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KKT OPTIMALITY CONDITIONS IN NON-SMOOTH, NON-CONVEX OPTIMIZATION

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Abstract. This paper is devoted to the study of non-smooth optimization problems with inequality constraints without the presence of convexity of objective function, of constraint functions and of feasible set. We present necessary and sufficient KKT optimality conditions for these problems in terms of tangential subdifferentials. Our results contain and improve some recent ones in the literature. Many examples are also given to explain the advantages of our main results.

1. Introduction

Optimization is an amazing subject in areas as diverse as accounting, computer science and engineering for identifying better solutions by utilizing a scientific and mathematical technique. A standard mathematical formulation of an optimization problem is :

$$\text{(P)} \qquad \min f(x), \qquad \text{subject to } x \in K,$$

where f is a real valued function on $\mathbb{R}^n$, and $K \subseteq \mathbb{R}^n$ is a feasible set. A special class of mathematical optimization problems is *convex optimization* which simply means the problem of minimizing a convex function f over a convex feasible set K, which often described by convex inequality constraints

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \dots, m\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \dots, m$, are convex functions.

It is known that the Karush-Kuhn-Tucker (KKT) conditions are first-order necessary conditions for a solution in (P) to be optimal, provided that some constraint qualifications are satisfied, such as Slater's constraint qualification condition[1].

In 2010, Lasserre [6] obtained the interesting result that, in the case of differentiable problems fulfilling the Slater constraint qualification and a mild nondegeneracy condition and additionally if f is a convex function then a point $x \in K$ is a minimizer if and only if x is a KKT point. Motivated by this, Dutta and Lalitha [2] extended the mentioned result to a non-smooth scenario involving the locally Lipschitz function. They considered the convex feasible set K to be defined

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[1]Slater's condition holds if there exists $x \in \mathbb{R}^n$ such that $g_i(x) < 0$ for all $i = 1, \dots, m$

by inequality constraints involving non-smooth locally Lipschitz functions that are regular. In 2015, Martínez-Legaz [8] showed that the convexity assumption on the objective function, considered in [2, 6], can be relaxed to pseudoconvexity, a condition which is not required for the necessity of the KKT conditions, but only for their sufficiency. This observation has also been made, in a differentiable context, by Giorgi in a recent paper [3]. Admittedly, the class of tangentially convex functions collapses to the class of regularly locally Lipschitz functions and of differentiable functions. Very recently, Ho [4] have presented some necessary and sufficient KKT optimality conditions by using the convexity of the level sets of the given function f without the convexity of the feasible set and of the functions f and each g_i. More precisely, it is proved in [4], in the case of the differentiable problem fulfilling Slater's condition, a non-degeneracy condition at the point $x \in K$, and the following additional condition at x:

$$\forall y \in K,\ \exists t_n \downarrow 0 \text{ such that } x + t_n(y - x) \in K, \tag{1.1}$$

that x is a minimizer if and only if the following two conditions are satisfied:

(i) x is a *non-trivial KKT point*[2], and
(ii) the strict level sets of f at x, given by $L_f^<(x) := \{y \in \mathbb{R}^n : f(y) < f(x)\}$, is convex.

It is natural to ask whether Ho's result can be extended to non-differentiable case. The present paper provides an affirmative answer to this question. More precisely, we will show that when Slater's condition holds and a non-degeneracy condition holds at the feasible point x without both the convexity and differentiability of f and g_i as well as the convexity of the feasible set K, the KKT necessary optimality condition becomes globally sufficient provided the set $L_f^<(x)$ is convex and the additional condition $x \in \mathrm{cl}L_f^<(x)$ is satisfied. It is remarkable that the condition $x \in \mathrm{cl}L_f^<(x)$ can be absent in the differentiable case. Our results contain and improve some recent ones in the literature. Many examples are also given to explain the advantages of our main results.

The rest of the paper is organized as follows: in the next section, we give the basic concepts and notations. In Section 3, we study the non-smooth optimization problems with inequality constraints without the presence of convexity of objective function, of constraint functions and of feasible set. We also present necessary and sufficient KKT optimality conditions for these problems in terms of tangential subdifferentials.

2. Preliminaries

In this section, we briefly overview some notations, basic definitions, and preliminary results which will be used throughout this paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. The *open ball* in $\mathbb{R}^n$ with center a and radius $r > 0$ is denoted by $B(a, r) := \{x \in \mathbb{R}^n : \|x - a\| < r\}$. For a given set $A \subseteq \mathbb{R}^n$, int$A$, ri$A$, cl$A$, co$A$ and cone$A$ denote the *interior*, *relative interior*, *closure*, *convex hull*

[2] x is a non-trivial KKT point if the KKT optimality condition is non-trivial in the sense that the corresponding KKT multipliers are not all simultaneously zero.

and *convex conic hull* of A, respectively. In this setting, The *polar cones* of A is $A^- := \{u \in \mathbb{R}^n : \langle u, x\rangle \leq 0,\ \forall x \in A\}$. If $A = \emptyset$, we adopt the convention $\emptyset^- = \mathbb{R}^n$. Let $x \in \text{cl}A$. The *normal cone* at x to A, denoted by $N(A, x)$, is defined by

$$N(A, x) := \{u \in \mathbb{R}^n : \langle u, y - x\rangle \leq 0,\ \forall y \in A\}.$$

If $x \notin \text{cl}A$, then one puts $N(A, x) = \emptyset$. The *contingent cone* of A at $x \in \text{cl}A$, denoted by $T(A, x)$, is the following cone:

$$T(A, x) := \left\{ d \in \mathbb{R}^n : \exists\{r_k\} \subset (0, +\infty) \to 0,\ \exists\{x_k\} \subset A \to x,\ \frac{x_k - x}{r_k} \to d \right\}.$$

It is well-known that if A is convex then, for $x \in \text{cl}A$, $T(A, x) = N(A, x)^-$.

Let f be a function from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$. We say that f is *proper* if $\text{dom}f := \{x \in \mathbb{R}^n : f(x) \in \mathbb{R}\} \neq \emptyset$. Following in [7, 9], a proper function f is said to be *tangentially convex* at $x \in \text{dom}f$ if for every $d \in \mathbb{R}^n$ the right-sided directional derivative of f at x

$$f'(x, d) := \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}$$

exists, is finite, and is a convex function of d. It is important to note that for f is tangentially convex at $x \in \text{dom}f$, the function $f'(x, \cdot)$ is a sublinear function which follows that the *tangential subdifferential* of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $x \in \text{dom}f$, is the set $\partial_T f(x)$ given as

$$\partial_T f(x) := \{\xi \in \mathbb{R}^n : \langle \xi, d\rangle \leq f'(x, d),\ \ \forall d \in \mathbb{R}^n\},$$

is nonempty. Further, the function $f'(x, \cdot)$ is a support function of the tangential subdifferential, that is,

$$f'(x, d) = \max_{\xi \in \partial_T f(x)} \langle \xi, d\rangle.$$

Let f be tangentially convex at $x \in \text{dom}f$. The tangential subdifferentials enjoy nice calculus properties including positive homogeneous rule and sum rule, i.e.,

(i) For every $\lambda \geq 0$, $\partial_T(\lambda f)(x) = \lambda \partial_T f(x)$;

(ii) If g is tangentially convex at the same point $x \in \text{dom}f \cap \text{dom}g$, one has

$$\partial_T(f + g)(x) = \partial_T f(x) + \partial_T g(x).$$

We close the section by the following results which will be useful later in the paper.

Lemma 2.1 ([8, Lemma 8])**.** *Let $s : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a sublinear function. If s vanishes on an open set, then it is nonnegative everywhere.*

Lemma 2.2. *Let s be a real-valued sublinear function on $\mathbb{R}^n$. If there exists $x_0 \in \mathbb{R}^n$ such that $s(x_0) < 0$, then we have*

$$\text{cl}\{x \in \mathbb{R}^n : s(x) < 0\} = \{x \in \mathbb{R}^n : s(x) \leq 0\}.$$

Proof. By assumption, we can check that $\inf_{x \in \mathbb{R}^n} s(x) < 0$ and $x_0 \in \text{int}\{x \in \mathbb{R}^n : s(x) < 0\} \neq \emptyset$. We claim that

$$\text{int}\{x \in \mathbb{R}^n : s(x) = 0\} = \emptyset.$$

Suppose on contrary that there exists $\hat{x} \in \text{int}\{x \in \mathbb{R}^n : s(x) = 0\}$. Then $B(\hat{x}, \varepsilon) \subseteq \{x \in \mathbb{R}^n : s(x) = 0\}$ for some $\varepsilon > 0$, and hence, by Lemma 2.1, $s(x) \geq 0$ for all

$x \in \mathbb{R}^n$. This is a contradiction, and therefore the conclusion follows by applying Theorem 11(ii) in [11]. □

3. Main results

According to technical approach given in [4], we begin with an extension of Proposition 2.2.(i) in [5] which will play a key role to derive sufficient KKT optimality condition in our main result for non-differentiable problem.

Lemma 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and tangentially convex at $x \in \mathbb{R}^n$. Suppose that the set $L_f^<(x)$ is nonempty and convex. If $x \in \text{cl}L_f^<(x)$, then*

$$N(L_f^<(x), x)^- \subseteq \{d \in \mathbb{R}^n : f'(x, d) \leq 0\}.$$

If $0 \notin \partial_T f(x)$, then also $x \in \text{cl}L_f^<(x)$ and the above containment becomes equality.

Proof. Consider $x \in \text{cl}L_f^<(x)$. Firstly, we will show that

$$\partial_T f(x) \subseteq N(L_f^<(x), x).$$

Now take any $\xi \in \partial_T f(x)$. Given any y in $L_f^<(x)$. Let us notice that, by continuity of f at x, the set $L_f^<(x)$ is (relative) open. So, $y \in L_f^<(x) = \text{ri}L_f^<(x)$ and, by the line segment principle [10, Theorem 6.1],

$$x + t(y - x) \in \text{ri}L_f^<(x) = L_f^<(x) \text{ for } t \in (0, 1].$$

For values t sufficiently small, $f'(x, y - x) \leq 0$, and hence $\langle \xi, y - x\rangle \leq 0$ by the definition of the tangential subdifferential of f at x. Therefore, for any $y \in L_f^<(x)$, $\langle \xi, y - x\rangle \leq 0$, thereby showing that $\xi \in N(L_f^<(x), x)$. Thus, $\partial_T f(x) \subseteq N(L_f^<(x), x)$ as required, which gives that

$$N(L_f^<(x), x)^- \subseteq \partial_T f(x)^-.$$

Further, it can be checked that

$$\partial_T f(x)^- = \{d \in \mathbb{R}^n : f'(x, d) \leq 0\}.$$

Hence, the desired result is obtained.

Assuming $0 \notin \partial_T f(x)$, we now demonstrate that x is contained in $\text{cl}L_f^<(x)$. As $0 \notin \partial_T f(x)$, there exists $d_0 \in \mathbb{R}^n$ such that $f'(x, d_0) < 0$. By the definition of directional derivative, there exists a positive real number δ such that

$$f(x + td_0) < f(x) \text{ for all } t \in (0, \delta).$$

In particular, for each $k \in \mathbb{N}$, the vector $x_k := x + \frac{\delta}{k+1} d_0$ belongs to $L_f^<(x)$, and $x_k \to x$ as $k \to +\infty$. This means that $x \in \text{cl}L_f^<(x)$. To establish the remaining inclusion $\{d \in \mathbb{R}^n : f'(x, d) \leq 0\} \subseteq N(L_f^<(x), x)^-$, we argue first by applying Lemma 2.2 that

$$\{d \in \mathbb{R}^n : f'(x, d) \leq 0\} = \text{cl}\{d \in \mathbb{R}^n : f'(x, d) < 0\}.$$

Moreover, convexity of $L_f^<(x)$ asserts that $N(L_f^<(x), x)^- = T(L_f^<(x), x)$. Thus it is enough to show that, as $T(L_f^<(x), x)$ is closed,

$$\{d \in \mathbb{R}^n : f'(x, d) < 0\} \subseteq T(L_f^<(x), x).$$

Given any $d \in \mathbb{R}^n$ such that $f'(x,d) < 0$. As seen before, we can find two sequences $\{r_k\} \subset (0,+\infty)$ and $\{x_k\} \subset L_f^<(x)$ such that

$$x_k = x + r_k d \in L_f^<(x) \text{ and } r_k \to 0 \text{ as } k \to +\infty.$$

Consequently, $x_k \to x$ and $\frac{x_k - x}{r_k} \to d$ as $k \to +\infty$. Therefore, $d \in T(L_f^<(x), x)$, thereby establishing the requisite result. □

Remark 3.2. In Lemma 3.1, if f is differentiable at $x \in \mathbb{R}^n$ such that $\nabla f(x) \neq 0$, then $f'(x,d) = \langle \nabla f(x), d \rangle$ for all $d \in \mathbb{R}^n$. In this case, we obtain the following result.

Corollary 3.3 ([5, Proposition 2.2.(i)]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at x with $\nabla f(x) \neq 0$. Then*

$$N(L_f^<(x), x) = \{d \in \mathbb{R}^n : d = r\nabla f(x), \textit{ for some } r \geq 0\}$$

provided that $L_f^<(x)$ is convex.

Proof. As the set $N(L_f^<(x), x)$ and $\text{cone}\{\nabla f(x)\}$ are closed convex cone, by polar cone theorem [1, Proposition 3.1.1(b)], $(N(L_f^<(x), x)^-)^- = N(L_f^<(x), x)$ and $((\text{cone}\{\nabla f(x)\})^-)^- = \text{cone}\{\nabla f(x)\}$. Therefore, owing to Proposition 3.2.1(a) in [1] and Lemma 3.1,

$$\begin{aligned} N(L_f^<(x), x) &= (N(L_f^<(x), x)^-)^- \\ &= \{d \in \mathbb{R}^n : \langle \nabla f(x), d \rangle \leq 0\}^- \\ &= ((\text{cone}\{\nabla f(x)\})^-)^- \\ &= \text{cone}\{\nabla f(x)\} = \{d \in \mathbb{R}^n : d = r\nabla f(x), \text{ for some } r \geq 0\}. \end{aligned}$$

□

Remark 3.4. The condition $0 \notin \partial_T f(x)$ given in Lemma 3.1 ensuring $x \in \text{cl} L_f^<(x)$ can be replaced by $L_f(x) = \text{cl} L_f^<(x)$, where $L_f(x) := \{y \in \mathbb{R}^n : f(y) \leq f(x)\}$. However, the following two examples show that condition $0 \notin \partial_T f(x)$ does not necessarily imply the condition $L_f(x) = \text{cl} L_f^<(x)$ and vise versa.

Example 3.5. Consider a non-differentiable function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \max\{\frac{1}{2}x(x+1)(x+2), 2x\}, & \text{if } x \in [-1,+\infty); \\ \max\{-\frac{1}{2}(x+1)(x+2), 0\}, & \text{ohterwise.} \end{cases}$$

For $x = 0$, we have $\partial_T f(x) = [1,2]$, and hence $0 \notin \partial_T f(x)$. However, $\text{cl} L_f^<(x) = [-1,0] \neq (-\infty,-2] \cup [-1,0] = L_f(x)$. □

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \max\{-x^2, x\}$, $\forall x \in \mathbb{R}$ and $x = 0$. So, $L_f(x) = (-\infty, 0] = \text{cl} L_f^<(x)$, while $\partial_T f(x) = [0,1]$. □

Remark 3.7. In Example 3.5 tell us that the fulfillment of $x \in \text{cl} L_f^<(x)$ is not sufficient to ensure that the equality $L_f(x) = \text{cl} L_f^<(x)$.

Next, we will see how the condition $0 \notin \partial_T f(x)$ is not necessarily to be assumed when considering KKT optimality conditions. Before doing so let us formally state the notion of a KKT point of (P) in terms of tangential subdifferentials. A feasible point $\bar{x}$ of (P) is called a *KKT point* if there exist scalars $\lambda_i \geq 0$, $i = 1, \ldots, m$, such that

i) $0 \in \partial_T f(\bar{x}) + \sum_{i=1}^m \lambda_i \partial_T g_i(\bar{x})$,
ii) $\lambda_i g_i(\bar{x}) = 0, \quad i = 1, \ldots, m.$

Also, a non-trivial KKT point is defined in an analogous manner as in differentiable case.

We now turn our attention to state our main result.

Theorem 3.8. *Given the nonlinear programming problem* (P) *fulling Slater's condition. Let $\bar{x} \in K$ be a feasible solution, the functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous such that for every $i \in I(\bar{x})$ the functions g_i is tangentially convex at $\bar{x}$. Suppose that $\bar{x}$ satisfies the condition* (1.1), *and $0 \notin \partial_T g_i(\bar{x})$ for all $i \in I(\bar{x})$. Assume further that the functions $f : \mathbb{R}^n \to \mathbb{R}$, is continuous and tangentially convex at $\bar{x}$.*

(i) *If $\bar{x}$ is a global minimizer then $\bar{x}$ is a KKT point.*
(ii) *Conversely, if $\bar{x}$ is a non-trivial KKT point such that $\bar{x} \in \mathrm{cl}L_f^<(\bar{x})$, and $L_f^<(\bar{x})$ is convex then $\bar{x}$ is a global minimizer.*

Proof. (i) If $\bar{x}$ is a global minimizer, then, by the Fritz-John optimality conditions [9, p. 88, Corollary], there exist real numbers $\lambda_i \geq 0, i = 1, \ldots,$ not all zero, satisfying ii) and

$$\lambda_0 f'(\bar{x}, d) + \sum_{i=1}^m \lambda_i g_i'(\bar{x}, d) \geq 0, \ \forall d \in \mathbb{R}^n. \tag{3.1}$$

We shall now show that $\lambda_0 > 0$. Let us assume that $\lambda_0 = 0$. Hence there exists some $i \in \{1, \ldots, m\}$ such that $\lambda_i > 0$ which, by ii), implies the non-emptyness of $I(\bar{x})$. Taking (3.1) into account we actually have

$$\sum_{i \in I(\bar{x})} \lambda_i g_i'(\bar{x}, d) \geq 0, \ \forall d \in \mathbb{R}^n. \tag{3.2}$$

As $\sum_{i \in I(\bar{x})} \lambda_i > 0$, setting $\bar{\lambda}_i := \frac{\lambda_i}{\sum_{i \in I(\bar{x})} \lambda_i}$ for each $i \in I(\bar{x})$. Multiplying both sides of (3.2) by $\frac{1}{\sum_{i \in I(\bar{x})} \lambda_i}$ for each $d \in \mathbb{R}^n$, we would have from the obtained inequality that

$$0 \in \partial_T \left(\sum_{i \in I(\bar{x})} \bar{\lambda}_i g \right)(\bar{x}) = \sum_{i \in I(\bar{x})} \bar{\lambda}_i \partial_T g_i(\bar{x}).$$

Then there exist $\xi_i \in \partial_T g_i(\bar{x})$, $i \in I(\bar{x})$, such that

$$\sum_{i \in I(\bar{x})} \bar{\lambda}_i \xi_i = 0. \tag{3.3}$$

It follows from the non-degeneracy condition at $\bar{x}$ that $\xi_i \neq 0$, $\forall i \in I(\bar{x})$. This together with the fact that $\sum_{i\in I(\bar{x})} \bar{\lambda}_i = 1$, we get

$$0 < \sum_{i\in I(\bar{x})} \bar{\lambda}_i \|\xi_i\|. \tag{3.4}$$

On the one hand, as the functions g_i are continuous, the set of Slater points, which contained in K, is open. It means that, for a Slater point x_0, there exists a positive real number ρ such that $B(x_0, \rho) \subseteq K$. Thus, $x_0 + \frac{\rho}{2\|\xi_i\|}\xi_i \in B(x_0, \rho) \subseteq K$, $\forall i \in I(\bar{x})$. It is worth noting that

$$g_i'(\bar{x}, y - \bar{x}) \leq 0 \text{ for all } i \in I(\bar{x}),\ y \in K. \tag{3.5}$$

Otherwise, $g_i'(\bar{x}, y - \bar{x}) > 0$ for some $y \in K$ and for some $i \in I(\bar{x})$. Then by the definition of directional derivative, there exists $\delta > 0$ such that

$$\left| \frac{g_i(\bar{x} + t(y - \bar{x}))}{t} - g_i'(\bar{x}, y - \bar{x}) \right| < g_i'(\bar{x}, y - \bar{x}) \text{ whenever } 0 < t < \delta.$$

Subsequently, $g_i(\bar{x}+t(y-\bar{x})) > 0$ for all $t \in (0, \delta)$. This contradicts to the condition (1.1) that we can find some t_n small enough such that $\bar{x}+t_n(y-\bar{x}) \in K$. Therefore, by the definition of tangential subdifferentials, for each $i \in I(\bar{x})$ we have

$$\langle \xi_i, x_0 - \bar{x}\rangle + \frac{\rho}{2}\|\xi_i\| = \left\langle \xi_i, x_0 + \frac{\rho}{2\|\xi_i\|}\xi_i - \bar{x} \right\rangle \leq g_i'\left(\bar{x}, x_0 + \frac{\rho}{2\|\xi_i\|}\xi_i - \bar{x}\right) \leq 0.$$

Multiplying both sides of inequality above by $\bar{\lambda}_i$, $i \in I(\bar{x})$, and summing up the obtained inequalities together with (3.3) we get

$$\frac{\rho}{2} \sum_{i\in I(\bar{x})} \lambda_i \|\xi_i\| \leq 0,$$

which is in turn a contrast to (3.4). Hence $\lambda_0 > 0$, and without loss of generality we can set $\lambda_0 = 1$. Then, for every $d \in \mathbb{R}^n$ we have

$$\left(f + \sum_{i=1}^{m} \lambda_i g_i\right)' (\bar{x}, d) = f'(\bar{x}, d) + \sum_{i=1}^{m} \lambda_i g_i'(\bar{x}, d) \geq 0,$$

which is noting else than

$$0 \in \partial_T \left(f + \sum_{i=1}^{m} \lambda_i g_i\right)(\bar{x}) = \partial_T f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial_T g_i(\bar{x}),$$

i) as required.

(ii) Let $\bar{x} \in K$ be an arbitrary non-trivial KKT point. We see that for every $y \in L_f^<(\bar{x})$ we get the following inequality $\langle x, y - \bar{x}\rangle \leq 0$, $\quad \forall x \in N(L_f^<(\bar{x}), \bar{x})$, which means that $y - \bar{x} \in N(L_f^<(\bar{x}), \bar{x})^-$. By Lemma 3.1 (i) we obtain that

$$f'(\bar{x}, y - \bar{x}) \leq 0 \text{ for all } y \in L_f^<(\bar{x}). \tag{3.6}$$

On the other hand, employing i) and ii), we obtain

$$f'(\bar{x}, d) + \sum_{i\in I(\bar{x})} \lambda_i g_i'(\bar{x}, d) \geq 0 \text{ for any } d \in \mathbb{R}^n.$$

In particular, by using (3.6),

$$\sum_{i\in I(\bar{x})} \lambda_i g_i'(\bar{x}, y-\bar{x}) \geq 0 \text{ for any } y \in L_f^<(\bar{x}). \tag{3.7}$$

Next, we claim that $L_f^<(\bar{x}) \cap K = \emptyset$. Arguing by contradiction, suppose that there exists $w \in L_f^<(\bar{x}) \cap K$. Then, from (3.5) and (3.7),

$$\sum_{i\in I(\bar{x})} \lambda_i g_i'(\bar{x}, w-\bar{x}) = 0. \tag{3.8}$$

Furthermore, since $L_f^<(\bar{x})$ is open, for each $d \in \mathbb{R}^n$ there exists some $t > 0$ small enough such that $w + td \in L_f^<(\bar{x})$. Hence, using (3.8),

$$\begin{aligned} 0 &\leq \sum_{i\in I(\bar{x})} \lambda_i g_i'(\bar{x}, w+td-\bar{x}) \\ &\leq \sum_{i\in I(\bar{x})} \lambda_i g_i'(\bar{x}, w-\bar{x}) + t \sum_{i\in I(\bar{x})} \lambda_i g_i'(\bar{x}, d) \\ &= t \sum_{i\in I(\bar{x})} \lambda_i g_i'(\bar{x}, d). \end{aligned}$$

As $\bar{x}$ is a non-trivial KKT point, we have $\sum_{i\in I(\bar{x})} \lambda_i > 0$. In a similar manner of the first argument, the last inequality arrives at a contradiction. So, our claim $L_f^<(\bar{x}) \cap K = \emptyset$ holds. This means that $\bar{x}$ is a global minimizer, and the proof is completed. □

Remark 3.9. It is worth observing that the Slater's condition along with a non-degeneracy at $\bar{x}$ arrives at the assertion

$$0 \notin \sum_{i\in I(\bar{x})} \lambda_i \partial_T g_i(\bar{x}) \text{ whenever } \lambda_i \geq 0,\ i \in I(\bar{x}) \text{ such that } \sum_{i\in I(\bar{x})} \lambda_i = 1,$$

or equivalently, $0 \notin \text{co}\left(\bigcup_{i\in I(x)} \partial_T g_i(x)\right)$.

Let us recall that a proper tangentially convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in \text{dom} f$ is said to be *pseudoconvex at* $\bar{x}$ (see [8, Definition 7]) if

$$\forall y \in \mathbb{R}^n,\ f'(\bar{x}, y-x) \geq 0 \Longrightarrow f(y) \geq f(\bar{x}).$$

It can be seen that the condition $0 \notin \partial_T f(\bar{x})$ will follows from non-emptiness of $L_f^<(\bar{x})$ and pseudoconvexity of f at $\bar{x}$. In this context together with Lemma 3.1 (ii), pseudoconvexity of f at $\bar{x}$ provided sufficient condition for $\bar{x} \in \text{cl}L_f^<(\bar{x})$ whenever $L_f^<(\bar{x})$ is a nonempty and convex set. However, the following example shows that pseudoconvexity of f at $\bar{x}$ does not necessarily imply convexity of $L_f^<(\bar{x})$, and hence Theorem 3.8 cannot be applied in this situation.

Example 3.10. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-differentiable function defined by

$$f(x) = \begin{cases} \max\{x^3 + x, 2x\}, & \text{if } x \in [0, +\infty); \\ \frac{1}{2}x(x+1)(x+2), & \text{if } x \in (-\infty, 0). \end{cases}$$

Then, for $\bar{x} = 0$, $f'(\bar{x}, d) = \max\{d, 2d\}$ for any $d \in \mathbb{R}$, from which we can obtain that f is pseudoconvex at $\bar{x}$, while $L_f^<(\bar{x}) = (-\infty, -2) \cup (-1, 0)$ is not convex. □

Here we give an example to illustrate that Theorem 3.8 is indicated to be conveniently applied in some cases where Theorem 9 of [8] cannot be used even when the feasible set K is convex. Namely, the objective function is not pseudoconvex at considered point.

Example 3.11. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-differentiable function defined by

$$f(x) = \begin{cases} (x-1)(x-2)(x-3)+1, & \text{if } x \in [1, +\infty); \\ \max\{x^3, x\}, & \text{if } x \in [-\infty, 1), \end{cases}$$

and the feasible set $K = \{x \in \mathbb{R} : g_1(x) \leq 0\}$, where

$$g_1(x) = \max\{-x, -\frac{1}{2}(x-1)^2(x-2) - 1\}.$$

Evidently, the function f is not pseudoconvex at $\bar{x} = 0$. We can verify that $K = [0, +\infty)$ and the feasible point $\bar{x}$ satisfies non-trivial KKT conditions with $\lambda_1 = 1$, non-degeneracy condition, and $\bar{x} \in \mathrm{cl}L_f^<(\bar{x})$ in which $L_f^<(\bar{x}) = (-\infty, 0)$ is convex. Then, by Theorem 3.8, $\bar{x}$ is a global minimizer. □

Example 3.12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as in Example 3.11, and the feasible set $K = \{x \in \mathbb{R} : g_1(x) \leq 0\}$, where

$$g_1(x) = \max\{-x, -x(x-1)(x-2)\}.$$

We can check that $K = [0, 1] \cup [2, +\infty)$ and the feasible point $\bar{x} = 0$ satisfies non-trivial KKT conditions with $\lambda_1 = 1$, $L_f^<(\bar{x}) = (-\infty, 0)$ is convex such that $\bar{x} \in \mathrm{cl}L_f^<(\bar{x})$. In addition, a non-degeneracy condition and condition (1.1) hold at $\bar{x}$. Theorem 3.8 then indicates that $\bar{x}$ is a global minimizer. □

It is worth mentioning that from Remark 3.4, the following result can be deduced.

Corollary 3.13. *If we replace the condition $\bar{x} \in \mathrm{cl}L_f^<(\bar{x})$ by $\mathrm{cl}L_f^<(\bar{x}) = L_f(\bar{x})$ or $0 \notin \partial_T f(\bar{x})$, Theorem 3.8 is also true.*

As tangential convexity collapses to regularly locally Lipschitz setting and differentiability, the following two corollaries are immediately direct consequences as a special case of Theorem 3.8. We will also see how the condition $x \in \mathrm{cl}L_f^<(x)$ can be absent in differentiable case.

Corollary 3.14. *Given the nonlinear programming problem* (P) *and let the Slater's condition holds. Let $\bar{x} \in K$ be a feasible solution satisfying the condition* (1.1) *and the functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I(\bar{x})$, be locally Lipschitz and regular in the sense of Clarke at $\bar{x}$. Assume that $0 \notin \partial^o g_i(\bar{x})$ for all $i \in I(\bar{x})$.*

(i) *If $\bar{x}$ is a global minimizer then there exist $\lambda_i \geq 0$, $i = 1, \dots, m$ such that*
   i) $0 \in \partial^o f(\bar{x}) + \sum_{i=1}^m \lambda_i \partial^o g_i(\bar{x})$,
   ii) $\lambda_i g_i(\bar{x}) = 0, \quad i = 1, \dots, m.$

(ii) *Conversely, if $\bar{x}$ is a non-trivial KKT point such that $\bar{x} \in \mathrm{cl}L_f^<(\bar{x})$, and $L_f^<(\bar{x})$ is convex then $\bar{x}$ is a global minimizer.*

Corollary 3.15. *[4, Theorem 1] Given the nonlinear programming problem* (P) *and let the Slater's condition holds. Let* $\bar{x} \in K$ *be a feasible solution satisfying the condition* (1.1) *and the functions* $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ *be differentiable functions. Assume that* $\nabla g_i(\bar{x}) \neq 0$ *for all* $i \in I(\bar{x})$.

(i) *If* $\bar{x}$ *is a global minimizer then there exist* $\lambda_i \geq 0$, $i = 1, \ldots, m$ *such that*
   i) $\nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0,$
   ii) $\lambda_i g_i(\bar{x}) = 0, \quad i = 1, \ldots, m.$

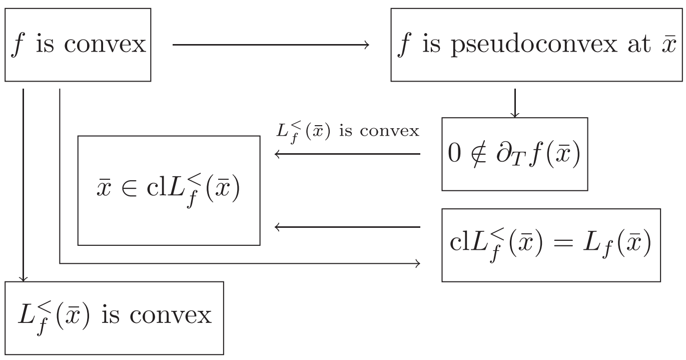
(ii) *Conversely, if* $\bar{x}$ *is a non-trivial KKT point, and* $L_f^{<}(\bar{x})$ *is convex then* $\bar{x}$ *is a global minimizer.*

Proof. Owing to $\bar{x}$ is a non-trivial KKT point,

$$-\frac{1}{\sum_{i \in I(\bar{x})} \lambda_i} \nabla f(\bar{x}) = \sum_{i \in I(\bar{x})} \frac{\lambda_i}{\sum_{i \in I(\bar{x})} \lambda_i} \nabla g_i(\bar{x}) \in \mathrm{co}\left(\bigcup_{i \in I(x)} \{\nabla g_i(x)\} \right).$$

In view of Remark 3.9, we obtain that $\nabla f(\bar{x}) \neq 0$. The desired result will follows by the virtue of Lemma 3.1. □

To this end, we would like to summarize the relationship of the several conditions, which were considered in this paper, for KKT optimality conditions whenever $f : \mathbb{R}^n \to \mathbb{R}$, $\bar{x} \in \mathbb{R}^n$ and $L_f^{<}(\bar{x}) \neq \emptyset$:



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