t \to 0^+$ in the case of the fixed point free mapping T and we may find a sequence (z_n) of γ for which $d(x_0, Tz_n) = n$. The existence of this sequence follows directly from the continuity of γ. Moreover, if $z_n = z_{t_n}$ for every $n \in \mathbb{N}$, then $t_n \to 0$.

Now let us notice that

$$\begin{aligned} d(x_0, z_t) + d(z_t, Tz_t) = d(x_0, Tz_t) &\leq d(x_0, Tx_0) + d(Tx_0, Tz_t) \\ &\leq d(x_0, Tx_0) + d(x_0, z_t), \end{aligned}$$

from which it follows directly that $d(z_t, Tz_t)$ must be uniformly bounded by $D := d(x_0, Tx_0)$.

The main goal of this step is to show that the Gromov product $(Tz_n|Tz_m)_{x_0}$ tends to infinity for $m, n \to \infty$.

Let us suppose that this is not the case. Then there exist two sequences $(z_{u(n)})$ and $(z_{v(n)})$ of (z_n) and a positive number R for which

$$(Tz_{u(n)}|Tz_{v(n)})_{x_0} \leq R. \tag{3.2}$$

Let us fix n and let us consider the triangle $\Delta(x_0, Tz_{u(n)}, Tz_{v(n)})$. There are three equiradial points a_n, b_n, c_n placed on the edges of this triangle such that $a_n \in [x_0, Tz_{u(n)}]$, $b_n \in [x_0, Tz_{v(n)}]$ and $d(x_0, a_n) = d(x_0, b_n) = (Tz_{u(n)}|Tz_{v(n)})_{x_0}$, $c_n \in [Tz_{u(n)}, Tz_{v(n)}]$ and $d(Tz_{u(n)}, a_n) = d(Tz_{u(n)}, c_n)$. Clearly, from the hyperbolicity of X it follows that

$$\max\{d(a_n, b_n), d(b_n, c_n), d(c_n, a_n)\} \leq \delta.$$

Since $d(z_{u(n)}, Tz_{u(n)})$ and $d(z_{v(n)}, Tz_{v(n)}) \leq D$, taking n to be large enough one may assume that $z_{u(n)} \in [Tz_{u(n)}, a_n]$, $z_{v(n)} \in [Tz_{v(n)}, b_n]$ and $d(Tz_{u(n)}, a_n)$, $d(Tz_{v(n)}, b_n) >> \delta$. Next, let us consider two triangles $\Delta(Tz_{u(n)}, a_n, c_n)$ and $\Delta(Tz_{v(n)}, b_n, c_n)$ and let us find two points u_n, v_n of the metric segment $[Tz_{u(n)}, Tz_{v(n)}]$ such that

$$d(Tz_{u(n)}, z_{u(n)}) = d(Tz_{u(n)}, u_n) \quad \text{and} \qquad d(Tz_{v(n)}, z_{v(n)}) = d(Tz_{v(n)}, v_n).$$

From the Busemann convexity for two geodesics $[Tz_{u(n)}, a_n]$ and $[Tz_{u(n)}, c_n]$ of the same length one may deduce that

$$d(z_{u(n)}, u_n) \leq d(a_n, c_n)\frac{d(Tz_{u(n)}, z_{u(n)})}{d(Tz_{u(n)}, a_n)} \leq d(Tz_{u(n)}, z_{u(n)})\frac{\delta}{d(Tz_{u(n)}, a_n)}.$$

In a similar way we get

$$d(z_{v(n)}, v_n) \leq d(Tz_{v(n)}, z_{v(n)})\frac{\delta}{d(Tz_{v(n)}, b_n)}.$$

Let us recall that we suppose that the Gromov products $(Tz_{u(n)}|Tz_{v(n)})_{x_0}$ were bounded by R. So $d(Tz_{u(n)}, a_n)$ and $d(Tz_{v(n)}, b_n)$ tend to ∞ and again for the large enough n we get

$$d(z_{u(n)}, u_n) < d(Tz_{u(n)}, z_{u(n)}) \qquad \text{and} \qquad d(z_{v(n)}, v_n) < d(Tz_{v(n)}, z_{v(n)}),$$

which yields

$$\begin{aligned} d(z_{u(n)}, z_{v(n)}) &\leq d(z_{u(n)}, u_n) + d(u_n, v_n) + d(z_{v(n)}, v_n) \\ &< d(Tz_{u(n)}, u_n) + d(u_n, v_n) + d(v_n, Tz_{v(n)}) \\ &= d(Tz_{u(n)}, Tz_{v(n)}), \end{aligned}$$

a contradiction. So it has been shown that the Gromov products $(Tz_n|Tz_m)_{x_0}$ tend to infinity for $n, m \to \infty$.

Step 2.

Next we will prove that the previous step guarantees that C cannot be geodesically bounded, i.e., C must contain a geodesic ray.

For each natural $n > D+1$ let us consider the projection $y_1^n = P_1(Tz_n) = P_1(z_n)$ onto a closed ball $\overline{B}(x_0, 1)$. We will show that (y_1^n) is the Cauchy sequence. Indeed, let us choose $\varepsilon > 0$. From the previous step it follows that there exists a natural number $N(\varepsilon)$ such that

$$(Tz_n|Tz_m)_{x_0} > \frac{\delta}{\varepsilon} \qquad \text{for all} \quad n, m \geq N(\varepsilon).$$

Clearly, from the Busemann convexity of X we obtain that

$$d(y_1^n, y_1^m) \leq \frac{\delta \cdot 1}{(Tz_n|Tz_m)_{x_0}} < \varepsilon$$

if n, m are both greater than $N(\varepsilon)$. Since ε can be arbitrarily small, this proves that (y_1^n) is the Cauchy sequence. Let x_1 be its limit. Since C is a closed subset of X, then $x_1 \in C$.

In the same way one may deduce the existence of points y_2^n, $n > 2$, with the distance to x_0 equal to 2. Their limit point x_2 also must be at distance 2 from x_0

and, moreover, from the uniqueness of metric segments it follows that $x_1 \in [x_0, x_2]$. Repeating our considerations we obtain a sequence (x_n) such that $d(x_0, x_n) = n$ and $x_n \in [x_0, x_m]$ for $m > n$, so this sequence forms a geodesic c in C issuing from x_0 and finally C cannot be geodesically bounded.

Step 3.

Now we assume that C contains a geodesic ray c and we intend to show that there exists a fixed point free nonexpansive mapping $T\colon C \to c$.

Let us define a Busemann type function $b\colon C \to \mathbb{R}$ as

$$b(x) = \liminf_{n\to\infty}(d(x, c(n)) - n).$$

First, let us notice that, similarly as in the case of CAT(0) spaces (see [3, Section II.1]), this function is nonexpansive. Indeed, we have

$$\liminf(d(x, c(n))-n)-\liminf(d(y, c(n)-n) \le \limsup(d(x, c(n))-d(y, c(n))) \le d(x, y).$$

Moreover, for each point of the geodesic ray c there must be $b(c(t)) = -t$, so we may define a nonexpansive mapping $P\colon C \to c$ by

$$P(x) = \begin{cases} c(t+1), & b(x) = -t \\ c(1), & b(x) \ge 0 \end{cases}.$$

It is easy to see that P is fixed point free. Otherwise, the fixed point set must be a subset of c, where $P(c(t)) = c(t+1)$, a contradiction.

Step 4.

Finally, we will show that properties b) and c) are equivalent. Clearly, b) implies c). On the other hand, let us recall the approximating curve from Step 1, i.e., for fixed $x \in C$ and $t \in (0, 1)$ one may find z_t as a fixed point of the strict contraction $G_t(y) = (1-t)Ty + tx$. Changing our viewpoint, one may define a mapping $U_t(x)$ by $U_t(x) = z_t$, i.e.,

$$U_t(x) = (1-t)TU_t(x) + tx.$$

Let us notice that from [2, Proposition 3.2] it follows that for each $t \in (0, 1)$ the mapping U_t is firmly nonexpansive and has the same fixed points as T. This fact completes our proof. □

Since each CAT(0) space is the Busemann space and simultaneously it is uniformly convex and its modulus of convexity possesses some nice properties, we get the following result for the complete hyperbolic CAT(0) spaces.

Corollary 3.2. *Let X be a complete CAT(0) space which is δ-hyperbolic for some positive δ. If C is a convex and closed subset of X, then the following facts are equivalent:*

a) *C is geodesically bounded.*
b) *Each nonexpansive mapping $T\colon C \to C$ has at least one fixed point.*
c) *Each firmly nonexpansive mapping $T\colon C \to C$ has at least one fixed point.*

Moreover, it is a well–known fact that each CAT(κ) space with negative κ is a δ-hyperbolic one, where δ depends only on κ. Since these spaces are also Busemann and uniformly convex, as a special case of Theorem 3.1 one may get the following result from [21].

Corollary 3.3. *Let X be a complete CAT(κ) space, where $\kappa < 0$.*
Each nonexpansive mapping $T\colon C \to C$, where C is a convex and closed subset of X, has at least one fixed point if and only if C is geodesically bounded.

4. Fixed point free nonexpansive mappings

In [13] Kazimierz Goebel and Simeon Reich did not only show when nonexpansive mappings, defined on the complex Hilbert ball with the hyperbolic metric must have at least one fixed point, but also considered how the approximating curves behave. Motivated by their results from Section 25 of [13] we will consider now more precisely the behaviour of the approximating curves defined by (3.1) in the general case of Busemann spaces.

Theorem 4.1. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us assume that X is δ-hyperbolic for some positive δ. If $T\colon C \to C$ is a fixed point free nonexpansive mapping, where C is a convex and closed subset of X, then there exists a point $\xi \in \partial^g X$ such that for each $x \in C$ the approximating curve, issuing from x and defined by* (3.1)*, converges to ξ with respect to the cone topology.*

Proof. Let us choose $x_0 \in C$. First, let us notice that from Step 2 of the proof of Theorem 3.1 it follows that there exists a sequence of points (z_n) of the approximating curve $\{z_t \mid t \in (0,1)\}$ such that $d(x_0, Tz_n) \to \infty$. Moreover, in view of the same step there is a geodesic ray c (issuing from x_0) and contained in C for which the following property holds:
Let us select a natural number N. Then for all z_n with $d(z_n, x_0) > N$ the projections $P_N(z_n)$ onto the closed ball $\overline{B}(x_0, N)$ converge to $c(N)$. Clearly, from the Busemann convexity it follows directly that for each positive number R the same property is true, i.e., the projections $P_R(z_n)$ must tend to $c(R)$.

Let us fix $D := \max\{d(x_0, Tz_0) + 1, \delta + 1\}$. For each $t \in (0,1)$, such that $d(x_0, z_t) > 2D$, we will show that

$$\operatorname{dist}(Tz_t, c) \leq D + \delta,$$

where c is a geodesic ray defined above.

First we prove that

$$\sup_{n > N_0} (x_0 | Tz_n)_{Tz_t} \leq D, \tag{4.1}$$

where N_0 is the first natural number not smaller than $d(x_0, Tz_t)$. Let us suppose that this is not true. Then there exists a natural number $n \geq d(x_0, Tz_t)$ such that

$$(x_0 | Tz_n)_{Tz_t} > D. \tag{4.2}$$

In the triangle $\Delta(x_0, Tz_t, Tz_n)$ we choose three equiradial points a, b, c, similarly as in Step 1. of the proof of Theorem 3.1, i.e., $a \in [x_0, Tz_t]$, $b \in [x_0, Tz_t]$, $c \in [Tz_n, Tz_t]$ and the respective distances between these points are equal.

Since $d(x_0, Tz_t) < d(x_0, Tz_n)$ and because (4.2) holds, we get that $d(b, Tz_n) > D$. Hence, from the Busemann convexity it follows that there exists two points u from

$[Tz_t, c]$ and v from $[c, Tz_n]$ such that $d(Tz_t, u) = d(Tz_t, z_t)$, $d(Tz_n, v) = d(Tz_n, z_n)$ and, moreover,

$$d(z_t, u) \leq \delta \cdot \frac{d(z_t, Tz_t)}{D} < d(z_t, Tz_t).$$

Similarly, we get

$$d(z_n, u) \leq \delta \cdot \frac{d(z_n, Tz_n)}{D} < d(z_n, Tz_n).$$

So, again as in Step 1, we obtain

$$d(z_t, z_n) \leq d(z_t, u) + d(u, v) + d(v, z_n) < d(Tz_t, Tz_n),$$

a contradiction. This completes the proof of (4.1).

Now let us fix a positive number ε and let $R = d(x_0, Tz_t) - D$. Clearly, there is the large enough n such that $d(x_0, Tz_n) > d(x_0, Tz_t)$ and the projection $P_R(z_n) = P_R(Tz_n)$ onto the closed ball $\overline{B}(x_0, R)$ satisfies the inequality

$$d(P_R(z_n), c(R)) < \varepsilon.$$

In view of (4.1), R cannot be greater than $(Tz_t|Tz_n)_{x_0}$, therefore from the Busemann convexity it follows that

$$d(P_R(Tz_t), c(R)) \leq d(P_R(Tz_t), P_R(z_n)) + d(P_R(z_n), c(R)) < \delta + \varepsilon.$$

Since ε was an arbitrarily small number, finally we have

$$\operatorname{dist}(P_R(Tz_t), c) \leq \delta$$

and

$$\operatorname{dist}(z_t, c) < \operatorname{dist}(Tz_t, c) \leq D + \delta. \tag{4.3}$$

Moreover, the point $c(r)$ from c, which satisfies (4.3), can be chosen in such a way that

$$d(x_0, Tz_t) - D \leq r \leq d(x_0, Tz_t) + D. \tag{4.4}$$

Next, let us notice that $d(x_0, z_t) \to \infty$ when $t \to 0^+$, so from the Busemann convexity, (4.3), (4.4) and the fact that projections $P_s(Tz_t)$ and $P_s(z_t)$ onto $\overline{B}(x_0, s)$ are equal, it follows that

$$\begin{aligned} d(P_s(Tz_t), c(s)) &\leq \frac{s}{d(x_0, Tz_t)} \cdot d(Tz_t, c(d(x, Tz_t))) \\ &\leq \frac{s}{d(x_0, Tz_t)} \cdot (d(Tz_t, c(r)) + D + \delta) \\ &\leq \frac{s}{d(x_0, Tz_t)} \cdot (2D + \delta) \ < \ \varepsilon \end{aligned}$$

for all t close enough to 0, thus almost the whole approximating curve $\{z_t \mid t \in (0, 1)\}$ is contained in $U(c, s, \varepsilon)$ for all positive s and ε.

Finally, let us choose $x, y \in C$. Then for each $t \in (0, 1)$ from the Busemann convexity we have

$$d(x_t, y_t) \leq td(x, y) + (1 - t)d(Tx_t, Ty_t) \leq td(x, y) + (1 - t)d(x_t, y_t)$$

and two curves c_x and c_y must be asymptotic, which completes the proof. □

Motivated by Theorem 25.4 from [13] we also consider mappings $T\colon X \cup \partial^g X \to X \cup \partial^g X$ – continuous with respect to the cone topology.

Corollary 4.2. *Let X be a complete Busemann space with the unique asymptotic center property. Moreover, let us suppose that X is δ-hyperbolic for some positive δ.*
If $T\colon X\cup\partial^g X\to X\cup\partial^g X$ is continuous with respect to the cone topology and there exists a nonempty closed and convex $C\subset X$ such that $T|_C\colon C\to C$ is nonexpansive, then T has at least one fixed point in $X\cup\partial^g X$.

Proof. If T has a fixed point in C, then there is nothing to prove. Otherwise, from the previous theorem it follows that for any $x_0\in C$ the approximating curve $\{z_t\mid t\in(0,1)\}$ converges to the point $\xi\in\partial^g X$ with respect to the cone topology. Since the distances $d(z_t,Tz_t)$ are uniformly bounded this implies that $\{Tz_t\mid t\in(0,1)\}$ must also tend to the same point ξ. The continuity of T on $X\cup\partial^g X$ completes the proof. □

As a special case of these results one may consider the hyperbolic CAT(κ) spaces with $\kappa\le 0$ which leads to the natural generalizations of Theorem 25.2 and Theorem 25.4 from [13].

Corollary 4.3. *Let X be a complete CAT(κ) space with nonpositive κ. Moreover, let us suppose that X is δ-hyperbolic for some positive δ. Then the following property is true:*
if $T\colon C\to C$ is a fixed point free nonexpansive mapping, where C is a convex and closed subset of X, then there exists a point $\xi\in\partial^g X$ such that for each $x\in X$ the approximating curve issuing from x and defined by (3.1) *converges to ξ with respect to the cone topology.*

Corollary 4.4. *Let X be a complete CAT(κ) space with nonpositive κ. Moreover, let us assume that X is δ-hyperbolic for some positive δ.*
If $T\colon X\cup\partial^g X\to X\cup\partial^g X$ is continuous with respect to the cone topology and there exists a nonempty closed and convex $C\subset X$ such that $T|_C\colon C\to C$ is nonexpansive, then T has at least one fixed point in $X\cup\partial^g X$.

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