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COUPLING POPOV'S ALGORITHM WITH SUBGRADIENT EXTRAGRADIENT METHOD FOR SOLVING EQUILIBRIUM PROBLEMS

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Abstract. Based on the recent works by Censor et al. [The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. 148, 318-335 (2011)], Malitsky and Semenov [An extragradient algorithm for monotone variational inequalities. Cybern. Syst. Anal. 50, 271-277 (2014)], and Lyashko and Semenov [A new two-step proximal algorithm of solving the problem of equilibrium programming, In: Optimization and Applications in Control and Data Sciences (ed. B.Goldengorin), Springer Optimization and Its Applications, volume 115, 315-326 (2016)], we propose a new scheme for solving pseudomonotone equilibrium problems in real Hilbert spaces. Weak and strong convergence results are suitably established. Our algorithm improves the recent one announced by Lyashko and Semenov not only from computational point of view, but also in some assumptions imposed on their main result. A comparative numerical study is carried out between the algorithms of Quoc-Muu-Nguyen [Extragradient algorithms extended to equilibrium problems. Optimization 57, 749-776 (2008)], Lyashko-Semenov, and the new one. Some numerical examples are given to illustrate the efficiency and performance of the proposed method.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and $f : C \times C \to \mathbb{R}$ be a bifunction. Following [7], an *equilibrium problem* (EP) is a problem of the form:

$$\text{find } x \in C \text{ such that } f(x,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

If $f(x,x) = 0$, for all $x \in C$, then the bifunction f is called an *equilibrium bifunction*. We denote by $EP(f)$ the set of solutions of the EP (1.1), i.e.,

$$EP(f) := \{x \in C : f(x,y) \geq 0, \forall y \in C\},$$

and note that the solution set of such problems may be empty. It is well known that several problems arising in nonlinear analysis such as variational inequalities, optimization problems, fixed-point problems, complementarity problems, Nash equilibria, minimax problems, traffic networks are special cases of equilibrium problems. For example, if $f(x,y) = \langle Ax, y-x\rangle$, where $A : C \to H$ is a mapping, then the

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problem (1.1) is the following classical variational inequality initially investigated by Kinderlehrer and Stampacchia in [27] (see also Malitsky [31] for recent advances):

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in C. \tag{1.2}$$

The set of solutions of the variational inequality problem (1.2) is denoted by $VI(C, A)$. Due to its importance, the EP (1.1) has been generalized and extensively studied in many directions in the last two decades; see, for example [6, 15, 16, 20, 21, 22, 25] and the references therein.

Some iterative methods have been proposed to solve various classes of equilibrium problems, see, for example [1, 2, 3, 4, 12, 20, 21, 22, 40, 45] and the references therein. In these papers, the authors introduced general iterative schemes based on the proximal method, which must solve a regularized equilibrium problem at each iteration. This task is not easy, and to overcome it, Antipin [5] replaced the regularized equilibrium problem by two strongly convex optimizations, which seem computationally easier than solving the regularized equilibrium problem in the proximal method. His method is known under the name of the extragradient method. The reason is that when the problem (EP) is a variational inequality problem, this method reduces to the classical extragradient method introduced by Korpelevich [29]:

$$\begin{cases} x_0 \in C, \\ y^n = \underset{y \in C}{\operatorname{argmin}} \Big\{ \lambda f(x^n, y) + \dfrac{1}{2} \|y - x^n\|^2 \Big\}, \\ x^{n+1} = \underset{y \in C}{\operatorname{argmin}} \Big\{ \lambda f(y^n, y) + \dfrac{1}{2} \|y - x^n\|^2 \Big\}, \ n \geq 0. \end{cases} \tag{1.3}$$

In 2010, the authors of [42] studied the algorithm (1.3) for pseudomonotone equilibrium problems in Hilbert spaces. Under mild conditions, they obtained the weak convergence of the sequences generated by (1.3). Subsequently, the algorithm (1.3) and its variants have been extensively studied, see for example [10, 14, 17, 18, 19, 41, 44, 46] and the references therein.

Recently, Lyashko and Semenov [30] proposed a Popov type algorithm for pseudomonotone equilibrium problems. The algorithm in [30] (called two-step proximal algorithm) is summarized as follows: choose $x^0 = y^0 \in C$, $\epsilon > 0$ and $0 < \lambda < \frac{1}{2(2c_1 + c_2)}$, where c_1, c_2 are positive constants (see (B5) in Section 3).

Step 1. For x^n and y^n, compute

$$x^{n+1} = \underset{y \in C}{\operatorname{argmin}} \Big\{ \lambda f(y^n, y) + \frac{1}{2} \|y - x^n\|^2 \Big\}.$$

Step 2. If $\max\{\|x^{n+1} - x^n\|, \|y^n - x^n\|\} \leq \epsilon$ then stop, else compute

$$y^{n+1} = \underset{y \in C}{\operatorname{argmin}} \Big\{ \lambda f(y^n, y) + \frac{1}{2} \|y - x^{n+1}\|^2 \Big\}.$$

Step 3. Set $n := n + 1$, and go to Step 1.

The weak convergence of their algorithm is proved under usual assumptions imposed on bifunctions in which they required that for all bounded sequences $\{x^n\}$, $\{y^n\}$ satisfying $\|x^n - y^n\| \to 0$ one has $f(x^n, y^n) \to 0$ (condition (A6)). It is worth mentioning that their condition (A6) is rather strong. On the other hand, in Lyashko and Semenov's algorithm and most other algorithms, it must either solve two strongly convex programming problems or solve one strongly convex programming problem and compute one projection onto the feasible set. Therefore, their computations are expensive if the bifunctions and the feasible sets have complicated structures. For more details, see for instance [10, 14, 30, 44, 46]. These observations lead us to the following concern.

Question. *Can we improve Lyashko and Semenov's algorithm such that the subprogram in Step 1 to be solved over a halfspace instead of over C, and furthermore, some assumptions on f to be weakened?*

In this paper, we give a positive answer to this question. Motivated and inspired by the algorithms in [9, 30, 32], we will introduce an efficient new algorithm for solving (EP). The advantage of our method is that in Step 1 it only requires solving the subprogram over a halfspace instead of over the feasible set as Lyashko and Semenov's algorithm in [30]. Besides, the assumptions on f can be relaxed and the convergence is still guaranteed. Numerical examples are presented to describe the efficiency of the proposed approach.

The rest of the paper is organized as follows. After collecting some definitions and basic results in Section 2, we prove in Section 3 the weak and (under some further assumptions) strong convergence of the proposed algorithm. In Section 4, we apply the proposed algorithm to variational inequalities. Finally, in Section 5 we provide some numerical results to illustrate the convergence of our algorithm and compare it with the previous algorithms.

2. Preliminaries

In this section, we present some preliminary results that we will use in our upcoming results. From now on, we will assume that H is a real Hilbert space endowed with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. When $\{x^n\}$ is a sequence in H, we denote strong convergence of $\{x^n\}$ to $x \in H$ by $x^n \to x$ and weak convergence by $x^n \rightharpoonup x$. Let C be a nonempty closed convex subset of H. For every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$||x - P_C x|| = \inf\{||x - y|| : \ y \in C\}.$$

P_C is called the metric projection of H onto C. It is well known that P_C has the following basic properties:

(a) $\langle x - P_C x, y - P_C x \rangle \leq 0$, for all $x \in H$ and all $y \in C$;
(b) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2$, for all $x \in H$ and all $y \in C$;
(c) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$, for every $x, y \in H$;
(d) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$, for all $x, y \in H$.

We now recall the concept of proximity operator introduced by Moreau in 1962 [36]. Let $g : C \subseteq H \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function and $\gamma > 0$. The Moreau envelope of g is the function

$${}^{\gamma}g(x) = \inf_{y \in C} \Big\{ g(y) + \frac{1}{2\gamma}\|y - x\|^2 \Big\}, \quad \forall x \in H.$$

For all $x \in H$, the function

$$y \mapsto g(y) + \frac{1}{2\gamma}\|y - x\|^2$$

is proper, strongly convex and lower semicontinuous, thus the infimum is attained, i.e., ${}^{\gamma}g : H \to \mathbb{R}$. The unique minimizer of

$$y \mapsto g(y) + \frac{1}{2\gamma}\|y - x\|^2 \tag{2.1}$$

is called proximal point of g at x and it is denoted by $\text{prox}_g(x)$. The operator

$$\begin{aligned} \text{prox}_g(x) &: H \to C \\ x &\mapsto \underset{y \in C}{\text{argmin}}\left\{ g(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\} \end{aligned}$$

is well-defined and is said to be the proximity operator of g. In practice, the problem of computing $\text{prox}_g(x)$ can be implemented by the Matlab Optimization Toolbox. It is clear that the proximity operator is a generalization of the projection operator.

We also recall that the subdifferential of $g : H \to (-\infty, \infty]$ at $x \in H$ is defined as the set of all subgradient of g at x:

$$\partial g(x) := \{w \in H : g(y) - g(x) \geq \langle w, y - x\rangle, \ \forall y \in H\}.$$

The normal cone of C at $x \in C$ is defined by

$$N_C(x) := \{q \in H : \langle q, y - x\rangle \leq 0, \ \forall y \in C\}.$$

We now recall classical concepts of monotonicity of nonlinear operators.

Definition 2.1. An operator $A : C \to H$ is said to be

(1) (see Minty [35]) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(2) (see Karamardian [23]) pseudomonotone if

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, x - y\rangle \leq 0, \quad \forall x, y \in C.$$

Analogous to Definition 2.1, we have the following concepts for equilibrium problems.

Definition 2.2. A bifunction $f : H \times H \to \mathbb{R}$ is said to be

(1) (see Mosco [37], Blum and Oettli [7]) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C.$$

(2) (see Bianchi and Schaible [6]) pseudomonotone on C if for any $x, y \in C$

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0.$$

(3) (see [13]) γ-strongly pseudomonotone on C if there exists $\gamma > 0$ such that for any $x, y \in C$

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\gamma\|x - y\|^2.$$

Remark 2.3. It is obvious that if $A : H \to H$ is monotone (pseudomonotone) in the sense of Definition 2.1 then the corresponding bifunction defined by $f(x, y) = \langle Ax, y - x\rangle$ is monotone (pseudomonotone) in the sense of Definition 2.2.

The following lemmas will be useful for proving the convergence results of this paper.

Lemma 2.4 (Opial [38])**.** *Let $\{x^n\}$ be a sequence of elements of the Hilbert space H which converges weakly to $x \in H$. Then we have*

$$\liminf_{n\to\infty}\|x^n - x\| < \liminf_{n\to\infty}\|x^n - y\|, \quad \forall y \in H \setminus \{x\}.$$

Lemma 2.5 ([8])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $g : H \to \mathbb{R}$ be a lower semicontinuous convex function. Then, x^* is a solution of the following convex problem*

$$\min\{g(x) : x \in C\}$$

if and only if

$$0 \in \partial g(x^*) + N_C(x^*).$$

3. A new projection algorithm for equilibrium problems

Unless otherwise specified, we assume that the following assumptions are satisfied from now on.

Assumption 3.1. Let C be a nonempty closed convex subset of H and $f : H\times H \to \mathbb{R}$ be a bifunction such that

(B1) f is pseudomonotone on C;

(B2) For any arbitrary sequence $\{z^k\}$ such that $z^k \rightharpoonup z$, if $\limsup_{k\to\infty} f(z^k, y) \geq 0$, for all $y \in C$ then $z \in EP(f)$.

(B3) $f(x, \cdot)$ is convex and lower semicontinuous for every $x \in H$;

(B4) There exist positive numbers c_1 and c_2 such that the Lipschitz type condition

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \tag{3.1}$$

holds for all $x, y, z \in H$.

(B5) For all bounded sequences $\{x^n\}, \{y^n\} \subset C$ such that $\|x^n - y^n\| \to 0$, the inequality

$$\limsup_{n\to\infty} f(x^n, y^n) \geq 0$$

holds.

(B6) $EP(f) \neq \emptyset$.

It follows from (B1) and (B4) that $f(x, x) = 0$, for all $x \in C$.

Remark 3.2. The condition (B2) was first introduced by Khatibzadeh and Mohebbi in [26]. It is easy to see that if $f(., y)$ is weakly upper semicontinuous for all $y \in C$ then f satisfies the condition (B2). However, the converse is not true in general.

This remark is illustrated by the following counterexample modified from Remark 2.1 of [26].

Example 3.3. Let $H = l^2$, $C = \{\xi = (\xi_1, \xi_2, ...) \in l^2 : \xi_i \geq 0 \ \ \forall i = 1, 2, ...\}$ and

$$f(x, y) = (y_1 - x_1) \sum_{i=1}^{\infty} (x_i)^2.$$

Take $x^k = (0, ..., 0, \underbrace{1}_{k \text{ th}}, 0, ...)$, we have $x^k \rightharpoonup x = (0, ..., 0, ...)$ and $x \in EP(f)$. Obviously, there is a $y \in C$ such that

$$\limsup_{k \to \infty} f(x^k, y) > 0 = f(x, y).$$

Then $f(., y)$ is not weakly upper semicontinuous. We now show that f satisfies the condition (B2). If $z^k = (z_1^k, z_2^k, ...) \rightharpoonup z = (z_1, z_2, ...)$ is an arbitrary sequence and $\limsup\limits_{k \to \infty} f(z^k, y) \geq 0$, for all $y \in C$, then we have

$$\limsup_{k \to \infty} (y_1 - z_1^k) \sum_{i=1}^{\infty} (z_i^k)^2 \geq 0.$$

Since $\lim\limits_{k \to \infty} (y_1 - z_1^k) = y_1 - z_1$, we get

$$(y_1 - z_1) \limsup_{k \to \infty} \sum_{i=1}^{\infty} (z_i^k)^2 \geq 0,$$

thus $y_1 \geq z_1$. Hence, $f(z, y) \geq 0$, for all $y \in C$, i.e., f satisfies the condition (B2).

From the above observations, it is clear that our conditions (B2) and (B5) are weaker than the conditions (A4) and (A6) in [30], respectively.

Remark 3.4. Besides the basic conditions (B1)-(B4) and (B6), the authors in [10, 17, 18, 19, 44, 46] (see also the references therein) considered the bifunction f satisfying a rather strong assumption, namely:

(B5') f is jointly weakly lower semicontinuous on the product $C \times C$.

We will now show that the assumption (B5') implies (B5). Indeed, let $\{x^n\}$, $\{y^n\}$ be bounded sequences in C with $\|x^n - y^n\| \to 0$. Thus there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ converging weakly to $\bar{x} \in C$. By the assumption, the subsequence $\{y^{n_k}\}$ converges weakly to the same $\bar{x}$. Hence,

$$\limsup_{n \to \infty} f(x^n, y^n) \geq \limsup_{k \to \infty} f(x^{n_k}, y^{n_k}) \geq \liminf_{k \to \infty} f(x^{n_k}, y^{n_k}) \geq f(\bar{x}, \bar{x}) = 0.$$

In [17], the author assumed that f is convex and subdifferentiable on H with respect to the second argument (the condition (A4)). On the other hand, from his condition (A3), we have the weak lower semicontinuity (which implies the lower semicontinuity) of f on the second argument. Therefore, the assumption of subdifferentiability is indeed not necessary.

Remark 3.5. We will hereafter consider two important particular cases of the equilibrium problem, namely the optimization problem and the variational inequality problem in which (B5) is satisfied under mild conditions.

(1) Let $f(x,y) = F(y) - F(x)$, where C is a nonempty closed convex subset of H and $F : H \to \mathbb{R}$ is a *uniformly continuous* function on C. Then f satisfies (B5). Indeed, if $\|x^n - y^n\| \to 0$, then by uniform continuity we have $F(x^n) - F(y^n) \to 0$, as $n \to \infty$, hence

$$\limsup_{n\to\infty} f(x^n, y^n) = \lim_{n\to\infty} f(x^n, y^n) = 0.$$

(2) In case of variational inequalities (1.2), the assumption (B5) is satisfied if A is bounded on bounded sets. Indeed,

$$\begin{aligned} f(x^n, y^n) = \langle Ax^n, y^n - x^n\rangle &\geq -\|Ax^n\|\|x^n - y^n\| \\ &\geq -M\|x^n - y^n\|, \end{aligned}$$

where M is a positive constant such that $\|Ax^n\| \leq M$. Hence, by the assumption

$$\limsup_{n\to\infty} f(x^n, y^n) \geq -\lim_{n\to\infty} M\|x^n - y^n\| = 0.$$

Remark 3.6. Condition (3.1) was introduced by Mastroeni [33] to prove the convergence of the Auxiliary Principle Method for equilibrium problems. Moreover, we see that

(1) If $f(x,y) = \langle Ax, y - x\rangle$, where $A : C \to H$ is Lipschitz continuous with constant $L > 0$ then f satisfies the inequality (3.1) with constants $c_1 = c_2 = \frac{L}{2}$. Indeed, for each $x, y, z \in C$, we have

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) &= \langle Ax, y - x\rangle + \langle Ay, z - y\rangle - \langle Ax, z - x\rangle \\ &= -\langle Ay - Ax, y - z\rangle \\ &\geq -\|Ax - Ay\|\|y - z\| \\ &\geq -L\|x - y\|\|y - z\| \\ &\geq -\frac{L}{2}\|x - y\|^2 - \frac{L}{2}\|y - z\|^2 \\ &= -c_1\|x - y\|^2 - c_2\|y - z\|^2. \end{aligned}$$

Thus f satisfies the inequality (3.1).

(2) If there exists $\Lambda > 0$ such that

$$|f(v,w) - f(x,w) - f(v,y) + f(x,y)| \leq \Lambda\|v - x\|\|w - y\|, \quad \forall v, w, x, y \in C, \tag{3.2}$$

then it is easy to see that f also satisfies the inequality (3.1). The inequality (3.2) is called Lipschitz type inequality and has been introduced by Antipin [5]. In the framework of a finite dimensional space, he showed that if f is a differentiable function whose partial derivative with respect to the first variable satisfies the Lipschitz type inequality, then the inequality (3.2) holds. Therefore, the class of these functions also satisfies the inequality (3.1).

Next, inspired by [9, 30, 39, 32], we introduce a new projection algorithm for solving the pseudomonotone equilibrium problems and prove a weak convergence theorem for this algorithm.

Algorithm 3.7 (Subgradient extragradient type Algorithm for Equilibrium Problems)**.**

Step 1: Specify $x^0, y^0 \in C$ and $\lambda > 0$;

Compute

$$x^1 = \operatorname*{argmin}_{y \in C} \left\{ \lambda f(y^0, y) + \frac{1}{2}\|y - x^0\|^2 \right\},$$

$$y^1 = \operatorname*{argmin}_{y \in C} \left\{ \lambda f(y^0, y) + \frac{1}{2}\|y - x^1\|^2 \right\}.$$

Step 2: Given x^n, y^n and y^{n-1} $(n \geq 1)$, let $w^n \in \partial f(y^{n-1}, .)(y^n)$ such that there exists an element $q^n \in N_C(y^n)$ satisfying

$$0 = \lambda w^n + y^n - x^n + q^n, \tag{3.3}$$

and construct the halfspace

$$H_n = \{z \in H : \langle x^n - \lambda w^n - y^n, z - y^n \rangle \leq 0\}.$$

Compute

$$x^{n+1} = \operatorname*{argmin}_{y \in H_n} \left\{ \lambda f(y^n, y) + \frac{1}{2}\|y - x^n\|^2 \right\},$$

$$y^{n+1} = \operatorname*{argmin}_{y \in C} \left\{ \lambda f(y^n, y) + \frac{1}{2}\|y - x^{n+1}\|^2 \right\}.$$

Step 3: If $x^{n+1} = x^n$ and $y^n = y^{n-1}$ then stop. Otherwise, set $n := n + 1$, and return to Step 2.

Remark 3.8. The existence of $w^n \in \partial f(y^{n-1}, .)(y^n)$ and $q^n \in N_C(y^n)$ satisfying (3.3) is guaranteed by Lemma 2.5. Hence, Algorithm 3.7 is well-defined.

The following lemmas are quite helpful to analyze the convergence of Algorithm 3.7.

Lemma 3.9. *$C \subseteq H_n, \ \forall n \geq 1$.*

Proof. From (3.3), we obtain

$$q^n = x^n - \lambda w^n - y^n, \quad \forall n \geq 1,$$

where $w^n \in \partial f(y^{n-1}, .)(y^n)$ and $q^n \in N_C(y^n)$. Moreover, we have

$$N_C(y^n) = \{q \in H : \langle q, y - y^n \rangle \leq 0, \ \forall y \in C\}.$$

Therefore, we infer that

$$\langle x^n - \lambda w^n - y^n, y - y^n \rangle \leq 0, \ \forall y \in C \ \ \forall n \geq 1.$$

This shows that $C \subseteq H_n$, $\forall n \geq 1$. □

Lemma 3.10. *If $x^{n+1} = x^n$ and $y^n = y^{n-1}$ then $y^n \in EP(f)$.*

Proof. If $x^n = x^{n+1}$ we have

$$0 \in \lambda\partial f(y^n,.)(x^{n+1}) + x^{n+1} - x^n + N_{H_n}(x^{n+1}) = \lambda\partial f(y^n,.)(x^n) + N_{H_n}(x^n),$$

thus there exists $w_1^n \in \partial f(y^n,.)(x^n)$ such that $-\lambda w_1^n \in N_{H_n}(x^n)$, that is, $\langle w_1^n, z - x^n\rangle \geq 0$, for all $z \in H_n$. Hence,

$$\lambda(f(y^n,z) - f(y^n,x^n)) \geq \lambda\langle w_1^n, z - x^n\rangle \geq 0, \quad \forall z \in H_n. \tag{3.4}$$

With $z \in H_n$ and $y^n = y^{n-1}$ we get

$$\langle x^n - \lambda w^n - y^n, z - y^n\rangle \leq 0, \tag{3.5}$$

where $w^n \in \partial f(y^n,.)(y^n)$ is the chosen element satisfying (3.3). We deduce that

$$\lambda(f(y^n,z) - f(y^n,y^n)) \geq \lambda\langle w^n, z - y^n\rangle \geq \langle x^n - y^n, z - y^n\rangle.$$

Taking into account that $x^n = x^{n+1} \in H_n$ we obtain

$$\lambda f(y^n,x^n) \geq \lambda\langle w^n, x^n - y^n\rangle \geq \langle x^n - y^n, x^n - y^n\rangle \geq 0. \tag{3.6}$$

Combining (3.4) and (3.6) we have

$$\lambda f(y^n,z) \geq \lambda(f(y^n,z) - f(y^n,x^n)) \geq 0, \quad \forall z \in H_n.$$

Since $C \subseteq H_n$, we arrive at

$$f(y^n,z) \geq 0, \quad \forall z \in C.$$

This means that $y^n \in EP(f)$. □

Remark 3.11. If $x^{n+1} = y^{n+1} = y^n$ then we also obtain $y^n \in EP(f)$.

Proof. If $x^{n+1} = y^{n+1} = y^n$ then from the fact that y^n is the unique solution to the strongly convex problem

$$\min_{y\in C}\left\{\lambda f(y^n,y) + \frac{1}{2}\|y - y^n\|^2\right\},$$

and Lemma 2.5, we deduce that

$$0 = \lambda w^n + y^n - y^n + q,$$

where $w^n \in \partial f(y^n,.)(y^n)$ and $q \in N_C(y^n)$. Since

$$N_C(y^n) = \{q \in H : \langle q, z - y^n\rangle \leq 0,\ \forall z \in C\},$$

we obtain

$$\langle -\lambda w^n, z - y^n\rangle \leq 0,\ \forall z \in C.$$

Moreover, we have

$$\lambda(f(y^n,z) - f(y^n,y^n)) \geq \lambda\langle w^n, z - y^n\rangle \geq 0, \quad \forall z \in C.$$

Taking into account that $f(y^n,y^n) = 0$ we arrive at

$$f(y^n,z) \geq 0, \quad \forall z \in C.$$

This means that $y^n \in EP(f)$. □

The next statement plays a crucial role in the proof of our main results.

Lemma 3.12. *Let* $\{x^n\}$ *and* $\{y^n\}$ *be the sequences generated by Algorithm 3.7 and* $z \in EP(f)$. *Then*

$$\|x^{n+1} - z\|^2 \leq \|x^n - z\|^2 - (1 - 4\lambda c_1)\|x^n - y^n\|^2 - (1 - 2\lambda c_2)\|x^{n+1} - y^n\|^2 + 4\lambda c_1\|x^n - y^{n-1}\|^2.$$

Proof. From $x^{n+1} = \operatorname*{argmin}_{y \in H_n}\left\{\lambda f(y^n, y) + \frac{1}{2}\|y - x^n\|^2\right\}$ and Lemma 2.5, we have

$$0 = \lambda w_1^n + x^{n+1} - x^n + q_1^n,$$

where $w_1^n \in \partial f(y^n, .)(x^{n+1})$ and $q_1^n \in N_{H_n}(x^{n+1})$. From the definition

$$N_{H_n}(x^{n+1}) = \{q \in H : \langle q, y - x^{n+1}\rangle \leq 0, \ \forall y \in H_n\},$$

and Lemma 3.9, it follows that

$$\langle x^n - x^{n+1} - \lambda w_1^n, z - x^{n+1}\rangle \leq 0.$$

Consequently,

$$\langle x^n - x^{n+1}, z - x^{n+1}\rangle \leq \lambda\langle w_1^n, z - x^{n+1}\rangle \leq \lambda(f(y^n, z) - f(y^n, x^{n+1})).$$

We have

$$\begin{aligned}
\|x^{n+1} - z\|^2 &= \|x^n - z\|^2 + \|x^{n+1} - x^n\|^2 + 2\langle x^{n+1} - x^n, x^n - z\rangle \\
&\leq \|x^n - z\|^2 - \|x^{n+1} - x^n\|^2 + 2\langle x^{n+1} - x^n, x^{n+1} - z\rangle \\
&\leq \|x^n - z\|^2 - \|x^{n+1} - x^n\|^2 + 2\lambda(f(y^n, z) - f(y^n, x^{n+1})) \qquad (3.7)\\
&= \|x^n - z\|^2 - \|x^{n+1} - x^n\|^2 + 2\lambda[f(y^{n-1}, y^n) - f(y^{n-1}, x^{n+1})] + \\
&\quad + 2\lambda[f(y^{n-1}, x^{n+1}) - f(y^{n-1}, y^n) - f(y^n, x^{n+1})] + 2\lambda f(y^n, z) \\
&= \|x^n - z\|^2 - \|x^{n+1} - x^n\|^2 + A + B + 2\lambda f(y^n, z),
\end{aligned}$$

where

$$\begin{aligned}
A &= 2\lambda[f(y^{n-1}, y^n) - f(y^{n-1}, x^{n+1})], \\
B &= 2\lambda[f(y^{n-1}, x^{n+1}) - f(y^{n-1}, y^n) - f(y^n, x^{n+1})].
\end{aligned}$$

From $x^{n+1} \in H_n$ we obtain $\langle x^n - \lambda w^n - y^n, x^{n+1} - y^n\rangle \leq 0$, where $w^n \in \partial f(y^{n-1}, .)(y^n)$. Using the definition of the subdifferential we arrive at

$$f(y^{n-1}, y) - f(y^{n-1}, y^n) \geq \langle w^n, y - y^n\rangle, \quad \forall y \in H.$$

Therefore,

$$\begin{aligned}
2\lambda[f(y^{n-1}, x^{n+1}) - f(y^{n-1}, y^n)] &\geq 2\lambda\langle w^n, x^{n+1} - y^n\rangle \\
&\geq 2\langle x^n - y^n, x^{n+1} - y^n\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
A &\leq 2\langle y^n - x^n, x^{n+1} - y^n\rangle \\
&= \|x^{n+1} - x^n\|^2 - \|x^n - y^n\|^2 - \|x^{n+1} - y^n\|^2.
\end{aligned}$$

By the assumption (B4), we get

$$\begin{aligned}B &= 2\lambda[f(y^{n-1},x^{n+1}) - f(y^{n-1},y^n) - f(y^n,x^{n+1})]\\ &\le 2\lambda[c_1\|y^{n-1}-y^n\|^2 + c_2\|y^n - x^{n+1}\|^2].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|y^{n-1}-y^n\|^2 &= \|y^n-x^n\|^2 + \|x^n-y^{n-1}\|^2 + 2\langle y^n-x^n, x^n-y^{n-1}\rangle\\ &\le 2(\|y^n-x^n\|^2 + \|x^n-y^{n-1}\|^2).\end{aligned}$$

So, we obtain

$$B \le 2\lambda[2c_1\|y^n-x^n\|^2 + 2c_1\|x^n-y^{n-1}\|^2 + c_2\|y^n-x^{n+1}\|^2].$$

It implies that

$$\begin{aligned}\|x^{n+1}-z\|^2 \le \|x^n-z\|^2 &- (1-4\lambda c_1)\|x^n-y^n\|^2 - (1-2\lambda c_2)\|x^{n+1}\\ &- y^n\|^2 + 4\lambda c_1\|x^n-y^{n-1}\|^2 + 2\lambda f(y^n,z).\end{aligned} \tag{3.8}$$

It follows from $z \in EP(f)$ and the pseudomonotonicity of f that $f(y^n,z) \le 0$. Then the inequality (3.8) implies

$$\begin{aligned}\|x^{n+1}-z\|^2 \le \|x^n-z\|^2 &- (1-4\lambda c_1)\|x^n-y^n\|^2\\ &- (1-2\lambda c_2)\|x^{n+1}-y^n\|^2 + 4\lambda c_1\|x^n-y^{n-1}\|^2.\end{aligned} \tag{3.9}$$

The proof is complete. □

We are now in a position to give the convergence of the sequence generated by Algorithm 3.7.

Theorem 3.13. *Let C be a nonempty closed convex subset of H. Let $f : H \times H \to \mathbb{R}$ satisfying Assumption 3.1. Assume that $\lambda \in \left(0, \frac{1}{2(2c_1+c_2)}\right)$. Then the sequence $\{x^n\}$ generated by Algorithm 3.7 converges weakly to a solution of the EP (1.1).*

Proof. We split the proof into several steps:

Step 1: We first show the boundedness of the sequence $\{x^n\}$. Let $z \in EP(f)$. The inequality (3.9) can be rewritten as

$$\begin{aligned}\|x^{n+1}-z\|^2 \le \|x^n-z\|^2 &- (1-4\lambda c_1)\|x^n-y^n\|^2\\ &- (1-2\lambda c_2-4\lambda c_1)\|x^{n+1}-y^n\|^2 +\\ &+ 4\lambda c_1\|x^n-y^{n-1}\|^2 - 4\lambda c_1\|x^{n+1}-y^n\|^2.\end{aligned} \tag{3.10}$$

We fix a number $N \in \mathbb{N}$ and consider the inequality (3.10) for all the numbers $N, N+1, ..., M$, where $M > N$. Adding these inequalities, we obtain

$$\|x^{M+1} - z\|^2 \leq \|x^N - z\|^2 - (1 - 4\lambda c_1) \sum_{n=N}^{M} \|x^n - y^n\|^2$$

$$- (1 - 2\lambda c_2 - 4\lambda c_1) \sum_{n=N}^{M} \|x^{n+1} - y^n\|^2$$

$$+ 4\lambda c_1 \|x^N - y^{N-1}\|^2 - 4\lambda c_1 \|x^{M+1} - y^M\|^2 \tag{3.11}$$

$$\leq \|x^N - z\|^2 + 4\lambda c_1 \|x^N - y^{N-1}\|^2. \tag{3.12}$$

The inequality (3.12) leads to the boundedness of $\{x^n\}$. Hence, there exists $\bar{x} \in H$ and a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ such that $x^{n_k} \rightharpoonup \bar{x}$. Moreover, from the inequality (3.11), we obtain the convergence of the series

$$\sum_{n=1}^{\infty} \|x^{n+1} - y^n\|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} \|x^n - y^n\|^2.$$

Thus, we have

$$\lim_{n\to\infty} \|x^{n+1} - y^n\| = \lim_{n\to\infty} \|x^n - y^n\| = 0. \tag{3.13}$$

Step 2: Let us show that $\bar{x} \in EP(f)$. It follows from (3.13) that $y^{n_k} \rightharpoonup \bar{x} \in C$. We have

$$\|y^n - y^{n+1}\| \leq \|y^n - x^{n+1}\| + \|x^{n+1} - y^{n+1}\|. \tag{3.14}$$

Combining (3.13) and (3.14) we deduce that

$$\lim_{n\to\infty} \|y^n - y^{n+1}\| = 0. \tag{3.15}$$

Therefore, we get $y^{n_k+1} \rightharpoonup \bar{x}$. It follows from

$$y^{n+1} = \underset{y\in C}{\operatorname{argmin}} \left\{ \lambda f(y^n, y) + \frac{1}{2}\|y - x^{n+1}\|^2 \right\}$$

and Lemma 2.5 that there exist $w^{n+1} \in \partial f(y^n, .)(y^{n+1})$ and $q^{n+1} \in N_C(y^{n+1})$ such that

$$0 = \lambda w^{n+1} + y^{n+1} - x^{n+1} + q^{n+1}.$$

From the definition of $N_C(y^{n+1})$, we deduce that

$$\langle x^{n+1} - y^{n+1} - \lambda w^{n+1}, y - y^{n+1} \rangle \leq 0, \quad \forall y \in C,$$

or

$$\langle x^{n+1} - y^{n+1}, y - y^{n+1} \rangle \leq \langle \lambda w^{n+1}, y - y^{n+1} \rangle, \quad \forall y \in C.$$

On the other hand, since $w^{n+1} \in \partial f(y^n, .)(y^{n+1})$, we get

$$\langle w^{n+1}, y - y^{n+1} \rangle \leq f(y^n, y) - f(y^n, y^{n+1}), \ \forall y \in C.$$

Hence, we arrive at

$$\frac{\langle x^{n+1} - y^{n+1}, y - y^{n+1} \rangle}{\lambda} \leq f(y^n, y) - f(y^n, y^{n+1}), \quad \forall y \in C. \tag{3.16}$$

Since the left-hand side converges to zero, replacing n in (3.16) by n_k we have by (3.15) and the assumption (B5) that

$$0 \le \limsup_{k\to\infty} f(y^{n_k}, y^{n_k+1}) = \limsup_{k\to\infty} \left(\frac{\langle x^{n_k+1} - y^{n_k+1}, y - y^{n_k+1}\rangle}{\lambda} + f(y^{n_k}, y^{n_k+1}) \right)$$
$$\le \limsup_{k\to\infty} f(y^{n_k}, y), \quad \forall y \in C.$$

Now under the condition (B2), we obtain, $\bar{x} \in EP(f)$.

Step 3: We claim that $x^n \rightharpoonup \bar{x}$. On the contrary, assume that there is a subsequence $\{x^{m_k}\}$ such that $x^{m_k} \rightharpoonup \widetilde{x}$ as $k \to \infty$ and $\bar{x} \neq \widetilde{x}$. Arguing as in Step 2, we also obtain $\widetilde{x} \in EP(f)$. It follows from the inequality (3.10) and the condition $0 < \lambda < \frac{1}{2(2c_1+c_2)}$ that

$$\|x^{n+1} - z\|^2 + 4\lambda c_1 \|x^{n+1} - y^n\|^2 \le \|x^n - z\|^2 + 4\lambda c_1 \|x^n - y^{n-1}\|^2, \quad \forall z \in EP(f).$$

Thus, for all $z \in EP(f)$, the sequence $\{\|x^n - z\|^2 + 4\lambda c_1 \|x^n - y^{n-1}\|^2\}$ must be convergent. From (3.13) we have

$$\lim_{n\to\infty} \|x^n - z\|^2 \in \mathbb{R}, \quad \forall z \in EP(f).$$

By the Lemma 2.4 (Opial lemma), we have

$$\begin{aligned}
\lim_{n\to\infty} \|x^n - \bar{x}\|^2 &= \lim_{k\to\infty} \|x^{n_k} - \bar{x}\|^2 \\
&= \liminf_{k\to\infty} \|x^{n_k} - \bar{x}\|^2 < \liminf_{k\to\infty} \|x^{n_k} - \widetilde{x}\|^2 \\
&= \lim_{k\to\infty} \|x^{n_k} - \widetilde{x}\|^2 = \lim_{n\to\infty} \|x^n - \widetilde{x}\|^2 \\
&= \lim_{k\to\infty} \|x^{m_k} - \widetilde{x}\|^2 = \liminf_{k\to\infty} \|x^{m_k} - \widetilde{x}\|^2 \\
&< \liminf_{k\to\infty} \|x^{m_k} - \bar{x}\|^2 \\
&= \lim_{k\to\infty} \|x^{m_k} - \bar{x}\|^2 = \lim_{n\to\infty} \|x^n - \bar{x}\|^2,
\end{aligned}$$

which is a contradiction. Thus, $\bar{x} = \widetilde{x}$ and this completes the proof. □

Remark 3.14. Concerning Theorem 3.13, we see that

(1) In comparison with [5, 30, 42], the proposed algorithm has the advantage that at each iteration, we have to solve a subproblem over a halfspace (one constraint) instead of the feasible set C which is, in general, complicated (for example, C is described by a finite number of constraints). The efficiency of our method is examined by experimental results in Section 5. Especially for variational inequalities, the proposed algorithm requires only one projection onto the feasible set.

(2) The sequence $\{x^n\}$ generated by Algorithm 3.7 is not Fejér monotone and the proof techniques are based on the papers of Malitsky and Semenov [32] and Lyashko and Semenov [30]. Therefore, our proof techniques are different from those in [5, 18, 19, 42].

(3) An obvious drawback of the studied algorithm is that the condition (B5) is rather strong and finding the constants c_1 and c_2 is not an easy task (analogously, finding the Lipschitz constant L in case of variational inequalities).

Therefore, obtaining a result for the proposed algorithm without using the condition (B5) seems to be more delicate and further investigations are necessary.

From Theorem 3.13, we obtain only weak convergence for the sequence $\{x^n\}$ generated by Algorithm 3.7. If, in addition, $\operatorname{int} EP(f) \neq \emptyset$, then we will hereafter prove that this sequence converges strongly to a solution of EP.

Theorem 3.15. *Suppose that beside the assumptions of Theorem 3.13,* $\operatorname{int} EP(f) \neq \emptyset$ *holds. Then the sequence* $\{x^n\}$ *generated by Algorithm 3.7 converges strongly to a solution of the EP (1.1).*

Proof. Take the sequence $\alpha_n := 4\lambda c_1 \|x^n - y^{n-1}\|^2$. Then from the inequality (3.10) we deduce that

$$\|x^{n+1} - z\|^2 \leq \|x^n - z\|^2 + \alpha_n - \alpha_{n+1} \quad \forall z \in EP(f). \tag{3.17}$$

Now fix an element $u \in \operatorname{int} EP(f)$ and choose $r > 0$ such that $\|v - u\| \leq r$ implies $v \in EP(f)$. Then for any $x^{n+1} \neq x^n$ we have

$$\left\| x^{n+1} - \left(u - r\frac{x^{n+1} - x^n}{\|x^{n+1} - x^n\|} \right) \right\|^2 \leq \left\| x^n - \left(u - r\frac{x^{n+1} - x^n}{\|x^{n+1} - x^n\|} \right) \right\|^2 + \alpha_n - \alpha_{n+1}. \tag{3.18}$$

Simplifying the inequality (3.18), we obtain

$$2r\|x^{n+1} - x^n\| \leq \|x^n - u\|^2 - \|x^{n+1} - u\|^2 + \alpha_n - \alpha_{n+1}. \tag{3.19}$$

Let $M > N$ be arbitrary positive integers. By summing up the inequality (3.19) from N to $M - 1$ we obtain

$$2r\|x^M - x^N\| \leq \|x^N - u\|^2 - \|x^M - u\|^2 + \alpha_N - \alpha_M. \tag{3.20}$$

Taking into account that the sequence $\beta_n := \|x^n - u\|^2 + \alpha_n$ converges, we conclude that the sequence $\{x^n\}$ is Cauchy, therefore, (strongly) convergent.

Since we already showed that each weak cluster point of $\{x^n\}$ is a solution of EP (Step 2 in the proof of Theorem 3.13), the proof is complete. □

In the following theorem we will show that our method has at least R-linear rate of convergence under a strong pseudomonotonicity assumption of f.

Theorem 3.16. *Let C be a nonempty closed convex subset of H. Let $f : H \times H \to \mathbb{R}$ be a bifunction satisfying conditions (B3), (B4), (B6) and be γ-strongly pseudomonotone on C. Assume that $\lambda \in \left(0, \min\left\{\frac{1}{16\gamma}, \frac{3}{16c_1}, \frac{1}{4c_1+2c_2+\gamma}\right\}\right)$. Then the sequence $\{x^n\}$ generated by Algorithm 3.7 converges strongly to the unique solution x^* of the EP (1.1). Moreover, there exist $M > 0$ and $\mu \in (0,1)$ such that*

$$\|x^{n+1} - x^*\| \leq M\mu^n \quad \forall n \geq 1. \tag{3.21}$$

Proof. The uniqueness follows by the strong pseudomonotonicity. Using similar arguments as in the proof of Lemma 3.12, it follows from (3.8) that

$$\begin{aligned}\|x^{n+1}-x^*\|^2 &\le \|x^n-x^*\|^2-(1-4\lambda c_1)\|x^n-y^n\|^2-(1-2\lambda c_2)\|x^{n+1}-y^n\|^2\\ &\quad+4\lambda c_1\|x^n-y^{n-1}\|^2+2\lambda f(y^n,x^*)\\ &\le \|x^n-x^*\|^2-(1-4\lambda c_1)\|x^n-y^n\|^2-(1-2\lambda c_2)\|x^{n+1}-y^n\|^2\\ &\quad+4\lambda c_1\|x^n-y^{n-1}\|^2-2\gamma\lambda\|y^n-x^*\|^2\\ &=(1-2\gamma\lambda)\|x^n-x^*\|^2-(1-4\lambda c_1+2\gamma\lambda)\|x^n-y^n\|^2\\ &\quad-(1-2\lambda c_2)\|x^{n+1}-y^n\|^2+4\lambda c_1\|x^n-y^{n-1}\|^2\\ &\quad-4\gamma\lambda\langle y^n-x^n,x^n-x^*\rangle.\end{aligned}$$

Applying the inequality $-4\gamma\lambda\langle y^n-x^n,x^n-x^*\rangle\le 16\gamma^2\lambda^2\|x^n-x^*\|^2+\frac{1}{4}\|y^n-x^n\|^2$, we obtain

$$\begin{aligned}\|x^{n+1}-x^*\|^2 \le (1-2\gamma\lambda+16\gamma^2\lambda^2)\|x^n-x^*\|^2-\Big(\frac{3}{4}-4\lambda c_1+2\gamma\lambda\Big)\|x^n-y^n\|^2\\ +4\lambda c_1\|x^n-y^{n-1}\|^2-(1-2\lambda c_2)\|x^{n+1}-y^n\|^2.\end{aligned}$$

Taking into account the fact that $\lambda\le\min\left\{\frac{1}{16\gamma},\frac{3}{16c_1}\right\}$, we have

$$\begin{aligned}\|x^{n+1}-x^*\|^2 &\le (1-\gamma\lambda)\|x^n-x^*\|^2+4\lambda c_1\|x^n-y^{n-1}\|^2-(1-2\lambda c_2)\|x^{n+1}-y^n\|^2\\ &=(1-\gamma\lambda)\left(\|x^n-x^*\|^2+\frac{4\lambda c_1}{1-\gamma\lambda}\|x^n-y^{n-1}\|^2\right) \qquad (3.22)\\ &\quad-\frac{4\lambda c_1}{1-\gamma\lambda}\|x^{n+1}-y^n\|^2-\left(1-2\lambda c_2-\frac{4\lambda c_1}{1-\gamma\lambda}\right)\|x^{n+1}-y^n\|^2.\end{aligned}$$

From the condition $\lambda<\frac{1}{4c_1+2c_2+\gamma}$, it is easy to see that $1-2\lambda c_2-\frac{4\lambda c_1}{1-\gamma\lambda}>0$. Hence

$$\|x^{n+1}-x^*\|^2+\frac{4\lambda c_1}{1-\gamma\lambda}\|x^{n+1}-y^n\|^2\le(1-\gamma\lambda)\left(\|x^n-x^*\|^2+\frac{4\lambda c_1}{1-\gamma\lambda}\|x^n-y^{n-1}\|^2\right).$$

Let $\mu:=1-\gamma\lambda\in(0,1)$, $M:=\|x^1-x^*\|^2+\frac{4\lambda c_1}{1-\gamma\lambda}\|x^1-y^0\|^2$. From the last inequality we arrive at

$$\|x^{n+1}-x^*\|^2\le\|x^{n+1}-x^*\|^2+\frac{4\lambda c_1}{1-\gamma\lambda}\|x^{n+1}-y^n\|^2\le M\mu^n.$$

This finishes the proof of Theorem 3.16. □

4. Applications to variational inequalities

If the equilibrium bifunction f is defined by $f(x,y)=\langle Ax,y-x\rangle$ for every $x,y\in C$, with $A:H\to H$, then the equilibrium problem (1.1) reduces to the *variational inequality problem* (VIP):

$$\text{find } x^*\in C \text{ such that } \langle Ax^*,y-x^*\rangle\ge 0,\quad \forall y\in C. \qquad (4.1)$$

The set of solutions of the problem (4.1) is denoted by $VI(C,A)$. In this situation, Algorithm 3.7 reduces to Algorithm 1 of [32], which is known as a subgradient extragradient type algorithm for monotone variational inequalities.

Algorithm 4.1 (Subgradient extragradient type algorithm for the VIP)**.**

Step 1: Specify $x^0, y^0 \in C$ and $\lambda > 0$;

Compute

$$\begin{cases} x^1 = P_C(x^0 - \lambda A y^0), \\ y^1 = P_C(x^1 - \lambda A y^0). \end{cases}$$

Step 2: Given x^n, y^n and y^{n-1}, construct a halfspace

$$H_n = \{z \in H : \langle x^n - \lambda A y^{n-1} - y^n, z - y^n \rangle \leq 0\}.$$

Step 3: Compute

$$\begin{cases} x^{n+1} = P_{H_n}(x^n - \lambda A y^n), \\ y^{n+1} = P_C(x^{n+1} - \lambda A y^n). \end{cases}$$

Step 4: If $x^{n+1} = x^n$ and $y^n = y^{n-1}$ then stop. Otherwise, set $n := n + 1$, and return to Step 2.

To guarantee that f is jointly weakly lower semicontinuous on $H \times H$, the authors in [46] required the weak-to-strong continuity of $A : H \to H$, i.e., A is such that for any sequence $\{x^n\} \subset H$,

$$x^n \rightharpoonup x \Longrightarrow Ax^n \to Ax. \tag{4.2}$$

As indicated in Section 3, we do not need anymore the joint weak lower semicontinuity of f, but only the weak upper semicontinuity of $f(\cdot, y)$. To this end, we remind the following concept for single-valued operators (called F-hemicontinuity in [34]).

Definition 4.2. Let X be a normed space with X^* its dual space and K a closed convex subset of X. The mapping $A : K \to X^*$ is called F-*hemicontinuous* iff for all $y \in K$, the function $x \mapsto \langle A(x), x - y \rangle$ is weakly lower semicontinuous on K (or equivalently, $x \mapsto \langle A(x), y - x \rangle$ is weakly upper semicontinuous on K).

Clearly, any weak-to-strong continuous mapping is also F-hemicontinuous, but vice-versa not, as the following example shows.

Example 4.3 ([24])**.** Consider the Hilbert space $l^2 = \{x = (x^i)_{i \in \mathbf{N}} : \sum_{i=1}^{\infty} |x^i|^2 < \infty\}$ and A: $l^2 \to l^2$ be the identity operator. Take an arbitrary sequence $\{x_n\} \subseteq l^2$ converging weakly to $\overline{x}$. Since the function $x \longmapsto \|x\|^2$ is continuous and convex, it is weakly lower semicontinuous. Hence,

$$\|\overline{x}\|^2 \leq \liminf_{n \to \infty} \|x_n\|^2,$$

which clearly implies

$$\langle \overline{x}, \overline{x} - y \rangle \leq \liminf_{n \to \infty} \langle x_n, x_n - y \rangle,$$

for all $y \in l^2$, i.e., A is F-hemicontinuous.

On the other hand, we take $x_n = e_n = (0, 0, ..., 0, 1, 0, ...)$ with 1 in the n^{th} position. It is obvious that $e_n \rightharpoonup 0$, but $\{e_n\}$ does not have any strongly convergent subsequence, as $\|e_n - e_m\| = \sqrt{2}$ for $m \neq n$. Therefore, A is not weak-to-strong continuous.

Corollary 4.4. *Let C be a nonempty closed convex subset of H. Let $A : H \to H$ be a pseudomonotone, F-hemicontinuous, Lipschitz continuous mapping with constant $L > 0$ such that $VI(C, A) \neq \emptyset$. Let $\{x^n\}$, $\{y^n\}$ be the sequences generated by Algorithm 4.1 with $0 < \lambda < \frac{1}{3L}$. Then the sequences $\{x^n\}$ and $\{y^n\}$ converge weakly to the same point $x^* \in VI(C, A)$.*

Proof. For each pair $x, y \in C$, we define

$$f(x, y) := \langle Ax, y - x\rangle.$$

From the assumptions, it is easy to check that all the conditions of Assumption 3.1 are satisfied. Note that Step 3 of Algorithm 4.1 can be written as

$$\begin{cases} x^{n+1} = \underset{y\in H_n}{\operatorname{argmin}}\left\{\lambda\langle Ay^n, y - y^n\rangle + \dfrac{1}{2}\|y - x^n\|^2\right\}, \\ y^{n+1} = \underset{y\in C}{\operatorname{argmin}}\left\{\lambda\langle Ay^n, y - y^n\rangle + \dfrac{1}{2}\|y - x^{n+1}\|^2\right\}. \end{cases}$$

It follows that

$$\begin{cases} x^{n+1} = \underset{y\in H_n}{\operatorname{argmin}}\left\{\dfrac{1}{2}\|y - (x^n - \lambda Ay^n)\|^2\right\} = P_{H_n}(x^n - \lambda Ay^n), \\ y^{n+1} = \underset{y\in C}{\operatorname{argmin}}\left\{\dfrac{1}{2}\|y - (x^{n+1} - \lambda Ay^n)\|^2\right\} = P_C(x^{n+1} - \lambda Ay^n). \end{cases}$$

By Theorem 3.13, the sequences $\{x^n\}$ and $\{y^n\}$ converge weakly to $x^* \in EP(f)$. It means that the sequences $\{x^n\}$ and $\{y^n\}$ converge weakly to $x^* \in VI(C, A)$. Hence, the result is true and the proof is complete. □

Note that in order to apply Corollary 4.4 one needs to know the Lipschitz constant L. When A is Lipschitz continuous but the Lipschitz constant is unknown, or cannot be calculated easily, we propose the following self-adaptive algorithm.

Algorithm 4.5 (Self-adaptive algorithm for the VIP)**.**

Step 1: Take $x^0, y^0 \in C$, $\mu \in (0, \frac{1}{3})$. Set $n = 0$.
Compute

$$\begin{cases} x^1 = P_C(x^0 - Ay^0), \\ y^1 = P_C(x^1 - Ay^0). \end{cases}$$

Step 2: Given x^n, y^n and y^{n-1} ($n \geq 1$), define

$$\lambda_n = \begin{cases} \mu\frac{\|y^n - y^{n-1}\|}{\|Ay^n - Ay^{n-1}\|}, & \text{if } \|Ay^n - Ay^{n-1}\| \neq 0; \\ 1, & \text{otherwise,} \end{cases} \tag{4.3}$$

and construct a halfspace

$$\mathcal{H}_n = \{z \in H : \langle x^n - \lambda_n Ay^{n-1} - y^n, z - y^n\rangle \leq 0\}.$$

Compute

$$\begin{cases} x^{n+1} = P_{\mathcal{H}_n}(x^n - \lambda_n A y^n), \\ y^{n+1} = P_C(x^{n+1} - \lambda_n A y^n). \end{cases} \tag{4.4}$$

Step 3: If $x^{n+1} = x^n$ and $y^n = y^{n-1}$ then stop. Otherwise, set $n := n + 1$, and return to Step 2.

Remark 4.6. Since A is Lipschitz continuous with constant L, from (4.3), we have

$$\lambda_n \geq \min\{1; \frac{\mu}{L}\}. \tag{4.5}$$

In what follows, we assume that

(C1) A is pseudomonotone on H;
(C2) A is F-hemicontinuous, Lipschitz continuous but the Lipschitz constant L is unknown;
(C3) $VI(C, A) \neq \emptyset$.

We are now in a position to give the weak convergence of the sequence generated by Algorithm 4.5.

Theorem 4.7. *Assume that $A : H \to H$ is a nonlinear mapping satisfying conditions (C1)-(C3). Then the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to the same element of $VI(C, A)$.*

Proof. The proof is divided into two steps:
Step 1: We show that for each $z \in VI(C, A)$, the following inequality holds:

$$\begin{aligned} \left\|x^{n+1} - z\right\|^2 &\leq \|x^n - z\|^2 - (1 - 2\mu) \left\|x^{n+1} - y^n\right\|^2 \\ &\quad - (1 - \mu) \|x^n - y^n\|^2 + \mu \left\|x^n - y^{n-1}\right\|^2. \end{aligned} \tag{4.6}$$

Indeed, since $z \in VI(C, A) \subset \mathcal{H}_n$ and $x^{n+1} = P_{\mathcal{H}_n}(x^n - \lambda_n A y^n)$, the characterization (b) of the metric projection provides

$$\begin{aligned} \left\|x^{n+1} - z\right\|^2 &\leq \|x^n - \lambda_n A y^n - z\|^2 - \left\|x^n - \lambda_n A y^n - x^{n+1}\right\|^2 \\ &= \|x^n - z\|^2 - \left\|x^n - x^{n+1}\right\|^2 - 2\lambda_n \left\langle A y^n, x^{n+1} - z\right\rangle. \end{aligned} \tag{4.7}$$

It follows from the pseudomonotonicity of A and $z \in VI(C, A)$ that $\langle A y^n, y^n - z\rangle \geq 0$. Adding this term to the right hand side of (4.7) we obtain

$$\begin{aligned} \left\|x^{n+1} - z\right\|^2 &\leq \|x^n - z\|^2 - \left\|x^n - x^{n+1}\right\|^2 - 2\lambda_n \left\langle A y^n, x^{n+1} - y^n\right\rangle \\ &= \|x^n - z\|^2 - \|x^n - y^n\|^2 - \left\|x^{n+1} - y^n\right\|^2 \\ &\quad - 2 \left\langle x^n - y^n, y^n - x^{n+1}\right\rangle - 2\lambda_n \left\langle A y^n, x^{n+1} - y^n\right\rangle \\ &= \|x^n - z\|^2 - \|x^n - y^n\|^2 - \left\|x^{n+1} - y^n\right\|^2 \\ &\quad + 2\lambda_n \left\langle A y^{n-1} - A y^n, x^{n+1} - y^n\right\rangle \\ &\quad + 2 \left\langle x^n - \lambda_n A y^{n-1} - y^n, x^{n+1} - y^n\right\rangle. \end{aligned}$$

Since $x^{n+1} \in \mathcal{H}_n$, we have $\left\langle x^n - \lambda_n A y^{n-1} - y^n, x^{n+1} - y^n \right\rangle \leq 0$. The fourth term of the above inequality is estimated as follows:

$$\begin{aligned}2\lambda_n \left\langle Ay^{n-1} - Ay^n, x^{n+1} - y^n \right\rangle &\leq 2\mu \left\|y^{n-1} - y^n\right\| \left\|x^{n+1} - y^n\right\| \\ &\leq 2\mu \left(\left\|y^{n-1} - x^n\right\| + \|x^n - y^n\|\right) \left\|x^{n+1} - y^n\right\| \\ &\leq \mu \left(\left\|y^{n-1} - x^n\right\|^2 + 2\left\|x^{n+1} - y^n\right\|^2 + \|x^n - y^n\|^2\right).\end{aligned}$$

Therefore, we get the desired inequality (4.6).

Step 2: We prove that the sequence $\{x^k\}$ converges weakly to $\bar{x} \in VI(C, A)$.

First, it follows from (4.6) that

$$\begin{aligned}\left\|x^{n+1} - z\right\|^2 &\leq \|x^n - z\|^2 - (1 - 3\mu)\left\|x^{n+1} - y^n\right\|^2 \\ &\quad - (1-\mu)\|x^n - y^n\|^2 + \mu\left\|x^n - y^{n-1}\right\|^2 - \mu\left\|x^{n+1} - y^n\right\|^2. \end{aligned} \tag{4.8}$$

We fix a number $N \in \mathbb{N}$ and consider the inequality (4.8) for all the numbers $N, N+1, ..., M$, where $M > N$. Adding these inequalities, we obtain

$$\begin{aligned}\|x^{M+1} - z\|^2 &\leq \|x^N - z\|^2 - (1-\mu)\sum_{n=N}^{M} \|x^n - y^n\|^2 \\ &\quad - (1-3\mu)\sum_{n=N}^{M} \|x^{n+1} - y^n\|^2 \\ &\quad + \mu\|x^N - y^{N-1}\|^2 - \mu\|x^{M+1} - y^M\|^2 \\ &\leq \|x^N - z\|^2 + \mu\|x^N - y^{N-1}\|^2.\end{aligned}$$

Thus we obtain the boundedness of the sequences $\{x^n\}$, $\{y^n\}$, and then, as in Step 1 in the proof of Theorem 3.13,

$$\lim_{n\to\infty} \|x^{n+1} - y^n\| = \lim_{n\to\infty} \|x^n - y^n\| = 0. \tag{4.9}$$

Consequently,

$$\lim_{n\to\infty} \|y^{n+1} - y^n\| = 0. \tag{4.10}$$

We now choose a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ such that $x^{n_k} \rightharpoonup \bar{x}$. By (4.9), $y^{n_k} \rightharpoonup \bar{x}$ and $\bar{x} \in C$. For all $x \in C$ with allowance for characterization (a) of the metric projection and (4.4), we get

$$\left\langle y^{n_k+1} - x^{n_k+1} + \lambda_{n_k} A y^{n_k}, y - y^{n_k+1} \right\rangle \geq 0, \quad \forall y \in C.$$

Hence,

$$\begin{aligned}0 \leq \left\langle y^{n_k+1} - x^{n_k+1} + \lambda_{n_k} A y^{n_k}, y - y^{n_k+1} \right\rangle &= \left\langle y^{n_k+1} - x^{n_k+1}, y - y^{n_k+1} \right\rangle \\ + \lambda_{n_k} \left\langle A y^{n_k}, y^{n_k} - y^{n_k+1} \right\rangle &+ \lambda_{n_k} \left\langle A y^{n_k}, y - y^{n_k} \right\rangle, \quad \forall y \in C.\end{aligned}$$

Dividing both sides of the last inequality by λ_{n_k} we get

$$\begin{aligned}0 \leq &\frac{\left\langle y^{n_k+1} - x^{n_k+1}, y - y^{n_k+1} \right\rangle}{\lambda_{n_k}} \\ &+ \left\langle A y^{n_k}, y^{n_k} - y^{n_k+1} \right\rangle + \left\langle A y^{n_k}, y - y^{n_k} \right\rangle, \quad \forall y \in C.\end{aligned}$$

Passing to the limit for k tending to ∞ in the above inequality and using weak lower semicontinuity of the function $x \mapsto \langle Ax, x-y\rangle$ together with (4.5), (4.9) and (4.10) we obtain

$$\langle A\bar{x}, y-\bar{x}\rangle \geq 0, \quad \forall y \in C,$$

i.e., $\bar{x} \in VI(C, A)$.

Now, in view of (4.8) we have

$$\left\|x^{n+1}-z\right\|^2 + \mu\left\|x^{n+1}-y^n\right\|^2 \leq \|x^n - z\|^2 + \mu\left\|x^n - y^{n-1}\right\|^2. \tag{4.11}$$

Therefore the sequence $\{\|x^n - z\|^2 + \mu\left\|x^n - y^{n-1}\right\|^2\}$ is convergent. From (4.9) we have

$$\lim_{n\to\infty} \|x^n - z\|^2 \in \mathbb{R}.$$

We claim that $x^n \rightharpoonup \bar{x}$. On the contrary, assume that there is a subsequence $\{x^{m_k}\}$ such that $x^{m_k} \rightharpoonup \widetilde{x}$ as $k \to \infty$ and $\bar{x} \neq \widetilde{x}$. Using the same argument as above, we also obtain $\widetilde{x} \in VI(C, A)$. Finally, as in Step 3 of Theorem 3.13, we obtain $x^n \rightharpoonup \bar{x}$ by Lemma 2.4 (Opial's lemma). The convergence of $\{y^n\}$ to $\bar{x}$ is guaranteed by (4.9). This finishes the proof of Theorem 4.7. □

5. Numerical results

In this section, we present some numerical results to test Algorithm 3.7. The MATLAB codes are run on a PC (with Intel®Core2$^{\text{TM}}$ Quad Processor Q9400 2.66Ghz 4GB Ram) under MATLAB Version 7.11 (R2010b). Some comparisons are also reported.

Example 5.1 (Nash-Cournot equilibrium models of electricity markets)**.** In this example, we apply the proposed algorithm to a Cournot-Nash equilibrium model of electricity markets (see [11, 41] for more details). In this model, it is assumed that there are three electricity companies i ($i = 1, 2, 3$). Each company i owns several generating units with index set I_i. In this example, suppose that $I_1 = \{1\}$, $I_2 = \{2, 3\}$, $I_3 = \{4, 5, 6\}$. Let x_j be the power generation of unit j ($j = 1, \dots, 6$) and assume that the electricity price p can be expressed by:

$$p = 378.4 - 2\sum_{j=1}^{6} x_j.$$

The cost of a generating unit j is defined as $c_j(x_j) := \max\{\hat{c}_j(x_j), \bar{c}_j(x_j)\}$ with

$$\hat{c}_j(x_j) := \frac{\hat{\alpha}_j}{2}x_j^2 + \hat{\beta}_j x_j + \hat{\gamma}_j$$

and

$$\bar{c}_j(x_j) := \bar{\alpha}_j x_j + \frac{\bar{\beta}_j}{\bar{\beta}_j + 1}\bar{\gamma}_j^{-1/\bar{\beta}_j}(x_j)^{(\bar{\beta}_j+1)/\bar{\beta}_j},$$

where the parameters $\hat{\alpha}_j$, $\hat{\beta}_j$, $\hat{\gamma}_j$, $\bar{\alpha}_j$, $\bar{\beta}_j$ and $\bar{\gamma}_j$ are given in Table 1.

Suppose that the profit of the company i is given by

$$f_i(x) := p\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j) = \left(378.4 - 2\sum_{l=1}^{6} x_l\right)\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j),$$

j	$\hat{\alpha}_j$	$\hat{\beta}_j$	$\hat{\gamma}_j$	$\bar{\alpha}_j$	$\bar{\beta}_j$	$\bar{\gamma}_j$
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

TABLE 1. The parameters used in Example 5.1

where $x = (x_1, \dots, x_6)^T$ subject to the constraint $x \in C$,

$$C := \left\{x \in \mathbb{R}^6 : x_j^{\min} \le x_j \le x_j^{\max}\right\},$$

with $x_j^{\min}$ and $x_j^{\max}$ given in Table 2.

j	1	2	3	4	5	6
$x_j^{\min}$	0	0	0	0	0	0
$x_j^{\max}$	80	80	50	55	30	40

TABLE 2. The parameters used in Example 5.1

We define the equilibrium bifunction f by

$$f(x,y) := \sum_{i=1}^{3} \left(\varphi_i(x,x) - \varphi_i(x,y)\right),$$

where

$$\varphi_i(x,y) := \left[378,4 - 2\left(\sum_{j\notin I_i} x_j + \sum_{j\in I_i} y_j\right)\right] \sum_{j\in I_i} y_j - \sum_{j\in I_i} c_j(y_j).$$

Then the Nash-Cournot equilibrium models of electricity markets can be reformulated as an equilibrium problem (see [28]):

$$(EP(f,C)) \qquad \text{find } x^* \in C \text{ such that } f(x^*,y) \ge 0,\ \forall y \in C.$$

Similar to [41], we rewrite the function f as

$$f(x,y) = \langle (A+B)x + By + a, y - x\rangle + c(y) - c(x), \tag{5.1}$$

where

$$A := 2\sum_{i=1}^{3} \bar{q}^i (q^i)^T,\ B := 2\sum_{i=1}^{3} q^i (q^i)^T,\ a := -378.4\sum_{i=1}^{3} q^i,\ c(x) := \sum_{j=1}^{6} c_j(x_j).$$

Here the vectors $q^i := (q_1^i, \dots, q_6^i)$ and $\bar{q}^i := (\bar{q}_1^i, \dots, \bar{q}_6^i)$ are defined by

$$q_j^i = \begin{cases} 1, & \text{if } j \in I_i \\ 0, & \text{if } j \notin I_i \end{cases}$$

and $\bar{q}^i_j = 1 - q^i_j$ for all $i = 1, 2, 3$ and $j = 1, \ldots, 6$. However, the function f defined by (5.1) is not pseudomonotone. Thanks to Lemma 7 in [41], the problem $EP(f, C)$ is equivalent to the problem $EP(f_1, C)$ where the function f_1 is given by

$$f_1(x, y) = \langle A_1 x + B_1 y + a, y - x\rangle + c(y) - c(x),$$

with $A_1 := A + \frac{3}{2}B$ and $B_1 := \frac{1}{2}B$. It is easy seen that the function f_1 satisfies all conditions in Assumption 3.1. We will apply the algorithm 3.7 to solve the problem $EP(f_1, C)$. Choose $\lambda = 0.02$, $x^0 = (0, \ldots, 0)^T$ and the stopping criteria $\|x^{n-1} - x^n\| < 10^{-4}$. The results are tabulated in Table 3.

Iter.	x_1^n	x_2^n	x_3^n	x_4^n	x_5^n	x_6^n
0	0	0	0	0	0	0
1	7.2329	6.9704	6.9729	6.6977	6.6976	6.6976
2	11.1446	10.4950	10.4936	9.8546	9.8519	9.8519
3	14.8503	13.7060	13.6949	12.6240	12.6166	12.6166
4	17.7731	16.0636	16.0387	14.5041	14.4906	14.4906
5	20.2529	17.9295	17.8874	15.8785	15.8578	15.8578
6	22.3430	19.3752	19.3134	16.8342	16.8056	16.8056
7	24.1385	20.5089	20.4254	17.4901	17.4531	17.4531
8	25.6973	21.3988	21.2920	17.9217	17.8760	17.8760
9	27.0678	22.1005	21.9693	18.1894	18.1347	18.1347
...	...	...	...	...	...	...
3568	46.6551	32.1196	15.0304	23.4718	11.6675	11.6675

TABLE 3. The results of Algorithm 3.7 in Example 5.1

The approximate solution obtained after 3568 iterations is

$$x^* = (46.6551 \;\; 32.1196 \;\; 15.0304 \;\; 23.4718 \;\; 11.6675 \;\; 11.6675)^T$$

We note that this result is slightly different from the ones obtained by algorithms 1 and 2 in [41]. To check the accuracy of these algorithms, we will use the quantity $\|x^* - prox_{f_1}(x^*)\|$, where $prox_{f_1}$ is the proximity operator of f_1, i.e.,

$$prox_{f_1} : \mathbb{R}^6 \to \mathbb{R}^6, \; prox_{f_1}(x) := \operatorname{argmin}\left\{\lambda f_1(x, y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\}.$$

It is easy to see that x^* is a solution of problem $EP(f_1, C)$ if and only if $x^* = prox_{f_1}(x^*)$. Hence, the accuracy of the algorithm is higher once the value of $\|x^* - prox_{f_1}(x^*)\|$ is smaller. Choosing $\lambda = 0.05$, the comparison results are reported in Table 4.

	Algorithm 3.7	Algorithm 1 [41]	Algorithm 2 [41]
$\|x^* - prox_{f_1}(x^*)\|$	0.0026	0.0088	0.0915
No. iter.	3568	4416	6850

TABLE 4. The accuracy of the three algorithms in Example 5.1.

From Table 4, we observe that in this example, the result obtained by our algorithm is more accurate than by Algorithms 1 and 2 in [41] even the new algorithm requires fewer iterations.

Example 5.2. In this example, we compare the performance of the proposed algorithm with the extragradient algorithm given in [42] and with Popov's extragradient algorithm [30]. Let $H = \mathbb{R}^p$ and

$$f : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R},\ f(x, y) = \langle Ax + By, y - x\rangle\ \forall x, y \in \mathbb{R}^p,$$

where A and B are matrices defined by $B = M^T.M + pI$, $A = B + N^T.N + 2pI$; M and N are $p \times p$ matrices; $M(i, j)$ and $N(i, j) \in (0, 1)$ are randomly generated and I is the identity matrix. The feasible set C is $C := \{x \in \mathbb{R}^p : Dx \leq d\}$, where D is a $m \times p$ matrix and $d = (d_1, \ldots, d_m)^T$ with $D(i, j),\ d_i \in (0, 1)$ randomly generated for all $i = 1, \ldots, m, j = 1, \ldots, p$.

We can see that all conditions in Assumption 3.1 are satisfied and the equilibrium problem (1.1) has a unique solution $x^* = (0, \ldots, 0)^T$. We will apply Algorithm 3.7 (Alg. 3.7), the extragradient method described in (1.3) (EGM) and Popov's extragradient algorithm [30] (Popov's Alg.) to solve this problem. To run these three algorithms, we use the same parameter $\lambda = \frac{1}{2(\|A\|+\|B\|)+4}$, the same starting point x^0, which is randomly generated and the same stopping criteria $\|x^n - x^*\| < 10^{-3}$.

We have generated some random samples with different choices of m and p. The results are tabulated in Table 5.

	EGM		Popov's Alg.		Alg. 3.7	
	CPU times (s)	Iter.	CPU times (s)	Iter.	CPU times (s)	Iter.
$p = 30, m = 20$	1.2563	96	1.3752	96	1.1430	97
$p = 30, m = 30$	2.0369	100	1.9203	100	1.3180	102
$p = 50, m = 20$	3.6452	155	3.2950	155	2.6257	157
$p = 50, m = 30$	4.1871	154	3.7750	154	3.0050	156
$p = 50, m = 50$	5.9796	150	5.0016	150	4.1230	152
$p = 50, m = 100$	6.0657	148	5.6674	148	4.2408	151
$p = 50, m = 200$	9.3348	137	8.6526	138	6.1459	141
$p = 50, m = 500$	9.5166	135	8.8857	137	5.8242	142
$p = 100, m = 100$	30.0713	299	29.5453	299	20.7186	303
$p = 100, m = 200$	38.1460	294	37.3275	294	24.8415	299
$p = 100, m = 500$	55.2270	274	52.9948	275	33.5809	281
$p = 100, m = 1000$	70.3557	260	58.5535	263	34.5569	270

TABLE 5. Comparison of the three algorithms in Example 5.2.

From Table 5, we observe that the time consumed by Algorithm 3.7 is less than that of (EGM) and of Popov's extragradient algorithm, even when the proposed algorithm requires more iterations. It is clearly shown that our algorithm performs better than the two known algorithms, especially when the function f and the feasible set C are more complicated (when m and p are large). This happens because as mentioned in Remark 3.14, at each iteration of Algorithm 3.7, we solve one subprogram over a halfspace (one constraint) instead of the feasible set C (many constraints) as in the considered two algorithms.

It is worth noting that our proposed algorithm computes the value of the bifunction f in the first argument only one time at each iteration, and hence, the new method is very effective when the equilibrium bifunction f is complicated and computationally expensive. We will illustrate this advantage by the following example.

Example 5.3. Consider the problem (1.1) where

$$f : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R},\ f(x,y) = \langle Ax, y-x\rangle \tag{5.2}$$

and $C := \{(x_1, \dots, x_p) \in \mathbb{R}^p : \sum_{i=1}^{p} x_i = 0\}$. Here A is the proximity operator of the function $g(x) = \|x\|^4$, namely,

$$A : \mathbb{R}^p \to \mathbb{R}^p,\ A(x) := \operatorname{argmin}\left\{\|y\|^4 + \frac{1}{2}\|y-x\|^2 : y \in \mathbb{R}^p\right\},\ \forall x \in \mathbb{R}^p.$$

We note that $A(x)$ does not have a closed form and it can be computed efficiently, for example, by the MATLAB Optimization Toolbox, however, its computation is expensive. It is easy to see that all the conditions of Assumption 3.1 are satisfied and the problem (1.1) has a unique solution $x^* = (0, \dots, 0)^T$. We will apply Algorithm 3.7 (Alg. 3.7), the extragradient method (1.3) (EGM) and the subgradient extragradient method (SEM) in [9] to solve this problem. As in Example 5.2, we use the same parameter $\lambda = 0.1$, the stopping criteria $\|x^n - x^*\| < 10^{-4}$ and the same starting point x^0, which is randomly generated.

	Ex. 1		Ex. 2		Ex. 3	
	CPU times (s)	Iter.	CPU times (s)	Iter.	CPU times (s)	Iter.
EGM	9.9265	136	9.7181	136	9.5595	134
SEM	8.8113	136	8.5276	136	8.6867	134
Alg. 3.7	4.3703	136	3.3658	89	3.5988	133

TABLE 6. Comparison of the three algorithms in Example 5.3: the case $p = 100$

In our experiments, we test the different choices of x^0 for both cases $p = 100$ and $p = 500$. The comparison results are given in Tables 6 and 7. As shown is

	Ex. 1		Ex. 2		Ex. 3	
	CPU times (s)	Iter.	CPU times (s)	Iter.	CPU times (s)	Iter.
EGM	65.1697	171	65.5159	171	68.6349	173
SEM	64.4563	172	66.3343	171	68.5598	173
Alg. 3.7	31.8188	172	32.6529	171	33.0386	173

TABLE 7. Comparison of the three algorithms in Example 5.3: the case $p = 500$

this example, our algorithm has much lower time consumption than the other two, although the number of iterations in all of them is almost the same.

To investigate the effect of the step size λ, we implement the three methods, using different step sizes. The results are reported in Table 8. We can see that the number of iterations and computational time depend crucially on the step size.

	Alg. 3.7		SEM		EGM	
	CPU times (s)	Iter.	CPU times (s)	Iter.	CPU times (s)	Iter.
$\lambda = 0.01$	88.1636	1438	180.1058	1439	180.2931	1439
$\lambda = 0.05$	17.8215	289	34.9525	289	35.0651	289
$\lambda = 0.1$	8.3393	144	17.1418	145	16.9643	145
$\lambda = 0.2$	4.9168	79	9.7717	80	9.4802	80
$\lambda = 0.3$	3.0769	54	6.5293	56	6.3676	56

TABLE 8. Performance of the three algorithms with different step sizes.

Next, we compare the performance of Algorithm 4.5 with the modified projection-type method (MPM) (Algorithm 3.2 in [43]). The parameters are chosen as follows.

- In Algorithm 4.5, $\mu = \frac{1}{4}$;
- In MPM, P is the identity matrix, $\theta = 1.5$, $\rho = 0.5$, $\beta = 0.9$, $\alpha = 1$.

We apply the two algorithms for solving (5.2) with the same starting point x^0, which is randomly generated and use the same stopping rule $\|x^n - x^*\| < 10^{-4}$. The results are presented in Table 9. As we can see from this table, the computational time of MPM is much greater than that of our method. This happens because at each iteration of the modified projection method, to find the largest $\alpha \in \{\alpha_{i-1}, \alpha_{i-1}\beta, \alpha_{i-1}\beta^2, \ldots\}$ satisfying

	MPM		Algorithm 4.5	
	CPU tims (s)	Iter.	CPU times (s)	Iter.
n=3	2.5911	54	0.5307	38
n=10	11.1413	123	0.5954	38
n=50	16.3885	141	0.6245	38
n=100	41.1386	162	0.9394	39
n=200	79.7956	201	1.5275	40

TABLE 9. Comparison of Algorithm 4.5 with the Modified projection method in [43].

$$\alpha(x^i - z(\alpha))^T(A(x^i) - A(z^i(\alpha))) \leq (1-\rho)\|x^i - z^i(\alpha)\|^2,$$

we have to compute the value of the mapping A many times. As we noted, this procedure is computationally very expensive.

6. Conclusions

This paper deals with the convergence analysis and some numerical examples of a Popov type subgradient extragradient method for pseudomonotone equilibrium problems in Hilbert spaces. The proposed algorithm is an equilibrium version of a recent algorithm introduced by Malitsky and Semenov [32] (for variational inequalities) and an improved form of Lyashko and Semenov's algorithm in [30]. Moreover, our algorithm is convergent under a weaker condition than the joint weak lower semicontinuity of the bifunction, assumed in several papers before. Numerical results show that the algorithm performs better than some existing methods. An unsolved question is whether is possible to discard the equation (3.3) in the process of constructing the halfspaces and thus avoid the calculation of normal cones. Future research will focus on projection techniques in order to avoid this equation.

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References

[1] M. Abbas, M. AlShahrani, Q. H. Ansari, O. S. Iyiola and Y. Shehu, *Iterative methods for solving proximal split minimization problems*, Numer. Algor. **78** (2018), 193–215.

[2] S. Alizadeh and F. Moradlou, *A strong convergence theorem for equilibrium problems and generalized hybrid mappings*, Mediterr. J. Math. **13** (2016), 379–390.

[3] Q. H. Ansari, A. Rehan and C.-F. Wen, *Implicit and explicit algorithms for split common fixed point problems*, J. Nonlinear Convex Anal. **17** (2016), 1381–1397.

[4] Q. H. Ansari, A. Rehan and J.-C. Yao, *Split feasibility and fixed point problems for asymptotically k-strict pseudo-contractive mappings in intermediate sense*, Fixed Point Theory **18** (2017), 57–68.

[5] A. S. Antipin, *The convergence of proximal methods to fixed points of extremal mappings and estimates of their rate of convergence*, Comput. Math. Math. Phys. **35** (1995), 539–551.

[6] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl. **90** (1996), 31–43.

[7] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[8] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[9] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2011), 318–335.

[10] T. Chamnarnpan, S. Phiangsungnoen and P. Kumam, *A new hybrid extragradient algorithm for solving the equilibrium and variational inequality problems*, Afr. Mat. **26** (2015), 87-98.

[11] J. Contreras, M. Klusch and J. B. Krawczyk, *Numerical solutions to Nash-Cournot equilibria in coupled constraint electricity markets*, IEEE Trans. Power Syst. **19** (2004), 195–206.

[12] L. P. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[13] B. V. Dinh and L. D. Muu, *A projection algorithm for solving pseudomonotone equilibrium problems and it's application to a class of bilevel equilibria*, Optimization **64** (2015), 559–575.

[14] N. T. P. Dong, J. J. Strodiot, N. T. T. Van and V. H. Nguyen, *A family of extragradient methods for solving equilibrium problems*, J. Ind. Manag. Optim. **11** (2015), 619–630.

[15] F. Giannessi, *Vector variational inequalities and vector equilibria*, Mathematical Theories. Nonconvex Optimization and its Applications, 38. Kluwer Academic Publishers, Dordrecht, 2000.

[16] N. Hadjisavvas, S. Komlósi and S. Schaible, *Handbook of Generalized Convexity and Generalized Monotonicity*, Springer, Berlin, 2005.

[17] D. V. Hieu, *Cyclic subgradient extragradient methods for equilibrium problems*, Arab. J. Math. **5** (2016), 159–175.

[18] D. V. Hieu, *Weak and strong convergence of subgradient extragradient methods for pseudomonotone equilibrium problems*, Commun. Korean Math. Soc. **31** (2016), 879–893.

[19] D. V. Hieu, L. D. Muu and P. K. Anh, *Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings*, Numer. Algor. **73** (2016), 197–217.

[20] Z. Jouymandi and F. Moradlou, *Extragradient methods for solving equilibrium problems, variational inequalities and fixed point problems*, Numer. Funct. Anal. Optim. **38** (2017), 1391–1409.

[21] Z. Jouymandi and F. Moradlou, *Retraction algorithms for solving variational inequalities, pseudomonotone equilibrium problems and fixed point problems in Banach spaces*, Numer. Algor. (2017), doi: 10.1007/s11075-017-0417-7

[22] Z. Jouymandi and F. Moradlou, *Extragradient methods for split feasibility problems and generalized equilibrium problems in Banach spaces*, Math. Meth. Appl. Sci. **41** (2018), 826–838.

[23] S. Karamardian, *Complementarity problems over cones with monotone and pseudomonotone maps*, J. Optim. Theory Appl. **18** (1976), 445–454.

[24] G. Kassay and M. Miholca, *Existence results for variational inequalities with surjectivity consequences related to generalized monotone operators*, J. Optim. Theory Appl. **159** (2013), 721–740.

[25] G. Kassay, M. Miholca and N. T. Vinh, *Vector quasi-equilibrium problems for the sum of two multivalued mappings*, J. Optim. Theory Appl. **169** (2016), 424–442.

[26] H. Khatibzadeh and V. Mohebbi, *Proximal point algorithm for infinite pseudo-monotone bifunctions*, Optimization **65** (2016), 1629–1639.

[27] D. Kinderlehrar and G. Stampacchia, *An Introduction to Variational Inequality and Their Application*, Academic Press: New York, 1980.

[28] I. V. Konnov, *Combined relaxation methods for variational inequalities*, Springer, Berlin, 2000.

[29] G. M. Korpelevich, *An extragradient method for finding saddle points and for other problems*, Ekonom. i Mat. Metody **12** (1976), 747–756.

[30] S. I. Lyashko and V. V. Semenov, *A new two-step proximal algorithm of solving the problem of equilibrium programming*, In: Optimization and Applications in Control and Data Sciences (ed. B.Goldengorin), Springer Optimization and Its Applications, vol. 115, 2016, pp. 315–326.

[31] Y. Malitsky, *Projected reflected gradient methods for monotone variational inequalities*, SIAM J. Optim. **25** (2015), 502–520.

[32] Y. V. Malitsky and V. V. Semenov, *An extragradient algorithm for monotone variational inequalities*, Cybern. Syst. Anal. **50** (2014), 271–277.

[33] G. Mastroeni, *On auxiliary principle for equilibrium problems.* in: Equilibrium problems and variational models, P. Daniele, F. Giannessi, A. Maugeri (eds.), Norwell: Kluwer Academic, 2003, pp. 289–298.

[34] A. Maugeri and F. Raciti, *On existence theorems for monotone and nonmonotone variational inequalities*, J. Convex Anal. **16** (2009), 899–911.

[35] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341-346.

[36] J. J. Moreau, *Fonctions convexes duales et points proximaux dans un espace hilbertien*, C. R. Acad. Sci. Paris **255** (1962), 2897–2899.

[37] U. Mosco, *Implicit variational problems and quasi variational inequalities*, in: Nonlinear Operators and the Calculus of Variations, Lecture Notes in Mathematics, J. P. Gossez, E. J. L. Dozo, J. Mawhin, L, Waelbroeck (eds.), vol. 543, Springer, Berlin, 1976, pp. 83–156.

[38] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Am. Math. Soc. **73** (1967), 591–597.

[39] L. Popov, *A modification of the Arrow-Hurwicz method for search for saddle points*, Math. Notes **28** (1980), 845–848.

[40] S. Plubtieng and R. Punpaeng, *A general iterative method for equilibrium problems and fixed points problems in Hilbert spaces*, J. Math. Anal. Appl. **336** (2007), 455–469.

[41] T. D. Quoc, P. N. Anh and L. D. Muu, *Dual extragradient algorithms to equilibrium Problems*, J. Glob. Optim. **52** (2012), 139–159.

[42] T. D. Quoc, L. D. Muu and V. H. Nguyen, *Extragradient algorithms extended to equilibrium problems*, Optimization **57** (2008), 749–776.

[43] M. V. Solodov and P. Tseng, *Modified projection-type methods for monotone variational inequalities*, SIAM J. Control Optim. **34** (1996), 1814–1830.

[44] J. J. Strodiot, P. T. Vuong and N. T. T. Van, *A class of shrinking projection extragradient methods for solving non-monotone equilibrium problems in Hilbert spaces*, J. Global Optim. **64** (2016), 159–178.

[45] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), 1025–1033.

[46] P. T. Vuong, J. J. Strodiot and V. H. Nguyen, *On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space*, Optimization **64** (2015), 429–451.

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