Journal of Nonlinear and Convex Analysis
Volume 25, Number 1, 2024, 169–180

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

AN INERTIAL ITERATION FOR AN INFINITE FAMILY OF NONEXPANSIVE MAPPINGS AND A BIFUNCTION

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Abstract. The aim of this paper is to investigate common fixed points of an infinite family of nonexpasnive mappings and equilibriium points of an equilibrium problem. This paper introduces an inertial acceleration method via a viscosity approximation and obtains a strong convergence theorem of common solutions under mild restrictions on operators and parameters. The main theorem improve and extend some related results announced recently in the literature.

1. Introduction-preliminaries

Fixed points of nonlinear operators play an important and useful role in several research fields, such as, applied mathematics, computer science, management, financial engineering, and so on. Nonexpansive mapping, which is defined as follows, has been extensively studied by many famous scholars around the world; see, e.g., [1, 12, 14, 20, 30, 37]. From now on, we suppose space H is Hilbert and its inner produce is $\langle\cdot,\cdot\rangle$. Beside, C is a convex and closed set in space H. To avoid the trivial case, C is also assumed to be nonempty throughout this paper. Recall a nonlinear operator S is a Lipschitz mapping if for all x and y in C, $\|Sx - Sy\| \leq L\|x - y\|$, where L is some positive constant. $F(S)$ always stands for the fixed point set. Recall S is called a contractive mapping if and only if $L < 1$. The celebrated Banach fixed point theorem asserts that every contractive mapping has a fixed point, which is also unique. Recall S is called a nonexpansive mapping if and only if $L = 1$. This slight change has a great impact. This mapping may has no fixed point. Even its fixed point set is nonempty, it may not be a singleton. To study fixed points of this mapping is meaningful. One powerful way to study such a mapping is the celebrated Mann iteration process

$$x_1 \in C,\ x_{n+1} = (1 - \alpha_n)\,Sx_n + \alpha_n x_n,$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$; see [17]. Mann's iteration is simple and powerful and it has been studied by many famous scholars; see, e.g., [3,9,15,24]. But, it is known this iteration is weakly convergent in the framework of infinite dimension spaces. This is challenging from viewpoint of real world applications. Recently, many scholars have focused on this iteration or its modified version in infinite dimensional spaces; see, e.g., [16, 27, 31].

One of the efficient and powerful method is to use contractions, that is, Halpern method:

$$x_1 \in C,\ x_{n+1} = (1 - \alpha_n)\,Sx_n + \alpha_n x,$$

2020 *Mathematics Subject Classification.* 47H09, 90C30.

Key words and phrases. Bifunction, equilibrium problem, fixed point; inertial term; nonexpansive mapping.

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and x is a fixed point. This iteration, which was proved is powerful for strong convergence in infinite dimensional spaces. Besides, no strong conditions are needed, such as compact conditions. We refer to [19, 22, 34] for some recent related results on this iteration.

In 2000, France mathematican Moudafi [18] presented the following iteration

$$x_1 \in C,\ x_{n+1} = (1-\alpha_n)\, Sx_n + \alpha_n f(x_n),$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $f : C \to C$ is a contractive operator. This iteration is now known as Moudafi's viscosity. Moudafi showed his sequence converges to S's fixed point in norm. In the past two decades, many scholars have investigated Moudafi's viscosity with various nonlinear mappings and in various spaces; see, e.g., [7, 26, 38].

The equilibrium problem with a bifunction F under Blum and Oettli [4] is to get a point y in C with $F(y, y') \geqslant 0$ for all y in C. $EP(F, C)$ stands for the solution set from now on. This problem covers many problems, such as saddle problems, complementary problems, variational inclusion problems and has many applications in traffic network, financial engineering, computer science and so on; see, e.g., [4, 8, 10]. The approximation solutions of the equilibrium problem are now in the spotlight of the academic research and the powerful approximation methods are projection and relaxed methods; see, e.g., [11, 25, 29] and the references therein.

For investigating the bifunction F on C, one always assumes, in this paper, that it satisfies:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) $F(x, y) + F(y, x) \leqslant 0$ for all $x, y \in C$;
(A3) $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leqslant F(x, y)$ for each $x, y, z \in C$;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and weakly lower semicontinuous.

About the rate of convergence, the speed of convergence of latest iteration algorithms for fixed point problems has been considered. The necessity of accelerating the convergence of iteration algorithms is under the spotlight of great interest; see, e.g., see [2, 5, 21]. Besides the Moudafi's viscosity, inertial extrapolation which was introduced by Polyak [23] is an powerful technique to accelerate the original algorithms. The main technique is to use two iterative information: the current and the last information. The inertial extrapolation open a new era recently; see, e.g., see [28, 35, 36].

Here, we mention the results in Shehu and Iyiola [28]. They studied an interesting alternative inertial term for finding fixed points of a nonexpansive mapping. Their algorithm is $\theta_n (x_n - x_{n-1}) + x_n$, where $0 \leq \theta_n \leq \bar{\theta}_n$, and $\bar{\theta}_n$ is chosen by:

$$\bar{\theta}_n = \begin{cases} \theta, & x_n = x_{n-1}, \\ \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}. \end{cases}$$

Let S be a nonexpansive mapping from C to H. Let f be a contractive mapping on H. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be some real sequences. Let $\{\lambda_n\}$ be a nonnegative real sequence and let F be a bifunction on C. Recently, in 2007, S. Takahashi and W. Takahashi [33] studied a strongly convergent solution method to get a common element in sets $F(S)$ and $EP(F, C)$. To be more precise, we list their algorithm as follows

$$\begin{cases} F(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ x_{n+1} = (1 - \alpha_n) S u_n + \alpha_n f(x_n), \end{cases}$$

They assumed that $\liminf_{n\to\infty} \lambda_n > 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. They got a theorem of strong convergence with no compact assumptions under mild conditions on the parameters constructed in their iteration algorithm.

This paper has received much attention since its publication. In this paper we based on this paper, re-consider equilibria with an infinite family nonexpansive mappings. Let $T_i : C \to C$ be a nonexpansive operator, and let $\{k_n\}$ be a positive real sequence with $k_n < 1$ for all $n \geq 1$. Recall the mapping W_n [32] defined by

$$\begin{aligned} &U_{n,n+1} = I, U_{n,n} = k_n T_n U_{n,n+1} + (1 - k_n) I, \\ &U_{n,n-1} = k_{n-1} T_{n-1} U_{n,n} + (1 - k_{n-1}) I, \\ &\cdots \\ &U_{n,j} = k_j T_j U_{n,j+1} + (1 - k_j) I, \\ &U_{n,j-1} = k_{j-1} T_{j-1} U_{n,j} + (1 - k_{j-1}) I, \\ &\cdots \\ &U_{n,2} = k_2 T_2 U_{n,3} + (1 - k_2) I, \\ &W_n = U_{n,1} = k_1 T_1 U_{n,2} + (1 - k_1) I. \end{aligned} \tag{1.1}$$

If $0 < k_n \leq \chi < 1$, where χ is some real number, for all $n \geq 1$, then W_n is nonexpansive for each n according to [32]. Besides $F(W_n) = \cap_{i=1}^{\infty} F(T_i)$; the mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$, is a nonexpansive operator with $F(W) = \cap_{i=1}^{\infty} F(T_i)$.

Next, we, motivated by the results in [28, 33], construct a strongly convergence inertial algorithm for the equilibrium problem with a family of infinite nonexpansive mappings defined in (1.1) We will give a strong convergence theorem in Hilbert spaces with no any compact assumption with the aid of alternative inertial extrapolation. To present our main theorem, we also need the following tools.

Let P_C stand for the metric projection of H onto its subset C

$$P_C(x) = \arg\min\{\|x - y\|, \, y \in C\}.$$

P_C has the following important properties:

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle, \, \forall x, y \in \mathcal{H}$$

and

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0, \, \forall x \in \mathcal{H}, \forall y \in C,$$

We know it is no hard to compute onto half-spaces, boxes, and balls.

Lemma 1.1 ([4]). *Let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying conditions (A1)-(A4) and let λ be a positive real number. There exists y in set C with $F(y, z) + \frac{1}{\lambda} \langle z - y, y - x \rangle \geqslant 0$, where x in H for all $z \in C$. Define a mapping $R_\lambda : H \to C$ as follows:*

$$R_\lambda(x) = \{z \in C : \lambda F(y, z) + \ \langle z - y, y - x \rangle \geqslant 0, \forall z \in C\}$$

for all $y \in H$. Then,

(1) R_λ *is single-valued;*
(2) $\|R_\lambda a - R_\lambda b\|^2 \leqslant \langle R_\lambda x - R_\lambda y, x-y\rangle$;
(3) $F(R_\lambda) = EP(F,C)$ *is convex and closed.*

Lemma 1.2 ([32])**.** *Let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let $\{k_i\}$ be a real sequence such that $0 < k_i \le l < 1$, where l is some real number for alli ≥ 1. Then*

(i) $F(W_n) = \cap_{i=1}^{\infty} F(T_i)$ *for each $n \ge 1$ and W_n is nonexpansive;*
(ii) *for each $x \in C$ and for each positive integer v, the limit $\lim_{n\to\infty} U_{n,v}$ exists.*
(iii) *the mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C,$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, \ldots$ and $k_1, k_2, \ldots$.

From Chang et al. [6], under some condition on $\{k_i\}$, $0 < k_i \le l < 1$, we have $lim\sup \|Wx - W_n x\| = 0$. In this paper, we further impose condition on $\{k_i\}$ such that it is also summable.

Lemma 1.3 ([13])**.** *Let $\{a_n\}$ be a nonnegative real number sequence with*

$$a_{n+1} \le (1-\alpha_n)\, a_n + \sigma_n + \gamma_n, \quad n \ge 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a real sequence. Assume $\sum \gamma_n < \infty$. Then, If $\sum \alpha_n = \infty$ and $\limsup \frac{\sigma_n}{\alpha_n} \le 0$, then $\lim a_n = 0$.

2. Main results

Theorem 2.1. *Let H be a real Hilbert space and let C be its some convex and closed subset. Let $T_i : C \to H$, for each positive integer $i \ge$, be a nonexpansive mapping with a nonempty fixed point set and such that their common fixed point set is also not empty, that is, $\cap_{i=1}^{\infty} F(T_i)$ is not empty. Let B be a bifunction on C satisfying the four conditions: (R1)-(R4). Let f on H be an α-contraction. Let $\{x_n\}$ and $\{u_n\}$ be vector sequences generated in the following iteration algorithm: x_0 and x_1 are two initials and*

$$y_n = \theta_n (x_n - x_{n-1}) + x_n,$$

compute a sequence $\{u_n\}$ such that

$$\lambda_n F(u_n, y) + \langle y - u_n, u_n - y_n\rangle \geqslant 0, \quad \forall y \in C,$$

and then compute

$$x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)\, W_n u_n,$$

where θ_n is chosen such that $0 \le \theta_n \le \bar{\theta}_n$ with

$$\overline{\theta_n} = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \ne x_{n-1} \\ \theta, & \textit{otherwise.} \end{cases}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ are sequences such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \quad \textit{and} \quad \liminf_{n\to\infty} \lambda_n > 0,$$

$$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad and \quad \sum^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$$

and $\{\epsilon_n\}_{n=1}^{\infty}$ *is a positive sequence such that* $\epsilon_n = o(\alpha_n)$, *i.e.,* $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$. *If* $\cap_{i=1} F(T_i) \cap EP(F,C)$ *is not empty, then the sequence* $\{x_n\}$ *generated by the algorithm above converges strongly to* $P_{F(S)\cap EP(F,C)} f(z)$, *where* $P_{F(S)\cap EP(F,C)} f(z)$ *denotes the nearest point projection from* H *onto* ${}_{F(S)\cap EP(F,C)} f(z)$.

Proof. From the construction of the inertial term on $\{\theta_n\}$, one has

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0. \tag{2.1}$$

and

$$\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0. \tag{2.2}$$

Next, we use j to denote some solution in $\cap_{i=1} F(T_i) \cap EP(F,C)$. In view of the bifunction, one finds from Lemma 1.1 that $u_n = Res_{\lambda_n} y_n$, yielding from the fact that the resolvet is firmly nonexpansive that

$$\|u_n - j\| = \|Res_{\lambda_n} y_n - Res_{\lambda_n} j\| \leqslant \|y_n - j\|. \tag{2.3}$$

From the construction of the inertial term on $\{\theta_n\}$, one has

$$\|y_n - j\| = \|\theta_n (x_n - x_{n-1}) + x_n - j\| \le \|x_n - j\| + \theta_n \|x_n - x_{n-1}\|. \tag{2.4}$$

Since W_n is nonexpansive for each n, then

$$\begin{aligned} \|x_{n+1} - j\| &\le \alpha_n \|f(y_n) - j\| + (1-\alpha_n)\|W_n u_n - j\| \\ &\le \alpha_n \|f(j) - j\| + \alpha_n \|f(y_n) - f(j)\| + (1-\alpha_n)\|W_n u_n - W_n j\| \\ &\le \alpha_n \|f(j) - j\| + \alpha_n \alpha \|y_n - j\| + (1-\alpha_n)\|u_n - j\|. \end{aligned} \tag{2.5}$$

Combing (2.3), (2.4), and (2.5) yields that

$$\begin{aligned} \|x_{n+1} - j\| &\le \alpha_n \|f(y_n) - j\| + (1-\alpha_n)\|W_n u_n - j\| \\ &\le \alpha_n \|f(j) - j\| + \alpha_n \|f(y_n) - f(j)\| + (1-\alpha_n)\|W_n u_n - W_n j\| \\ &\le \alpha_n \|f(j) - j\| + \alpha_n \alpha \|y_n - j\| + (1-\alpha_n)\|u_n - j\| \\ &\le \alpha_n \|f(j) - j\| + (1 - \alpha_n(1-\alpha))\|y_n - j\| \\ &\le (1 - \alpha_n(1-\alpha))\|x_n - j\| \\ &\quad + \alpha_n(1-\alpha)\left(\frac{(1-\alpha_n(1-\alpha))}{(1-\alpha)} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| + \frac{\|f(j) - j\|}{1-\alpha} \right). \end{aligned}$$

Since $\{\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|\}$ is bounded, we also have that

$$\left\{ \frac{(1-\alpha_n(1-\alpha))}{(1-\alpha)} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| + \frac{\|f(j) - j\|}{1-\alpha} \right\}$$

is also bounded. So,

$$\|x_{n+1} - j\| \le (1 - \alpha_n(1-\alpha))\|x_n - j\| + \alpha_n(1-\alpha)M,$$

where M is an appropriate constant. Next, we always use M to denote different appropriate constant. By mathematical induction, we easily conclude that $\|x_{n+1} - j\| \le \max\{M, \|j - x_0\|\}$, which indicates that $\{x_n\}$ is bounded, too. On account of the definitions of f and T_i, we also say that $\{u_n\}$ and $\{y_n\}$ are bounded as well. Since $\{x_n\}$ is bounded, one can assert there exits a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$.

Next, one observes that $u_n = Res_{\lambda_n} y_n$, which implies that

$$\langle y - u_n, u_n - y_n\rangle + \lambda_n F(u_n, y) \geqslant 0 \quad \text{for all } y \in C \tag{2.6}$$

Letting $y = u_{n+1}$ in (2.6), one sees that

$$\langle u_{n+1} - u_n, u_n - y_n\rangle + \lambda_n F(u_n, u_{n+1}) \geqslant 0. \tag{2.7}$$

Further observe that $u_{n+1} = R_{\lambda_{n+1}} y_{n+1}$, which indeed implies that and

$$\langle y - u_{n+1}, u_{n+1} - y_{n+1}\rangle + \lambda_{n+1} F(u_{n+1}, y) \geqslant 0 \quad \text{for all } y \in C. \tag{2.8}$$

Setting $y = u_n$ in (2.8) yields that

$$\langle u_n - u_{n+1}, u_{n+1} - y_{n+1}\rangle + \lambda_{n+1} F(u_{n+1}, u_n) \geqslant 0. \tag{2.9}$$

Combing (2.7) and (2.9) implies that

$$\begin{aligned}
&\frac{1}{\lambda_n}\langle u_{n+1} - u_n, u_n - y_n\rangle + \frac{1}{\lambda_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - y_{n+1}\rangle \\
&\qquad \geqslant \frac{1}{\lambda_n}\langle u_{n+1} - u_n, u_n - y_n\rangle + \frac{1}{\lambda_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - y_{n+1}\rangle \\
&\qquad\qquad + F(u_{n+1}, u_n) + F(u_n, u_{n+1}) \geqslant 0,
\end{aligned}$$

that is,

$$\left\langle u_n - u_{n+1} + u_{n+1} - y_n - \frac{\lambda_n}{\lambda_{n+1}}(u_{n+1} - y_{n+1}), u_{n+1} - u_n \right\rangle \geqslant 0.$$

Furthermore,

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leqslant \left\langle y_{n+1} - y_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(u_{n+1} - y_{n+1}), u_{n+1} - u_n \right\rangle \\
&\leqslant \|u_{n+1} - u_n| \left(\|y_{n+1} - y_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|u_{n+1} - y_{n+1}\| \right).
\end{aligned}$$

From $\liminf_{n\to\infty} \lambda_n > 0$, one without loss of generality assumes that there exists a real positive number λ such $\lambda \leq \lambda_n$, which shows

$$\|u_{n+1} - u_n\| \leqslant \|y_{n+1} - y_n\| + \frac{|\lambda_{n+1} - \lambda_n| M}{\lambda}, \tag{2.10}$$

Observe that

$$x_{n+1} - x_n = (1-\alpha_n)W_n u_n - (1-\alpha_{n-1})W_{n-1}u_{n-1} + \alpha_n f(y_n) - \alpha_{n-1} f(y_{n-1}).$$

Since W_n is nonexpanisve, we have that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1-\alpha_n)\|W_n u_n - W_{n-1}u_{n-1}\| \\
&\qquad + (\alpha_{n-1} - \alpha_n)\|W_{n-1}u_{n-1} - f(y_{n-1})\| + \alpha_n \alpha \|y_n - y_{n-1}\|.
\end{aligned} \tag{2.11}$$

Indeed, we also have

$$\begin{aligned}\|W_{n-1}u_{n-1}-W_nu_n\| &\leq \|Wu_n-W_nu_n\|+\|Wu_{n-1}-Wu_n\|\\ &\quad+\|W_{n-1}u_{n-1}-Wu_{n-1}\|\\ &\leq 2\sup_{y\in R}\|Wy-W_ny\|+\|u_{n-1}-u_n\|\end{aligned}$$

with R being an appropriate bounded subset of set C. Hence, it follows from (2.11) that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq 2\sup_{y\in R}\|Wy-W_ny\|+(1-\alpha_n)\|u_{n-1}-u_n\|\\ &\quad+(\alpha_{n-1}-\alpha_n)\|W_{n-1}u_{n-1}-f(y_{n-1})\|+\alpha_n\alpha\|y_n-y_{n-1}\|,\end{aligned}$$

which together (2.10) finds

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq 2\sup_{y\in R}\|Wy-W_ny\|+(1-\alpha_n(1-\alpha_n))\|y_{n+1}-y_n\|\\ &\quad+\frac{|\lambda_{n+1}-\lambda_n|\,M}{\lambda}+(\alpha_{n-1}-\alpha_n)\|W_{n-1}u_{n-1}-f(y_{n-1})\|\end{aligned}\tag{2.12}$$

On the other hand, one has

$$\|y_n-y_{n+1}\|\leqslant\theta_{n+1}\|x_{n+1}-x_n\|+\theta_n\|x_n-x_{n-1}\|+\|x_{n+1}-x_n\|.\tag{2.13}$$

Combing this together with (2.12) yields

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq 2\sup_{y\in R}\|Wy-W_ny\|+(1-\alpha_n(1-\alpha_n))\|y_{n+1}-y_n\|\\ &\quad+\frac{|\lambda_{n+1}-\lambda_n|\,M}{\lambda}+(\alpha_{n-1}-\alpha_n)\|W_{n-1}u_{n-1}-f(y_{n-1})\|\\ &\leq 2\sup_{y\in R}\|Wy-W_ny\|\\ &\quad+(1-\alpha_n(1-\alpha_n))(\theta_{n+1}\|x_{n+1}-x_n\|+\theta_n\|x_n-x_{n-1}\|)\\ &\quad+(1-\alpha_n(1-\alpha_n))\,\|x_{n+1}-x_n\|+\frac{|\lambda_{n+1}-\lambda_n|\,M}{\lambda}\\ &\quad+(\alpha_{n-1}-\alpha_n)\|W_{n-1}u_{n-1}-f(y_{n-1})\|.\end{aligned}$$

Thus $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. (2.10) and (2.13) yield that $\|u_{n+1}-u_n\|\to 0$ as $n\to\infty$. On account of

$$x_{n+1}=\alpha_nf(y_n)+(1-\alpha_n)\,W_nu_n,$$

one has

$$\begin{aligned}\|x_n-W_nu_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-W_nu_n\|\\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|f(y_n)-W_nu_n\|.\end{aligned}$$

Since $\alpha_n\to 0$ as $n\to\infty$, we have $\|x_n-W_nu_n\|\to 0$ as $n\to\infty$. From the construction of y_n, we get $\|y_n-x_n\|\leqslant\theta_n\|x_n-x_{n-1}\|\to 0$. Since

$$\|y_n-x_{n+1}\|\leqslant\|y_n-x_n\|+\|x_n-x_{n+1}\|$$

we have $\|y_n - x_{n+1}\| \to \infty$ as $n \to \infty$. Observe that the resolvent of bifunctions are firmly nonexpansive. For $j \in \cap_{i=1}F(T_i) \cap EP(F,C)$, we have

$$\begin{aligned}2\|u_n - j\|^2 &\leq 2\langle u_n - j, y_n - j\rangle \\ &= \|u_n - j\|^2 - \|y_n - u_n\|^2 + \|y_n - j\|^2,\end{aligned}$$

that is, $\|u_n - j\|^2 \leqslant \|y_n - j\|^2 - \|y_n - u_n\|^2$. On the other hand, the convexity of $\|\cdot\|^2$ yields

$$\begin{aligned}\|x_{n+1} - v\|^2 &\leqslant \alpha_n \|f(y_n) - j\|^2 + (1-\alpha_n)\|W_n u_n - W_n v\|^2 \\ &\leqslant \alpha_n \|f(y_n) - j\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\ &\leqslant (1-\alpha_n)\|y_n - j\|^2 - (1-\alpha_n)\|y_n - u_n\|^2 + \alpha_n M,\end{aligned}$$

which also mean

$$\begin{aligned}(1-\alpha_n)\|y_n - u_n\|^2 &\leqslant (1-\alpha_n)\|y_n - j\|^2 - \|x_{n+1} - v\|^2 + \alpha_n M \\ &\leqslant \|y_n - j\|^2 - \|x_{n+1} - v\|^2 + \alpha_n M \\ &\leqslant \|y_n - x_{n+1}\|(\|y_n - j\| + \|x_{n+1} - j\|) + \alpha_n M \\ &\leqslant (\|y_n - x_{n+1}\| + \alpha_n)M,\end{aligned}$$

which indicate by the restriction on $\{\alpha_n\}$ that $\|u_n - y_n\| \to 0$ when n tends to ∞. Observe $\|x_n - u_n\| \leqslant \|y_n - x_n\| + \|u_n - y_n\|$ and $\|y_n - x_n\| \to 0$. it results $\|x_n - u_n\| \to 0$ as $n \to \infty$. Notice

$$\|W_n u_n - u_n\| \leqslant \|W_n u_n - x_n\| + \|x_n - u_n\|,$$

and $\|x_n - W_n u_n\| \to 0$ as $n \to \infty$. It amounts to

$$\lim_{n\to\infty} \|W_n u_n - u_n\| = 0.$$

The following proof is split into three steps.

Step 1. Show that $p \in \cap_{i=1}^{F}(T_i)$.

Observe that

$$\|x_n - Wx_n\| \leq 2\|x_n - u_n\| + \|u_n - W_n u_n\| + \|W_n x_n - Wx_n\|.$$

It results in $\|x_n - Wx_n\| \to 0$ as $n \to \infty$. Since W is nonexpansive and it satisfies the Browder's demiclosed principle, one has q is a fixed point of W, which present that q is also a common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Step 2. Show that $p \in EP(F,C)$.

By the construction of u_n, that is, $u_n = Res_{\lambda_n} y_n$, we get

$$F(y, u_{n_i}) \leq \left\langle y - u_{n_i}, \frac{u_{n_i} - y_{n_i}}{\lambda_{n_i}} \right\rangle.$$

Noticing $\|u_{n_i} - x_{n_i}\| \to 0$ and $\lambda_{n_i} \nrightarrow 0$, we find $\frac{\|u_{n_i} - x_{n_i}\|}{\lambda_{n_i}} \to 0$. By $u_{n_i} \rightharpoonup p$, for all $y \in C$, $F(y,q) \leqslant 0$. We next suppose $y_q^z = zy + (1-z)q$, where $0 < z \leqslant 1$ and $y \in C$. We have y_q^z is in C, therefore $F(y_q^z, q) \leqslant 0$. It has

$$0 \leqslant (1-z)F(y_q^z, q) + zF(y_q^z, y) \leqslant zF(y_q^z, y)$$

Consequently, $F(q,y) \geqslant 0$ for all $y \in C$, that is, $p \in EP(F,C)$. It finishes the proof $p \in \cap_{i=1}^{\infty} F(T_i) \cap EP(F,C)$.

Step 3. $\{x_n\}$ converges strongly to z.

Notice there exists an (only) element z in H with the condition $z = P_{\cap_{i=1}^{\infty} F(T_i)\cap EP(F,C)} f(z)$ due to the projection and the contraction. It results

$$\begin{aligned}\|y_n - z\| &= \|x_n + \theta_n (x_n - x_{n-1}) - z\| \\ &\leqslant \sqrt{\|x_n - z\|^2 + 2\theta_n \langle x_n - x_{n-1}, x_n - z\rangle + \theta_n \|x_n - x_{n-1}\|^2}.\end{aligned}$$

On the other hand,

$$2\langle x_n - x_{n-1}, x_n - z\rangle = -\|x_{n-1} - z\|^2 + \|x_n - z\|^2 + \|x_n - x_{n-1}\|^2.$$

It yields

$$\|y_n - z\| \leqslant \sqrt{2\theta_n \|x_n - x_{n-1}\|^2 + \theta_n \left(\|x_n - z\|^2 - \|x_{n-1} - z\|^2\right) + \|x_n - z\|^2}.$$

We also have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leqslant (1-\alpha_n)^2 \|W_n u_n - W_n z\|^2 + 2\alpha_n \langle f(y_n) - z, x_{n+1} - z\rangle \\ &\leqslant (1-\alpha_n)^2 \|u_n - z\|^2 + 2\alpha_n \langle f(y_n) - f(z), x_{n+1} - z\rangle \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\ &\leqslant (1-\alpha_n)^2 \|y_n - z\|^2 + \alpha_n a(\|y_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\ &\leqslant \left(1 - \frac{2(1-a)\alpha_n}{1-\alpha_n a}\right) \|y_n - z\|^2 + \frac{2(1-a)\alpha_n}{1-\alpha_n a} \cdot \frac{\alpha_n}{2(1-a)} \|y_n - z\|^2 \\ &\quad + \frac{2(1-a)\alpha_n}{1-\alpha_n a} \cdot \frac{1}{1-a} \langle f(z) - z, x_{n+1} - z\rangle,\end{aligned}$$

It further gets

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leqslant (1-\beta_n) \|y_n - z\|^2 \\ &\quad + \beta_n \left\{\frac{\alpha_n M}{2(1-a)} + \frac{1}{1-a} \langle f(z) - z, x_{n+1} - z\rangle\right\} \\ &\leqslant (1-\beta_n) \|x_n - z\|^2 \\ &\quad + \theta_n (1-\beta_n) \left(\|x_n - z\|^2 - \|x_{n-1} - z\|^2\right) \\ &\quad + 2\theta_n (1-\beta_n) \|x_n - x_{n-1}\|^2 + \xi_n \beta_n,\end{aligned} \tag{2.14}$$

where $\xi_n = \frac{\alpha_n M}{2(1-a)} + \frac{1}{1-a} \langle f(z) - z, x_{n+1} - z\rangle$, $\beta_n = \frac{2(1-a)\alpha_n}{1-a\alpha_n}$ (noiced $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$). Thus

$$\begin{aligned}\Gamma_{n+1} &\leqslant (1-\beta_n) \Gamma_n + \theta_n (1-\beta_n)(\Gamma_n - \Gamma_{n-1}) \\ &\quad + 2\theta_n (1-\beta_n) \|x_n - x_{n-1}\|^2 + \beta_n b_n,\end{aligned} \tag{2.15}$$

where $\Gamma_n := \|x_n - z\|^2$.

The following proof is split into two sub-cases.

Case i. We consider the situation there exists positive integer n_0 such that $\{\Gamma_n\}$ is decreasing for all $n \geq n_0$ monotonically, so $\{\Gamma_n\}$ is a convergent sequence. Since H is a reflexive space, there exists a subsequence of $\{x_n\}$ such that, say $\{x_{n_k}\}$

with $\{x_{n_k}\} \rightharpoonup p \in H$. We obtain p is in $\cap_{i=1}^{\infty} F(T_i) \cap EP(F,C)$. Due to $z = P_{F(S)\cap EP(F)} f(z)$, we obtain

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle .$$

Hence, $\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \leqslant 0$ due to $\limsup_{n\to\infty} \xi_n \le 0$. From (2.15), one has

$$\begin{aligned} \Gamma_{n+1} &\leqslant (1-\beta_n)\Gamma_n + \theta_n(1-\beta_n)\|x_n - x_{n-1}\| \left(\sqrt{\Gamma_n} + \sqrt{\Gamma_{n-1}}\right) + \xi_n\beta_n \\ &\quad + 2\theta_n(1-\beta_n)\|x_n - x_{n-1}\|^2 \\ &\leqslant (1-\beta_n)\Gamma_n + \theta_n\|x_n - x_{n-1}\| M + \xi_n\beta_n, \end{aligned} \tag{2.16}$$

Using Lemma 1.3, we get that $x_n \to z$ as desired.

Case ii. If $\{\Gamma_n\}$ is not decreasing monotonically. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all n such $n \ge n_0$ for some n_0 large enough via

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\},$$

One asserts that τ is a non-decreasing sequence such that $0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ for all $n \ge n_0$ and $\lim_{n\to\infty}\tau(n) = \infty$. It is also not heard to show that $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$. By following Case i, $\limsup_{n\to\infty}\left\langle f(z) - z, x_{\tau(n)+1} - z\right\rangle \le 0$. It follows from (2.16) that

$$\Gamma_{\tau(n)} \le \frac{\theta_{\tau(n)}}{\beta_{\tau(n)}} \left\|x_{\tau(n)} - x_{\tau(n)-1}\right\| U + \xi_{\tau(n)}.$$

In addition, for $n \ge n_0$, from the definition of $\tau(n)$, $\Gamma_j \ge \Gamma_{j+1}$ for $\tau(n)+1 \le j \le n$. As a result, $\Gamma_{\tau(n)+1} \ge \Gamma_{\tau(n)+2} \ge \cdots \ge \Gamma_{n-1} \ge \Gamma_n$ Thus $0 \le \Gamma_n \le \Gamma_{\tau(n)+1}$. Furthermore, We have that

$$\lim_{n\to\infty} \left\|x_{\tau(n)} - z\right\| = \lim_{n\to\infty} \Gamma_{\tau(n)} = 0$$

which implies $\lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$ and $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, $\{x_n\}$ converges strongly to z. This completes the proof. □

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Manuscript received February 16, 2023
revised May 3, 2023

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