Journal of Nonlinear and Convex Analysis
Volume 25, Number 3, 2024, 567–578

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON APPROXIMATION METHODS FOR SOLUTIONS OF SPLIT FIXED POINTS AND VARIATIONAL INCLUSION PROBLEMS

LI ZHANG

Abstract. In this paper, we devote to find a solution of a split fixed point and variational inclusion problem in Hilbert spaces. Based on resolvent technique, fixed point method and self-adaptive rule, we present a new algorithm for solving the studied split problem. Under some mild assumptions, we prove the introduced algorithm converges strongly to some special solution of the split problem.

1. Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $\phi_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $\phi_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, $\psi_1, \varphi_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $\psi_2, \varphi_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Set $Fix(\varphi_1) := \{p_1 \in \mathcal{H}_1 : \varphi_1(p_1) = p_1\}$ and $Fix(\varphi_2) := \{p_2 \in \mathcal{H}_2 : \varphi_2(p_2) = p_2\}$.

The task of this paper is to seek a solution of the following split problem: find $p \in \mathcal{H}_1$ such that

$$p \in Fix(\varphi_1) \cap (\phi_1 + \psi_1)^{-1}(0) \text{ and } Ap \in Fix(\varphi_2) \cap (\phi_2 + \psi_2)^{-1}(0), \tag{1.1}$$

where $(\phi_i+\psi_i)^{-1}(0)(i = 1, 2)$ is the solution set of the following variational inclusion:

$$\text{Find } \hat{p} \in \mathcal{H}_i \text{ such that } 0 \in \phi_i(\hat{p}) + \psi_i(\hat{p}),\ i = 1, 2. \tag{1.2}$$

Special Cases. (i) The split fixed point problem is to find a point $\tilde{y} \in \mathcal{H}_1$ such that

$$\text{Find } \tilde{y} \in Fix(\varphi_1) \text{ such that } A\tilde{y} \in Fix(\varphi_2). \tag{1.3}$$

(ii) The split variational inclusion problem is to find a point $\tilde{z} \in \mathcal{H}_1$ such that

$$\text{Find } \tilde{z} \in (\phi_1 + \psi_1)^{-1}(0) \text{ such that } A\tilde{z} \in (\phi_2 + \psi_2)^{-1}(0). \tag{1.4}$$

Variational inequalities and variational inclusion problems have attracted so much attention and are central problems in nonlinear analysis and optimization ([7, 11, 13, 18, 19, 21, 24]). There are a large number of iterative methods and techniques for solving variational inequalities and variational inclusion problems. Among them, many common methods and techniques such as projection method, resolvent method, fixed point method and self-adaptive technique are to apply for solving VI, see [1, 4–6, 8]. On the other hand, the split fixed point problem has been investigated extensively due to it is a generalization of the split feasibility problem

2020 *Mathematics Subject Classification.* 47H10, 47J25, 65K15, 90C25.

Key words and phrases. Split problem, monotone mapping, pseudocontractive mapping, fixed point, variational inclusion.

This work was supported by the National Natural Science Foundation of China (grant number: NSFC 12201462).

([3, 9, 10, 12, 14, 16, 17, 22–24]). Several valuable methods to solve the split problem are presented and studied, see [2, 4, 5, 8].

Motivated and inspired by the work in the literature, the main purpose of this paper is to investigate the split problem (1.1) in which the consider operators ϕ and ϕ_2 are maximal monotone, ψ and ψ_2 are ismo and φ_1 and φ_2 are pseudocontractive. To solve the split problem (1.1), we construct a new algorithm which are based on resolvent technique, fixed point method and self-adaptive rule. Under some standard assumptions, we show that the constructed algorithm converges strongly to a point in the solution set Γ of the split problem (1.1) which is the nearest point projection of a fixed point $\hat{v}$ from $\mathcal{H}_1$ to Γ.

2. Notations and Lemmas

We include some related notations and lemmas in this section. Throughout, suppose $\mathcal{H}$ is a real Hilbert space. $\to$ and $\rightharpoonup$ mean strong convergence and weak convergence, respectively. $\forall p, \hat{p} \in \mathcal{H}$, the following results hold:

$$\|\delta p + (1-\delta)\hat{p}\|^2 = \delta\|p\|^2 + (1-\delta)\|\hat{p}\|^2 - \delta(1-\delta)\|p-\hat{p}\|^2, \ \forall \delta \in \mathbb{R}, \tag{2.1}$$

$$\|p+\hat{p}\|^2 = \|p\|^2 + 2\langle p, \hat{p}\rangle + \|\hat{p}\|^2, \tag{2.2}$$

and

$$\|p+\hat{p}\|^2 \le \|p\|^2 + 2\langle \hat{p}, p+\hat{p}\rangle. \tag{2.3}$$

A mapping $\varphi : \mathcal{H} \to \mathcal{H}$ is said to be pseudocontractive if

$$\langle \varphi(p) - \varphi(\hat{p}), p - \hat{p}\rangle \le \|p-\hat{p}\|^2, \ \forall p, \hat{p} \in \mathcal{H},$$

or,

$$\|\varphi(p) - \varphi(\hat{p})\|^2 \le \|p-\hat{p}\|^2 + \|(I-\varphi)p - (I-\varphi)\hat{p}\|^2, \ \forall p, \hat{p} \in \mathcal{H}. \tag{2.4}$$

A mapping $F : \mathcal{H} \to \mathcal{H}$ is said to be nonexpansive if

$$\|F(p) - F(\hat{p})\| \le \|p-\hat{p}\|, \ \forall p, \hat{p} \in \mathcal{H}.$$

A mapping $G : \mathcal{H} \to \mathcal{H}$ is said to be averaged if $G = (1-\mu)I + \mu F$, where $F : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping and $\mu \in (0,1)$.

A mapping $\psi : \mathcal{H} \to \mathcal{H}$ is said to be inverse strongly monotone if

$$\langle \psi(p) - \psi(\hat{p}), p-\hat{p}\rangle \ge \mu\|\psi(p) - \psi(\hat{p})\|^2, \ \forall p, \hat{p} \in \mathcal{H},$$

where $\mu > 0$ is a constant. For this case, ψ is called μ-ismo.

Let ϕ: $\mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued operator with domain $\mathrm{dom}(\phi)$. The graph of ϕ is defined by $\mathrm{G}(\phi) := \{(x,y) \in \mathcal{H} \times \mathcal{H} : x \in \mathrm{dom}(\phi), y \in \phi(x)\}$. Recall that ϕ is said to be

- monotone if the set $\mathrm{G}(\phi)$ is monotone, i.e.,

$$\langle p_1 - p_2, q_1 - q_2\rangle \ge 0, \ \forall (p_i, q_i) \in \mathrm{G}(\phi), i = 1, 2.$$

- maximal monotone if and only if ϕ is monotone and $(p_1, q_1) \in \mathcal{H} \times \mathcal{H}$,

$$\text{for all } (p_2, q_2) \in \mathrm{G}(\phi), \ \langle p_1 - p_2, q_1 - q_2\rangle \ge 0 \Rightarrow (p_1, q_1) \in \mathrm{G}(\phi). \tag{2.5}$$

Let $\phi : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. For any $\gamma > 0$, the resolvent $(I + \gamma\phi)^{-1}$ is a single-valued nonexpansive mapping.

For a nonempty closed convex set $\Gamma \subset \mathcal{H}$, there exists a unique nearest point $proj_\Gamma : \mathcal{H} \to \Gamma$ defined by for any $p^\dagger \in \mathcal{H}$, $proj_\Gamma(p^\dagger) := \arg\min_{p\in\Gamma}\{\|p - p^\dagger\|\}$ which satisfies

$$\langle p^\dagger - proj_\Gamma(p^\dagger), p - proj_\Gamma(p^\dagger)\rangle \leq 0,\ \forall p^\dagger \in \mathcal{H}, p \in \Gamma. \tag{2.6}$$

Lemma 2.1 ([10])**.** *Let $\phi : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $\psi : \mathcal{H} \to \mathcal{H}$ be κ-ismo. Then,*

(i) *$\hat{z} \in (\phi + \psi)^{-1}(0) \Leftrightarrow \hat{z} \in \mathrm{Fix}((I + \gamma\phi)^{-1}(I - \gamma\psi))$ for all $\gamma > 0$.*
(ii) *$(I + \gamma\phi)^{-1}(I - \gamma\psi)$ is averaged when $\gamma \in (0, 2\kappa)$.*

Lemma 2.2 ([20])**.** *Let $\varphi : \mathcal{H} \to \mathcal{H}$ be a continuous pseudocontractive mapping. If $Fix(\varphi) \neq \emptyset$, then $I - \varphi$ is demi-closed at zero.*

Lemma 2.3 ([15])**.** *Let $\{\zeta_n\}$, $\{\delta_n\}$ and $\{\mu_n\}$ be three real number sequences. Suppose that the following conditions are satisfied:*

(i) *$\zeta_n \in [0, +\infty)$, $\delta_n \in (0, 1)$, $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \mu_n \leq 0$ or $\sum_{n=1}^{\infty} |\mu_n\delta_n| < \infty$;*
(ii) *$\zeta_{n+1} \leq (1 - \delta_n)\zeta_n + \mu_n\delta_n$ for all $n \in \mathbb{N}$.*

Then $\lim_{n\to\infty} \zeta_n = 0$.

Lemma 2.4 ([8])**.** *Let $\{\tau_n\}$ be a real number sequence. Suppose that there is a subsequence $\{\tau_{n_k}\} \subset \{\tau_n\}$ such that for all $k \geq 0$, $\tau_{n_k} \leq \tau_{n_k+1}$. For every $n \geq m \in \mathbb{N}$, let $\{\nu(n)\}$ be an integer sequence generated by*

$$\nu(n) = \max\{m \leq k \leq n : \tau_{n_k} < \tau_{n_k+1}\}.$$

Then, $\lim_{n\to\infty} \nu(n) = \infty$ and for all $n \geq m$, $\max\{\tau_{\nu(n)}, \tau_n\} \leq \tau_{\nu(n)+1}$.

3. Iterative method and convergence result

In this section, we present an iterative algorithm and show it converges to a solution of the split problem (1.1). First, we give some conditions as follows:

(Con 1): Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a nonzero bounded linear operator. Let $\phi_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $\phi_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two maximal monotone operators and $\psi_1 : \mathcal{H}_1 \to \mathcal{H}_1$, $\psi_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be κ_1-ismo and κ_2-ismo, respectively. Let $\varphi_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $\varphi_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be two continuous pseudocontractive mappings.

(Con 2): Let $\beta \in (0, \frac{1}{\|A\|^2})$, $\omega_1 \in (0, 2\kappa_1)$, $\omega_2 \in (0, 2\kappa_2)$, $\delta_1 \in (0, 1), \delta_2 \in (0, 1), \epsilon \in (0, 1)$ and $\delta \in (0, \frac{1}{2}(1 - \max\{\delta_1, \delta_2\}))$ be seven constants. Let $\{\beta_n\}$ be a real number sequence in $(0, 1)$.

Set $\Gamma := \{\hat{x} \in \mathcal{H}_1 : \hat{x} \in Fix(\varphi_1) \cap (\phi_1 + \psi_1)^{-1}(0)$ and $A\hat{x} \in Fix(\varphi_2) \cap (\phi_2 + \psi_2)^{-1}(0)\}$. Next, we introduce our algorithm to solve the split problem (1.1).

Algorithm 3.1. *Let $\hat{v} \in \mathcal{H}_1$ be a fixed point and let $x_0 \in \mathcal{H}_1$ be a starting point. Set $n = 0$.*

Step 1. Assume the n-th iterate x_n is known. Calculate

$$u_n = (I + \omega_1\phi_1)^{-1}(x_n - \omega_1\psi_1(x_n)), \tag{3.1}$$

and

$$\begin{cases} s_n = (1-\alpha_n)u_n + \alpha_n\varphi_1(u_n), \\ w_n = (1-\frac{\alpha_n}{2})u_n + \frac{\alpha_n}{2}\varphi_1(s_n). \end{cases} \tag{3.2}$$

where $\alpha_n = \delta\epsilon^{m_i}$ *with* m_i *being the smallest nonnegative integer satisfying the relation*

$$\alpha_n^2\|\varphi_1(u_n) - \varphi_1(s_n)\|^2 \leq \delta_1\|u_n - s_n\|^2. \tag{3.3}$$

Step 2. Calculate

$$y_n = (I + \omega_2\phi_2)^{-1}(Aw_n - \omega_2\psi_2(Aw_n)), \tag{3.4}$$

and

$$\begin{cases} t_n = (1-\theta_n)y_n + \theta_n\varphi_2(y_n), \\ z_n = (1-\frac{\theta_n}{2})y_n + \frac{\theta_n}{2}\varphi_2(t_n). \end{cases} \tag{3.5}$$

where $\theta_n = \delta\epsilon^{m_j}$ *with* m_j *being the smallest nonnegative integer satisfying the relation*

$$\theta_n^2\|\varphi_2(y_n) - \varphi_2(t_n)\|^2 \leq \delta_2\|y_n - t_n\|^2. \tag{3.6}$$

Step 3. Compute

$$v_n = w_n - \beta A^*(Aw_n - z_n), \tag{3.7}$$

and

$$x_{n+1} = \beta_n\hat{v} + (1-\beta_n)v_n. \tag{3.8}$$

Let $n := n+1$ *and go back to Step 1.*

Theorem 3.2. *Suppose that* $\Gamma \neq \emptyset$ *and the following restrictions are satisfied:* $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^{\infty}\beta_n = \infty$. *Then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to* $proj_\Gamma(\hat{v})$.

Proof. Set $q^* = proj_\Gamma(\hat{v})$. Applying Lemma 2.1, we can write $(I+\omega_1\phi_1)^{-1}(I - \omega_1\psi_1) = (1-\zeta_1)I + \zeta_1F_1$ where $F_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is a nonexpansive mapping and $\zeta_1 \in (0,1)$. By (3.1), we obtain

$$u_n = (I+\omega_1\phi_1)^{-1}(x_n - \omega_1\psi_1(x_n)) = (1-\zeta_1)x_n + \zeta_1F_1(x_n). \tag{3.9}$$

Utilizing (2.1) to (3.9) to get

$$\begin{aligned} \|u_n - q^*\|^2 &= (1-\zeta_1)\|x_n - q^*\|^2 + \zeta_1\|F_1(x_n) - q^*\|^2 \\ &\quad - \zeta_1(1-\zeta_1)\|x_n - F_1(x_n)\|^2 \\ &\leq \|x_n - q^*\|^2 - \frac{1-\zeta_1}{\zeta_1}\|x_n - u_n\|^2. \end{aligned} \tag{3.10}$$

By Lemma 2.1, we can write $(I+\omega_2\phi_2)^{-1}(I-\omega_2\psi_2) = (1-\zeta_2)I + \zeta_2F_2$ where $\zeta_2 \in (0,1)$ and $F_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is a nonexpansive operator. It follows from (2.1) and

(3.4) that

$$\begin{aligned}\|y_n - Aq^*\|^2 &= (1-\zeta_2)\|Aw_n - Aq^*\|^2 + \zeta_2\|F_2(Aw_n) - Aq^*\|^2 \\ &\quad - \zeta_2(1-\zeta_2)\|Aw_n - F_2(Aw_n)\|^2 \\ &\leq \|Aw_n - Aq^*\|^2 - \frac{1-\zeta_2}{\zeta_2}\|Aw_n - y_n\|^2.\end{aligned} \tag{3.11}$$

From (2.1), (2.4) and (3.2), we have

$$\begin{aligned}\|s_n - q^*\|^2 &= (1-\alpha_n)\|u_n - q^*\|^2 + \alpha_n\|\varphi_1(u_n) - q^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|u_n - \varphi_1(u_n)\|^2 \\ &\leq (1-\alpha_n)\|u_n - q^*\|^2 - \alpha_n(1-\alpha_n)\|u_n - \varphi_1(u_n)\|^2 \\ &\quad + \alpha_n(\|u_n - q^*\|^2 + \|u_n - \varphi_1(u_n)\|^2) \\ &= \|u_n - q^*\|^2 + \alpha_n^2\|u_n - \varphi_1(u_n)\|^2,\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\|w_n - q^*\|^2 &= (1-\frac{\alpha_n}{2})\|u_n - q^*\|^2 + \frac{\alpha_n}{2}\|\varphi_1(s_n) - q^*\|^2 \\ &\quad - (1-\frac{\alpha_n}{2})\frac{\alpha_n}{2}\|u_n - \varphi_1(s_n)\|^2 \\ &\leq (1-\frac{\alpha_n}{2})\|u_n - q^*\|^2 - (1-\frac{\alpha_n}{2})\frac{\alpha_n}{2}\|u_n - \varphi_1(s_n)\|^2 \\ &\quad + \frac{\alpha_n}{2}(\|s_n - q^*\|^2 + \|s_n - \varphi_1(s_n)\|^2).\end{aligned} \tag{3.13}$$

Similarly, by (2.1), (3.2) and (3.3), we have

$$\begin{aligned}\|s_n - \varphi_1(s_n)\|^2 &= (1-\alpha_n)\|u_n - \varphi_1(s_n)\|^2 + \alpha_n\|\varphi_1(u_n) - \varphi_1(s_n)\|^2 \\ &\quad - (1-\alpha_n)\alpha_n\|u_n - \varphi_1(u_n)\|^2 \\ &\leq (1-\alpha_n)\|u_n - \varphi_1(s_n)\|^2 + \frac{\delta_1}{\alpha_n}\|u_n - s_n\|^2 \\ &\quad - (1-\alpha_n)\alpha_n\|u_n - \varphi_1(u_n)\|^2 \\ &= (1-\alpha_n)\|u_n - \varphi_1(s_n)\|^2 - \alpha_n(1-\delta_1-\alpha_n)\|u_n - \varphi_1(u_n)\|^2.\end{aligned} \tag{3.14}$$

Based on (3.12)-(3.14), we have

$$\begin{aligned}\|w_n - q^*\|^2 &\leq (1-\frac{\alpha_n}{2})\|u_n - q^*\|^2 + \frac{\alpha_n}{2}(1-\alpha_n)\|u_n - \varphi_1(s_n)\|^2 \\ &\quad + \frac{\alpha_n}{2}(\|u_n - q^*\|^2 + \alpha_n^2\|u_n - \varphi_1(u_n)\|^2) \\ &\quad - \frac{\alpha_n^2}{2}(1-\delta_1-\alpha_n)\|u_n - \varphi_1(u_n)\|^2 \\ &\quad - (1-\frac{\alpha_n}{2})\frac{\alpha_n}{2}\|u_n - \varphi_1(s_n)\|^2 \\ &= \|u_n - q^*\|^2 - \frac{\alpha_n^2}{2}(1-\delta_1-2\alpha_n)\|u_n - \varphi_1(u_n)\|^2 \\ &\quad - \frac{\alpha_n^2}{4}\|u_n - \varphi_1(s_n)\|^2.\end{aligned} \tag{3.15}$$

From (2.1), (2.4) and (3.5), we have

$$\begin{aligned}
\|t_n - Aq^*\|^2 &= (1-\theta_n)\|y_n - Aq^*\|^2 + \theta_n\|\varphi_2(y_n) - Aq^*\|^2 \\
&\quad - (1-\theta_n)\theta_n\|y_n - \varphi_2(y_n)\|^2 \\
&\le (1-\theta_n)\|y_n - Aq^*\|^2 - (1-\theta_n)\theta_n\|y_n - \varphi_2(y_n)\|^2 \\
&\quad + \theta_n(\|y_n - Aq^*\|^2 + \|y_n - \varphi_2(y_n)\|^2) \\
&= \|y_n - Aq^*\|^2 + \theta_n^2\|y_n - \varphi_2(y_n)\|^2,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\|z_n - Aq^*\|^2 &= (1-\frac{\theta_n}{2})\|y_n - Aq^*\|^2 + \frac{\theta_n}{2}\|\varphi_2(t_n) - Aq^*\|^2 \\
&\quad - (1-\frac{\theta_n}{2})\frac{\theta_n}{2}\|y_n - \varphi_2(t_n)\|^2 \\
&\le (1-\frac{\theta_n}{2})\|y_n - Aq^*\|^2 - (1-\frac{\theta_n}{2})\frac{\theta_n}{2}\|y_n - \varphi_2(t_n)\|^2 \\
&\quad + \frac{\theta_n}{2}(\|t_n - Aq^*\|^2 + \|t_n - \varphi_2(t_n)\|^2).
\end{aligned} \tag{3.17}$$

Again, by (2.1), (3.5) and (3.6), we have

$$\begin{aligned}
\|t_n - \varphi_2(t_n)\|^2 &= (1-\theta_n)\|y_n - \varphi_2(t_n)\|^2 + \theta_n\|\varphi_2(y_n) - \varphi_2(t_n)\|^2 \\
&\quad - (1-\theta_n)\theta_n\|y_n - \varphi_2(y_n)\|^2 \\
&\le (1-\theta_n)\|y_n - \varphi_2(t_n)\|^2 + \frac{\delta_2}{\theta_n}\|y_n - t_n\|^2 \\
&\quad - (1-\theta_n)\theta_n\|y_n - \varphi_2(y_n)\|^2 \\
&= (1-\theta_n)\|y_n - \varphi_2(t_n)\|^2 + \delta_2\theta_n\|y_n - \varphi_2(y_n)\|^2 \\
&\quad - (1-\theta_n)\theta_n\|y_n - \varphi_2(y_n)\|^2 \\
&= (1-\theta_n)\|y_n - \varphi_2(t_n)\|^2 - (1-\delta_2-\theta_n)\theta_n\|y_n - \varphi_2(y_n)\|^2.
\end{aligned} \tag{3.18}$$

Take into account of (3.16)-(3.18), we obtain

$$\begin{aligned}
\|z_n - Aq^*\|^2 &\le (1-\frac{\theta_n}{2})\|y_n - Aq^*\|^2 + \frac{\theta_n}{2}(1-\theta_n)\|y_n - \varphi_2(t_n)\|^2 \\
&\quad + \frac{\theta_n}{2}(\|y_n - Aq^*\|^2 + \theta_n^2\|y_n - \varphi_2(y_n)\|^2) \\
&\quad - \frac{\theta_n^2}{2}(1-\delta_2-\theta_n)\|y_n - \varphi_2(y_n)\|^2 \\
&\quad - (1-\frac{\theta_n}{2})\frac{\theta_n}{2}\|y_n - \varphi_2(t_n)\|^2 \\
&= \|y_n - Aq^*\|^2 - \frac{\theta_n^2}{2}(1-\delta_2-2\theta_n)\|y_n - \varphi_2(y_n)\|^2 \\
&\quad - \frac{\theta_n^2}{4}\|y_n - \varphi_2(t_n)\|^2.
\end{aligned} \tag{3.19}$$

Thanks to (3.7), we acquire

$$
\begin{aligned}
\|v_n - q^*\|^2 &= \|w_n - q^* - \beta A^*(Aw_n - z_n)\|^2 \\
&= \|w_n - q^*\|^2 + \beta^2 \|A^*(Aw_n - z_n)\|^2 \\
&\quad - 2\beta \langle w_n - q^*, A^*(Aw_n - z_n)\rangle \\
&= \|w_n - q^*\|^2 + \beta^2 \|A^*(Aw_n - z_n)\|^2 \\
&\quad - 2\beta \langle Aw_n - Aq^*, Aw_n - z_n\rangle .
\end{aligned} \tag{3.20}
$$

Combining (3.11) and (3.19), we have

$$
\begin{aligned}
\|z_n - Aq^*\|^2 \le \|Aw_n - Aq^*\|^2 &- \frac{1-\zeta_2}{\zeta_2}\|Aw_n - y_n\|^2 \\
&- \frac{\theta_n^2}{2}(1 - \delta_2 - 2\theta_n)\|y_n - \varphi_2(y_n)\|^2 \\
&- \frac{\theta_n^2}{4}\|y_n - \varphi_2(t_n)\|^2 .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\langle Aw_n - Aq^*, Aw_n - z_n\rangle &= \frac{1}{2}(\|Aw_n - Aq^*\|^2 + \|Aw_n - z_n\|^2 - \|z_n - Aq^*\|^2) \\
&\ge \frac{1}{2}\|Aw_n - z_n\|^2 + \frac{1-\zeta_2}{2\zeta_2}\|Aw_n - y_n\|^2 \\
&\quad + \frac{\theta_n^2}{4}(1 - \delta_2 - 2\theta_n)\|y_n - \varphi_2(y_n)\|^2 \\
&\quad + \frac{\theta_n^2}{8}\|\varphi_2(t_n) - y_n\|^2 .
\end{aligned}
$$

This together with (3.10), (3.15) and (3.20) implies that

$$
\begin{aligned}
\|v_n - q^*\|^2 &\le \|w_n - q^*\|^2 + \beta^2\|A\|^2\|Aw_n - z_n\|^2 - \beta\|Aw_n - z_n\|^2 \\
&\quad - \frac{\beta(1-\zeta_2)}{\zeta_2}\|Aw_n - y_n\|^2 - \frac{\beta\theta_n^2}{4}\|\varphi_2(t_n) - y_n\|^2 \\
&\quad - \frac{\beta\theta_n^2}{2}(1 - \delta_2 - 2\theta_n)\|y_n - \varphi_2(y_n)\|^2 \\
&\le \|x_n - q^*\|^2 - \frac{1-\zeta_1}{\zeta_1}\|x_n - u_n\|^2 - \frac{\beta\theta_n^2}{4}\|\varphi_2(t_n) - y_n\|^2 \\
&\quad - \frac{\alpha_n^2}{2}(1 - \delta_1 - 2\alpha_n)\|u_n - \varphi_1(u_n)\|^2 - \frac{\alpha_n^2}{4}\|u_n - \varphi_1(s_n)\|^2 \\
&\quad - \beta(1 - \beta\|A\|^2)\|Aw_n - z_n\|^2 - \frac{\beta(1-\zeta_2)}{\zeta_2}\|Aw_n - y_n\|^2 \\
&\quad - \frac{\beta\theta_n^2}{2}(1 - \delta_2 - 2\theta_n)\|y_n - \varphi_2(y_n)\|^2 .
\end{aligned} \tag{3.21}
$$

By virtue of (3.8) and (3.16), we attain

$$\begin{aligned}\|x_{n+1}-q^*\| &= \|\beta_n(\hat{v}-q^*)+(1-\beta_n)(v_n-q^*)\| \\ &\le \beta_n\|\hat{v}-q^*\|+(1-\beta_n)\|v_n-q^*\| \\ &\le \beta_n\|\hat{v}-q^*\|+(1-\beta_n)\|x_n-q^*\| \\ &\le \max\{\|x_0-q^*\|,\|\hat{v}-q^*\|\}.\end{aligned} \tag{3.22}$$

Hence, $\{x_n\}$ is bounded.

Now, we prove two cases. Case 1: there is an integer n_0 fulfilling $\|x_{n+1}-q^*\| \le \|x_n-q^*\|$ for all $n \ge n_0$. Case 2: for any n_0, there is an integer $m \ge n_0$ fulfilling $\|x_m-q^*\| \le \|x_{m+1}-q^*\|$.

Assume Case 1 holds. Then, $\lim_{n\to\infty}\|x_n-q^*\|$ exists. By (3.21) and (3.22), we have

$$\begin{aligned}\|x_{n+1}-q^*\|^2 &\le \beta_n\|\hat{v}-q^*\|^2+\|v_n-q^*\|^2 \\ &\le \beta_n\|\hat{v}-q^*\|^2+\|x_n-q^*\|^2-\frac{\alpha_n^2}{2}(1-\delta_1-2\alpha_n)\|u_n-\varphi_1(u_n)\|^2 \\ &\quad -\frac{1-\zeta_1}{\zeta_1}\|x_n-u_n\|^2-\frac{\beta\theta_n^2}{2}(1-\delta_2-2\theta_n)\|y_n-\varphi_2(y_n)\|^2 \\ &\quad -\frac{\alpha_n^2}{4}\|u_n-\varphi_1(s_n)\|^2-\beta(1-\beta\|A\|^2)\|Aw_n-z_n\|^2 \\ &\quad -\frac{\beta(1-\zeta_2)}{\zeta_2}\|Aw_n-y_n\|^2-\frac{\beta\theta_n^2}{4}\|\varphi_2(t_n)-y_n\|^2.\end{aligned}$$

It yields that

$$\begin{aligned}&\beta(1-\beta\|A\|^2)\|Aw_n-z_n\|^2+\frac{1-\zeta_1}{\zeta_1}\|x_n-u_n\|^2 \\ &\quad +\frac{\beta(1-\zeta_2)}{\zeta_2}\|Aw_n-y_n\|^2+\frac{\beta\theta_n^2}{4}\|\varphi_2(t_n)-y_n\|^2 \\ &\quad +\frac{\alpha_n^2}{2}(1-\delta_1-2\alpha_n)\|u_n-\varphi_1(u_n)\|^2+\frac{\alpha_n^2}{4}\|u_n-\varphi_1(s_n)\|^2 \\ &\quad +\frac{\beta\theta_n^2}{2}(1-\delta_2-2\theta_n)\|y_n-\varphi_2(y_n)\|^2 \\ &\le \beta_n\|\hat{v}-q^*\|^2+\|x_n-q^*\|^2-\|x_{n+1}-q^*\|^2 \to 0,\end{aligned}$$

which results in that

$$\lim_{n\to\infty}\|Aw_n-z_n\|=0, \tag{3.23}$$

$$\lim_{n\to\infty}\|x_n-u_n\|=0, \tag{3.24}$$

$$\lim_{n\to\infty}\|Aw_n-y_n\|=0, \tag{3.25}$$

$$\lim_{n\to\infty}\|\varphi_2(t_n)-y_n\|=0, \tag{3.26}$$

$$\lim_{n\to\infty}\|u_n-\varphi_1(s_n)\|=0, \tag{3.27}$$

$$\lim_{n\to\infty} \|u_n - \varphi_1(u_n)\| = 0, \tag{3.28}$$

and

$$\lim_{n\to\infty} \|y_n - \varphi_2(y_n)\| = 0. \tag{3.29}$$

By (3.7), we have $\|v_n - w_n\| \le \beta\|A\|\|z_n - Aw_n\|$. It follows from (3.23) that

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{3.30}$$

Note that $\|w_n - x_n\| \le \|w_n - u_n\| + \|u_n - x_n\| \le \frac{\alpha_n}{2}\|\varphi_1(s_n) - u_n\| + \|u_n - x_n\|$. From (3.24) and (3.32), we deduce

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.31}$$

Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{p}$ and

$$\limsup_{n\to\infty}\langle \hat{v} - q^*, x_n - q^*\rangle = \lim_{i\to\infty}\langle \hat{v} - q^*, x_{n_i} - q^*\rangle \tag{3.32}$$

Note that $Aw_{n_i} \rightharpoonup A\tilde{p}, u_{n_i} \rightharpoonup \tilde{p}, y_{n_i} \rightharpoonup A\tilde{p}, v_{n_i} \rightharpoonup \tilde{p}, w_{n_i} \rightharpoonup \tilde{p}$ and $z_{n_i} \rightharpoonup \tilde{p}$. So,

$$\left.\begin{array}{r}\|u_{n_i} - \varphi_1(u_{n_i})\| \to 0\\ u_{n_i} \rightharpoonup \tilde{p}\\ \text{Lemma 2.2}\end{array}\right\} \Rightarrow \tilde{p} \in Fix(\varphi_1) \text{ and } \left.\begin{array}{r}\|y_{n_i} - \varphi_2(y_{n_i})\| \to 0\\ y_{n_i} \rightharpoonup A\tilde{p}\\ \text{Lemma 2.2}\end{array}\right\} \Rightarrow A\tilde{p} \in Fix(\varphi_2).$$

At the same time, we have

$$\left.\begin{array}{r}\|(I + \omega_1\phi_1)^{-1}(I - \omega_1\psi_1)x_{n_i} - x_{n_i}\| \to 0\\ x_{n_i} \rightharpoonup \tilde{p}\\ \text{Lemmas 2.1 and 2.2}\end{array}\right\} \Rightarrow \tilde{p} \in (\phi_1 + \psi_1)^{-1}(0),$$

and

$$\left.\begin{array}{r}\|(I + \omega_2\phi_2)^{-1}(I - \omega_2\psi_2)Aw_{n_i} - Aw_{n_i}\| \to 0\\ Aw_{n_i} \rightharpoonup A\tilde{p}\\ \text{Lemmas 2.1 and 2.2}\end{array}\right\} \Rightarrow A\tilde{p} \in (\phi_2 + \psi_2)^{-1}(0).$$

Thus, $\tilde{p} \in \Gamma$. Applying (2.6), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \hat{v} - q^*, x_n - q^*\rangle &= \lim_{i\to\infty}\langle \hat{v} - q^*, x_{n_i} - q^*\rangle\\ &= \lim_{i\to\infty}\langle \hat{v} - q^*, \tilde{p} - q^*\rangle\\ &\le 0.\end{aligned}$$

Since $\|x_{n+1} - x_n\| \le \beta_n\|\hat{v} - x_n\| + (1 - \beta_n)\|v_n - x_n\| \to 0$, we have $\limsup_{n\to\infty}\langle \hat{v} - q^*, x_{n+1} - q^*\rangle \le 0$. Applying (2.3) to (3.8) to deduce

$$\begin{aligned}\|x_{n+1} - q^*\|^2 &= \|(1 - \beta_n)(v_n - q^*) + \beta_n(\hat{v} - q^*)\|^2\\ &\le (1 - \beta_n)\|v_n - q^*\|^2 + 2\beta_n\langle \hat{v} - q^*, x_{n+1} - q^*\rangle\\ &\le (1 - \beta_n)\|x_n - q^*\|^2 + 2\beta_n\langle \hat{v} - q^*, x_{n+1} - q^*\rangle,\end{aligned}$$

which together with Lemma 2.3 implies that $x_n \to q^*$.

Next, we prove Case 2. Set $\tau_n = \|x_n - q^*\|^2$ and $\nu(n) = \max\{i \in \mathbb{N} | m \le i \le n, \tau_i \le \tau_{i+1}\}$. Then, $\tau_m \le \tau_{m+1}$, $\nu(n) \le \nu(n+1)$, $\lim_{n\to\infty}\nu(n) = +\infty$ and $\tau_{\nu(n)} \le \tau_{\nu(n)+1}$ for all $n \ge m$.

By the similar statement as that of Case 1, we have

$$\limsup_{n\to\infty}\langle \hat{v}-q^*, x_{\nu(n)+1}-q^*\rangle \le 0, \tag{3.33}$$

and

$$\tau_{\nu(n)+1} \le (1-\beta_{\nu(n)})\tau_{\nu(n)} + 2\beta_{\nu(n)}\langle \hat{v}-q^*, x_{\nu(n)+1}-q^*\rangle, \tag{3.34}$$

this together with $\tau_{\nu(n)} \le \tau_{\nu(n)+1}$ implies that

$$\tau_{\nu(n)} \le \langle \hat{v}-q^*, x_{\nu(n)+1}-q^*\rangle. \tag{3.35}$$

Combining (3.33) and (3.35) to deduce $\limsup_{n\to\infty}\tau_{\nu(n)} \le 0$ and so

$$\lim_{n\to\infty}\tau_{\nu(n)} = 0. \tag{3.36}$$

Furthermore, from (3.34), we deduce $\limsup_{n\to\infty}\tau_{\nu(n)+1} \le \limsup_{n\to\infty}\tau_{\nu(n)}$. This together with (3.36) yields that $\lim_{n\to\infty}\tau_{\nu(n)+1} = 0$. Utilizing Lemma 2.4, we attain $0 \le \tau_n \le \max\{\tau_{\nu(n)}, \tau_{\nu(n)+1}\}$. Hence, $\tau_n \to 0$ and $x_n \to q^*$. □

Setting $\varphi_1 = I_{\mathcal{H}_1}$ and $\varphi_2 = I_{\mathcal{H}_2}$ in Algorithm 3.1 and Theorem 3.2, we have the following results:

Algorithm 3.3. *Let $\hat{v} \in \mathcal{H}_1$ be a fixed point and let $x_0 \in \mathcal{H}_1$ be a starting point. Set $n = 0$.*

Step 1. Assume the n-th iterate x_n is known. Calculate

$$u_n = (I+\omega_1\phi_1)^{-1}(x_n - \omega_1\psi_1(x_n)).$$

Step 2. Calculate

$$y_n = (I+\omega_2\phi_2)^{-1}(Au_n - \omega_2\psi_2(Au_n)).$$

Step 3. Compute

$$v_n = u_n - \beta A^*(Au_n - y_n),$$

and

$$x_{n+1} = \beta_n\hat{v} + (1-\beta_n)v_n.$$

Let $n := n+1$ and go back to Step 1.

Corollary 3.4. *Suppose that $\Gamma_1 := \{\hat{x} \in \mathcal{H}_1 : (\phi_1+\psi_1)^{-1}(0)$ and $A\hat{x} \in (\phi_2+\psi_2)^{-1}(0)\} \neq \emptyset$ and the following restrictions are satisfied: $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = \infty$. Then the sequence $\{x_n\}$ generated by Algorithm 3.3 converges strongly to $proj_{\Gamma_1}(\hat{v})$.*

References

[1] H. H. Bauschke and P. L. Combettes, *A weak-to-strong convergence principle for Fejer-monotone methods in Hilbert spaces*, Math. Oper. Res. **26** (2001), 248–264.

[2] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl. **20** (2004), 103–120.

[3] Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, *A unified approach for inversion problems intensitymodulated radiation therapy*, Phys. Med. Biol. **51** (2006), 2353–2365.

[4] Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal. **16** (2009), 587–600.

[5] Q. L. Dong, Y. Peng and Y. Yao, *Alternated inertial projection methods for the split equality problem*, J. Nonlinear Convex Anal. **22** (2021), 53–67.

[6] Y. P. Fang and N. J. Huang, *H-monotone operator and resolvent operator technique for variational inclusions*, Appl. Math. Comput. **145** (2003), 795–803.

[7] K. R. Kazmi, R. Ali, S. Yousuf and M. Shahzad, *A hybrid iterative algorithm for solving monotone variational inclusion and hierarchical fixed point problems*, Calcolo **56** (2019): Article number 34.

[8] P. E. Mainge, *Numerical approach to monotone variational inequalities by a one-step projected reflected gradient method with line-search procedure*, Comput. Math. Appl. **72** (2016), 720–728.

[9] A. Moudafi, *The split common fixed-point problem for demicontractive mappings*, Inverse Probl. **26** (2010): Article number 055007.

[10] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), 275–283.

[11] S. M. Robinson, *Generalized equations and their solutions-I: basic theory*, Math. Program. Stud. **10** (1979), 128–141.

[12] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.

[13] G. Stampacchi, *Formes bilineaires coercivites surles ensembles convexes*, C. R. Acad. Sci. **258** (1964), 4413–4416.

[14] A. Taiwo, T. O. Alakoya and O. T. Mewomo, *Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces*, Numer. Algor. **86** (2021), 1359–1389.

[15] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240–256.

[16] H. K. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Probl. **26** (2010): Art. No. 105018.

[17] Y. Yao, M. Postolache and J. C. Yao, *An approximation algorithm for solving a split problem of fixed point and variational inclusion*, Optim. DOI: 10.1080/02331934.2023.2256769.

[18] Y. Yu and T. C. Yin, *Weak convergence of a self-adaptive Tseng-type algorithm for solving variational inclusion problems*, U.P.B. Sci. Bull., Series A **85** (2023), 51–58.

[19] Y. Yu, Y. Zhao and T. C. Yin, *Convergence of extragradient-type methods for fixed point problems and quasimonotone variational inequalities*, J. Nonlinear Convex Anal. **24** (2023), 2225–2237.

[20] H. Zhou, *Strong convergence of an explicit iterative algorithm for continuous pseudocontractions in Banach spaces*, Nonlinear Anal. **70** (2009), 4039–4046.

[21] X. Zhao, J. C. Yao and Y. Yao, *A nonmonotone gradient method for constrained multiobjective optimization problems*, J. Nonlinear Var. Anal. **6(6)** (2022), 693–706.

[22] L. J. Zhu, J. C. Yao and Y. Yao, *Approximating solutions of a split fixed point problem of demicontractive operators*, Carpathian J. Math. **40** (2024), 195–206.

[23] L. J. Zhu and Y. Yao, *Algorithms for approximating solutions of split variational inclusion and fixed point problems*, Mathematics **11** (2023): Article number 641.

[24] L. J. Zhu and Y. Yao, *Modified splitting algorithms for approximating solutions of split variational inclusions in Hilbert spaces*, Fixed Point Theory, in press.

Manuscript received July 5, 2023
revised December 21, 2023

Li Zhang
School of Science, Tianjin University of Commerce, Tianjin 300134, China
E-mail address: zhangliyoumeng@163.com