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ON CONVERGENCE THEOREMS FOR THE GENERALIZED FUZZY HENSTOCK INTEGRALS

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Abstract. In the paper, the concept of weakly uniformly fuzzy Henstock integrability and some convergence theorems of generalized fuzzy Henstock integral are obtain on a infinite interval. Especially, the strong fuzzy Henstock integrable sequence of function on infinite interval can termwise integration if and only if the weak uniformly fuzzy Henstock integrability. As the application, the existence for global solutions of generalized fuzzy differential equations are discussed. Finally, several numerical examples are provide, which can visually demonstrate practical applications.

1. Introduction

In 1986, Puri and Ralescu [14] first put forward fuzzy integral to study the expectation about fuzzy random variable. This integral is generalized Aumann integral for set-valued mapping [2]. In 1987, by defining upper and lower integral, Matloka [12] defined (M) integral. In the same year, Kaleva [10] extended the Aumann integral of set-valued mapping to fuzzy sets and defined Kaleva integral in order to study fuzzy differential equations. Three years later, Kaleva [11] further considered the Cauchy problem for Kaleva integral. In 1989, Nanda [13] proposed the fuzzy Riemann-Stieltjes integral. Since the existence of infimum and supremum for a bounded set of fuzzy numbers is not as straightforward as it appears, pointed out by Wu et al. [17], fuzzy Riemann-Stieltjes integral proposed by Nanda was incorrect. And then, Wu [17] found that for a bounded fuzzy number sets, the level set of the infimum and supremum is not equal to the infimum and supremum of its level sets. That's means, using the upper and lower integral approximation to discuss the integral is limited. In order to overcome this shortcoming, Gong and Wu [8] used the Hausdorff distance for fuzzy numbers to define fuzzy MacShane integral, fuzzy Henstock integral and strong fuzzy Henstock integral in the sense of δ-fine partition, which avoids the problem that fuzzy sets can not be approximated arbitrarily when taking the supremum and infimum. According to the representation theorem of fuzzy numbers, one-dimensional fuzzy numbers are completely determined by the interval endpoint function, and the characterization theorems of these integrals are given. In 2005, Gong [9] gave the partial order relation, the representation of supremum and infimum. Meanwhile, by using the support function of fuzzy numbers, they also discuss the distance for n-dimensional fuzzy number.

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Due to the practicality of fuzzy analysis, it's theories has been systematically and widely used in deep learning [1], system analysis [18], network engineering [19], and so on. At present, most researchers studied the fuzzy differential equations(FDEs) under the fuzzy differential inclusion theory [3] and the sense of H-differentiability [10]. In addition, Bede et al. [4, 5] investigated the generalized differentiability of fuzzy-valued functions and first order linear FDEs. In 2019, based on Lipschitz condition, to solve nonlinear fuzzy Volterra integral equation, Bica et al. [6, 20] studied the error estimation of iterative numerical method. In the same years, Shao [16] proved for fuzzy Henstock integrals the generalized convergence theorem. As the applications, for discontinuous fuzzy systems, Shao et al. [15] considered it's Cauchy problems.

The outline of the rest of this paper is as follows. Some basic notions and preliminary results are introduced in Section 2. Some important lemmas and convergence theorems for the generalized fuzzy Henstock integral are given in Section 3. As the application we obtain the existence for global solutions of generalized FDES in Section 4. The conclusions of research is given in Section 5.

2. Preliminaries

A map $u : \mathbb{R} \to [0, 1]$ is a fuzzy number, if u is convex fuzzy set, $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, upper semicontinuous function and normal. And the fuzzy number space is represented by E^n.

Denote $[u]_\lambda = \{x \in R^n | u(x) \geq \lambda\}$, where $0 < \lambda \leq 1$. By Zadeh's extension principle, for $0 \leq \lambda \leq 1$ and $s, u \in E^n$, $[u + s]_\lambda = [u]_\lambda + [s]_\lambda, [ks]_\lambda = k[s]_\lambda$.

The distance of u and s is defined as

$$D(\tilde{s}, \tilde{u}) = \sup_{\lambda \in [0,1]} d([\tilde{s}]_\lambda, [\tilde{u}]_\lambda) = \sup_{\lambda \in [0,1]} \max\{| s_\lambda^- - u_\lambda^- |, | s_\lambda^+ - u_\lambda^+ |\}.$$

Then D is a metric in E^n. Using the results in [14] (E^n, D) is complete.

If $\exists m \in E^n$ for $u, s \in E^n$ s.t. $u = s + m$, m is the H-difference of u and s, then we note $u \ominus_H s$. When dealing with the subtraction operation for fuzzy numbers, the H-difference of them always exists in this paper.

Adding $-\infty, +\infty$ on R, we get $\overline{R}$. Suppose $\mho : [c, +\infty] \to E^n$, we set $\mho(+\infty) = \tilde{0}$, and $\tilde{0} \cdot (\infty) = \tilde{0}$.

Definition 2.1 ([15])**.** For $\mho : [c, d] \to E^n$. If for $\forall \epsilon > 0$, $\exists \eta > 0$ s.t. for $\{[c_i, d_i]\}$ which is finite sequence and $[c_i, d_i] \cap [c_j, d_j] = \emptyset$ for $i \neq j$ with $\sum_{i=1}^n |c_i - d_i| < \eta$,

$$\sum D(\mho(d_i), \mho(c_i)) < \epsilon.$$

Then $\mho$ is AC on $[c, d]$.

Definition 2.2 ([15])**.** For $\mho : [c, d] \to E^n$. If there are a sequence of sets $\{E_n\}$ s.t. $\cup_n E_n = [c, d]$ and F is AC on E_n. Then $\mho$ is ACG on $[c, d]$.

Definition 2.3 ([7])**.** Let $c \in (-\infty, +\infty)$, $\Pi = \{[t_{i-1}, t_i], \xi_i\}_{i=1}^n$ is δ-fine on $[c, +\infty)$, that is,

(1)$c = t_0 < t_1 < t_2 < \ldots < x_{n-1} = d < +\infty$;

(2)$\xi_i \in [t_{i-1}, t_i] \subset O(\xi_i), i = 1, \ldots n$.

Definition 2.4 ([7])**.** Let $\mho(t):[c,+\infty)\longrightarrow E^n$. $\mho$ is fuzzy Henstock integrable on $[c,+\infty)$, if $\exists\tilde{B}\in E^n$, for $\forall\varepsilon>0$ $\exists\delta(t)>0$, s.t. for $\Pi=\{[t_{i-1},t_i],\xi_i\}_{i=1}^n$ which is δ-fine,

$$D\bigg(\sum\mho(\xi_i)(t_i-t_{i-1}),\tilde{B}\bigg)<\varepsilon,$$

writen (FH)$\int_c^{+\infty}\mho(x)\mathrm{d}x=\tilde{B}$ or $\mho\in FH[c,+\infty)$.

3. Some convergence theorems for generalized fuzzy Henstock integrals

Now, we put forward the weakly uniformly fuzzy Henstock integrability and obtain some necessary and sufficient conditions about that strong fuzzy Henstock integrable function sequence defined on infinite interval can termwise integration.

Definition 3.1. Suppose $\{\mho_n(t)\in FH[c,+\infty)\}$. If for $\forall\varepsilon>0$, $\exists\Delta>c$, s.t. $d>\Delta$,

$$D\bigg(\tilde{0},\int_d^{+\infty}\mho_n(t)\mathrm{d}t\bigg)<\varepsilon.$$

Then $\{\mho_n(t)\}$ is (FH) equi-integrable.

Theorem 3.2. *Suppose* $\mho:[c,+\infty)\longrightarrow E^n$, $d>c$, *if*

(1) *for* $\mho_n(t)\in FH[c,d]$, *and* $\mho_n(t)\to\mho(t)$, *a.e. for* $c\le t<+\infty$*;*

(2) $\mho\in FH[c,d]$ *, with* $\int_c^d\mho(t)\mathrm{d}t=\lim\limits_{n\to\infty}\int_c^d\mho_n(t)\mathrm{d}t$;

(3) $\{\mho_n(t)\}$ *is* (FH) *equi-integrable.*

Then $\mho\in FH[c,+\infty)$ *with*

$$\lim_{n\to\infty}\int_c^{+\infty}\mho_n(t)\mathrm{d}t=\int_c^{+\infty}\mho(t)\mathrm{d}t.$$

Proof. Given any $\varepsilon>0$, since $\{\mho_n(t)\}$ are (FH) equi-integrable, $\exists\Delta_0>c$, for $d_2>d_1>\Delta_0$, according to above condition (2)

$$D\bigg(\tilde{0},\int_{d_1}^{d_2}\mho(t)\mathrm{d}t\bigg)=D\bigg(\tilde{0},\lim_{n\to\infty}\int_{d_1}^{d_2}\mho_n(t)\mathrm{d}t\bigg)\le\lim_{n\to\infty}D\bigg(\tilde{0},\int_{d_1}^{d_2}\mho_n(t)\mathrm{d}t\bigg)\le2\varepsilon.$$

Therefore, $\mho\in FH[c,+\infty)$. Besides, $\exists\Delta>\Delta_0$, s.t. for $d>\Delta$,

$$D\bigg(\tilde{0},\int_d^{+\infty}\mho_n(t)\mathrm{d}t\bigg)<\frac{\varepsilon}{3},\quad D\bigg(\tilde{0},\int_d^{+\infty}\mho(t)\mathrm{d}t\bigg)<\frac{\varepsilon}{3}.$$

Taking advantage of condition (2) again, $\exists N$, for $n>N$,

$$D\bigg(\int_c^d\mho_n(t)\mathrm{d}t,\int_c^d\mho(t)\mathrm{d}t\bigg)<\frac{\varepsilon}{3}.$$

So,

$$\begin{aligned}D\bigg(\int_c^{+\infty}\mho_n(t)\mathrm{d}t,\int_c^{+\infty}\mho(t)\mathrm{d}t\bigg)\ \le\ & D\bigg(\int_c^{+\infty}\mho_n(t)\mathrm{d}t,\int_c^d\mho_n(t)\mathrm{d}t\bigg)\\ &+D\bigg(\int_c^d\mho_n(t)\mathrm{d}t,\int_c^d\mho(t)\mathrm{d}t\bigg)\end{aligned}$$

$$+D\bigg(\int_c^d \mho(t)\mathrm{d}t, \int_c^{+\infty} \mho(t)\mathrm{d}t\bigg)$$
$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Thus,

$$\lim_{n\to\infty}\int_c^{+\infty} \mho_n(t)\mathrm{d}t = \int_c^{+\infty} \mho(t)\mathrm{d}t.$$

The proof is completed. □

Definition 3.3. Suppose $\{\mho_n \in FH[c, +\infty)\}$. If for $\forall\varepsilon > 0$, $\exists\Delta > c$ and N, s.t. for $d > \Delta$, $\exists N_d$, for $n > N_d$,

$$D\bigg(\int_d^{+\infty} \mho_n(t)\mathrm{d}t, \tilde{0}\bigg) < \varepsilon.$$

Then $\{\mho_n(t)\}$ is weak (FH) equi-integrable .

Remark 3.4. If $\{\mho_n(t)\}$ is (FH) equi-integrable, then it is weak (FH) equi-integrable.

Theorem 3.5. *In Theorem 3.2, condition (3) is weaken as "weak (FH) equi-integrable", the conclusion also holds.*

Definition 3.6. Suppose sequence $\{\mho_n \in FH[c, +\infty)\}$. If for $\forall\varepsilon > 0$, $\exists\delta(t) > 0$, s.t. for Π which is δ-fine on $[c, +\infty)$, for all n,

$$D\bigg(\int_c^{+\infty} \mho_n(t)\mathrm{d}t, \sum \mho_n(\xi)\Delta t\bigg) < \varepsilon,$$

where $\Delta t = v - u$, then $\{\mho_n(t)\}$ is (FH) uniformly integrable on $[c, +\infty)$.

Theorem 3.7. *Suppose* $\mho : [c, +\infty) \longrightarrow E^n, t \in [c, +\infty)$, *if*

(1) $\{\mho_n(t)\}$ *is (FH) uniformly integrable on* $[c, +\infty)$;
(2) $\mho_n(t) \to \mho(t), (n \to \infty)$.

Then $\mho \in FH[c, +\infty)$ *with*

$$\lim_{n\to\infty}\int_c^{+\infty} \mho_n(t)\mathrm{d}t = \int_c^{+\infty} \mho(t)\mathrm{d}t.$$

Proof. For $\forall\varepsilon > 0$, since $\{\mho_n(t)\}$ is uniformly (FH) integrable on $[c, +\infty)$, $\exists\delta(t) > 0$, s.t. for $\Pi = \{[u, v], \xi\}$ which is δ-fine on $[c, +\infty)$,

$$D\bigg((\Pi)\sum \mho_n(\xi)\Delta t, \int_c^{+\infty} \mho_n(t)\mathrm{d}t\bigg) < \frac{\varepsilon}{3},$$

where $\Delta t = v - u$. Let $\delta(+\infty) = \Delta$, for $d > \Delta$ and δ-fine on $[c, d]$ Π_0, let $\Pi = \Pi_0 \cup \{[d, +\infty], +\infty\}$ is δ-fine on $[c, +\infty)$. Since $\mho_n(t) \to \mho(t)$, for Π_0 which is δ-fine on $[c, d]$, $\exists N(\Pi_0)$, for $n, m > N(\Pi_0)$, for $\forall\xi \in \Pi_0$,

$$D\bigg(\mho_m(\xi), \mho_n(\xi)\bigg) < \frac{\varepsilon}{3(d-c)}.$$

Thus,

$$\begin{aligned} & D\Big(\int_c^{+\infty}\mho_n(t)\mathrm{d}t, \int_c^{+\infty}\mho_m(t)\mathrm{d}t\Big) \\ \leq\ & D\Big(\int_c^{+\infty}\mho_n(t)\mathrm{d}t, (\Pi_0)\sum\mho_n(\xi)\Delta t\Big) \\ & +D\Big((\Pi_0)\sum\mho_n(\xi)\Delta t, (\Pi_0)\sum\mho_m(\xi)\Delta t\Big) \\ & +D\Big((\Pi_0)\sum\mho_m(\xi)\Delta t, \int_c^{+\infty}\mho_m(t)\mathrm{d}t\Big) \\ <\ & \frac{\varepsilon}{3}+D\Big((\Pi_0)\sum_c^d\mho_n(\xi)\Delta t, (\Pi_0)\sum_c^d\mho_m(\xi)\Delta t\Big)+\frac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

That means $\{\int_c^{+\infty}\mho_n(t)\mathrm{d}t\}$ is a Cauchy-Sequence in E^n. Since E^n is complete and linear, $\{\int_c^{+\infty}\mho_n(t)\mathrm{d}t\}$ is convergent. Set $\tilde{x_0} = \lim\limits_{n\to\infty}\int_c^{+\infty}\mho_n(t)\mathrm{d}t$, $\exists N$, for $n > N$,

$$D\Big(\int_c^{+\infty}\mho_n(t)\mathrm{d}t, \tilde{x_0}\Big) < \varepsilon.$$

For given ε, $\exists k > N$, s.t.

$$D\Big(\sum\mho_k(\xi)\Delta t, \sum\mho(\xi)\Delta t\Big) < \varepsilon.$$

Then,

$$\begin{aligned} D\Big(\sum\mho(\xi)\Delta t, \tilde{x_0}\Big) \ \leq\ & D\Big(\sum\mho(\xi)\Delta t, \sum\mho_k(\xi)\Delta t\Big) \\ & +D\Big(\sum\mho_k(\xi)\Delta t, \int_c^{+\infty}\mho_k(t)\mathrm{d}t\Big) \\ & +D\Big(\int_c^{+\infty}\mho_k(t)\mathrm{d}t, \tilde{x_0}\Big) \\ <\ & \varepsilon+\varepsilon+\varepsilon=3\varepsilon. \end{aligned}$$

Therefore, $\mho \in FH[c, +\infty)$ with

$$\lim_{n\to\infty}\int_c^{+\infty}\mho_n(t)\mathrm{d}t = \int_c^{+\infty}\mho(t)\mathrm{d}t.$$

□

Remark 3.8. If $\{\mho_n \in FH[c, +\infty)\}$ uniformly, then it is (FH) equi-integrable.

Definition 3.9. Suppose $\mho : [c, +\infty) \longrightarrow E^n$ and $\{\mho_n \in FH[c, +\infty)\}$, if $\forall\varepsilon > 0$, $\exists\delta(t) > 0$, s.t. for $\Pi = \{[u, v], \xi\}$ which is δ-fine on $[c, +\infty)$, $\exists N(\Pi)$, for $n > N(\Pi)$,

$$D\Big(\int_c^{+\infty}\mho_n(t)\mathrm{d}t, (\Pi)\sum\mho_n(\xi)\Delta t\Big) < \varepsilon,$$

Then $\{\mho_n(t)\}$ is weak uniformly (FH) integrable on $[c, +\infty)$.

Remark 3.10. If $\{\mho_n(t)\}$ is (FH) integrable on $[c, +\infty)$ uniformly, then $\{\mho_n(t)\}$ is weak uniformly (FH) integrable. But, the opposite may not necessarily hold true.

Theorem 3.11. *Let $\mho : [c, +\infty) \longrightarrow E^n$, if $\mho_n \in FH[c, +\infty)$ and $\mho_n(t) \to \mho(t)$ for $t \in [c, +\infty)$, then $\mho \in FH[c, +\infty)$ iff $\{\mho_n(t)\}$ is weak uniformly (FH) integrable on $[c, +\infty)$.*

Proof. (Sufficiency)Proof of this part is similar to Theorem 3.7, so we omit it.

(Necessity) For $\forall\varepsilon > 0$, since $\mho \in FH[c, +\infty)$, $\exists\delta(t) > 0$, s.t. $\Pi = \{[u, v], \xi\}$ which is δ-fine on $[c, +\infty)$, and let $\Delta t = v - u$

$$D\bigg((\Pi)\sum \mho(\xi)\Delta t, \int_c^{+\infty} \mho(t)\mathrm{d}t\bigg) < \varepsilon.$$

Since $\lim\limits_{n\to\infty} \int_c^{+\infty} \mho_n(t)\mathrm{d}t = \int_c^{+\infty} \mho(t)\mathrm{d}t$, $\exists N$, for $n > N$,

$$D\bigg(\int_c^{+\infty} \mho_n(t)\mathrm{d}t, \int_c^{+\infty} \mho(t)\mathrm{d}t\bigg) < \varepsilon.$$

Since $\mho_n(t) \longrightarrow \mho(t)$ on $[c, +\infty)$, s.t. for $\Pi = \{[u, v], \xi\}$ which is δ-fine on $[c, +\infty)$, $\exists N(\Pi) > N$, for $n > N(\Pi)$,

$$D\bigg((\Pi)\sum \mho_n(\xi)\Delta t, (\Pi)\sum \mho(\xi)\Delta t\bigg) < \varepsilon.$$

Thus,

$$\begin{aligned} D\bigg((\Pi)\sum \mho_n(\xi)\Delta t, \int_c^{+\infty} \mho_n(t)\mathrm{d}t\bigg) \leq\ & D\bigg((\Pi)\sum \mho_n(\xi)\Delta t, (\Pi)\sum \mho(\xi)\Delta t\bigg) \\ & + D\bigg((\Pi)\sum \mho(\xi)\Delta t, \int_c^{+\infty} \mho(t)\mathrm{d}t\bigg) \\ & + D\bigg(\int_c^{+\infty} \mho(t)\mathrm{d}t, \int_c^{+\infty} \mho_n(t)\mathrm{d}t\bigg) \\ <\ & \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Therefore, $\{\mho_n(t)\}$ is weak uniformly (FH) integrable on $[c, +\infty)$. □

Theorem 3.12. *In the Theorem 3.11, If we replace "$\{\mho_n(t)\}$ is weak uniformly (FH) integrable on $[c, +\infty)$" by "$\{\mho_n(t)\}$ has weak uniformly (FH) integrable sub-series on $[c, +\infty)$", the conclusion still holds.*

Lemma 3.13. *Suppose $\mho : [c, +\infty) \longrightarrow E^n$, then $\mho \in FH[c, +\infty)$ iff for $d > c$, $\mho \in FH[c, d]$, and for $\forall\varepsilon > 0$, $\exists\Delta > c$, s.t. $d_1, d_2 > \Delta$,*

$$D\bigg(\int_{d_1}^{+\infty} \mho(t)\mathrm{d}t, \tilde{0}\bigg) < \varepsilon \quad or \quad D\bigg(\int_{d_1}^{d_2} \mho(t)\mathrm{d}t, \tilde{0}\bigg) < \varepsilon.$$

Theorem 3.14. *Suppose $\mho : [c, +\infty) \longrightarrow E^n$, if $\mho_n \in FH[c, +\infty)$, and $\tilde{g}, \tilde{h} \in FH[c, +\infty)$, s.t. $\tilde{g}(t) \preceq \mho_n(t) \preceq \tilde{h}(t)$, then $\{\mho_n(t)\}$ is (FH) equi-integrable on $[c, +\infty)$.*

Proof. For $\forall\varepsilon > 0$, since $\tilde{g}, \tilde{h} \in FH[c, +\infty)$, according to Lemma 3.13, $\exists\Delta > c$, for $d > \Delta$

$$D\Big(\int_d^{+\infty} \tilde{g}(t)\mathrm{d}t, \tilde{0}\Big) < \varepsilon \quad \text{or} \quad D\Big(\int_d^{+\infty} \tilde{h}(t)\mathrm{d}t, \tilde{0}\Big) < \varepsilon.$$

Then $\tilde{g}, \tilde{h} \in FH[d, +\infty)$

$$\int_d^{+\infty} \tilde{g}(t)\mathrm{d}t \preceq \int_d^{+\infty} \mho_n(t)\mathrm{d}t \preceq \int_d^{+\infty} \tilde{h}(t)\mathrm{d}t.$$

Through calculation,

$$\begin{aligned} D\Big(\int_d^{+\infty} \mho_n(t)\mathrm{d}t, \tilde{0}\Big) &\leq D\Big(\int_d^{+\infty} \tilde{g}(t)\mathrm{d}t, \tilde{0}\Big) + D\Big(\int_d^{+\infty} \mho_n(t)\mathrm{d}t, \int_d^{+\infty} \tilde{g}(t)\mathrm{d}t\Big) \\ &\leq D\Big(\int_d^{+\infty} \tilde{g}(t)\mathrm{d}t, \tilde{0}\Big) + D\Big(\int_d^{+\infty} \mho_n(t)\mathrm{d}t, \int_d^{+\infty} \tilde{h}(t)\mathrm{d}t\Big) \\ &\leq D\Big(\int_d^{+\infty} \tilde{g}(t)\mathrm{d}t, \tilde{0}\Big) + D\Big(\int_d^{+\infty} \tilde{h}(t)\mathrm{d}t, \tilde{0}\Big) \\ &\quad + D\Big(\int_d^{+\infty} \tilde{g}(t)\mathrm{d}t, \tilde{0}\Big) \\ &\leq 3\varepsilon. \end{aligned}$$

So, $\{\mho_n(t)\}$ is (FH) equi-integrable. □

Theorem 3.15. *Suppose* $\mho : [c, +\infty) \longrightarrow E^n$, *if*

(1) $\{\mho_n \in FH[c, +\infty)\}$ *and* $\mho_n(t) \to \mho(t), t \in [c, +\infty)$;

(2) $\tilde{g}, \tilde{h} \in FH[c, +\infty)$ *satisfied that* $\tilde{g}(t) \preceq \mho_n(t) \preceq \tilde{h}(t)$ *for* $t \in [c, +\infty)$, *then* $\mho \in FH[c, +\infty)$ *with*

$$\lim_{n\to\infty} \int_c^{+\infty} \mho_n(t)\mathrm{d}t = \int_c^{+\infty} \mho(t)\mathrm{d}t.$$

For $d > c$,

$$\lim_{n\to\infty} \int_c^{d} \mho_n(t)\mathrm{d}t = \int_c^{d} \mho(t)\mathrm{d}t.$$

Theorem 3.16. *Suppose* $\mho : [c, +\infty) \longrightarrow E^n$ *for* $d > c$, $\mho \in FH[c, d]$ *and* $\exists\tilde{g}, \tilde{h} \in FH[c, +\infty)$ *s.t.* $\tilde{g}(t) \preceq \mho(t) \preceq \tilde{h}(t)$, $t \in [c, +\infty)$, *then* $\mho \in FH[c, +\infty)$.

Proof. Suppose $d_0 = c, d_n \nearrow +\infty$, let

$$\mho_n(t) = \begin{cases} \mho(t), & c \leq t \leq d_n, \\ \tilde{0}, & \text{other}. \end{cases}$$

Then, for each n, the $\mho_n \in FH[c, +\infty)$ and $\mho_n(t) \to \mho(t), t \in [c, +\infty)$. Based on the above conditions, $\exists\tilde{g}, \tilde{h} \in FH[c, +\infty)$, s.t.

$$\tilde{g}(t) \preceq \mho(t) \preceq \tilde{h}(t), \quad t \in [c, +\infty).$$

By Theorem 3.15, $\mho \in FH[c, +\infty)$ with

$$\lim_{n\to\infty} \int_c^{+\infty} \mho_n(t)\mathrm{d}t = \lim_{n\to\infty} \int_c^{d_n} \mho(t)\mathrm{d}t = \int_c^{+\infty} \mho(t)\mathrm{d}t.$$

□

Example 3.17. Given $\tilde{g}, \tilde{h}, \mho : [2, +\infty) \longrightarrow E^n$ by

$$\tilde{g}(t) = \tilde{0}, \mho(t) = \begin{cases} (0, \frac{1}{2}, \frac{1}{2}), t \in \mathbb{Q}, \\ (0, 0, \frac{1}{e^{t^2}}), t \in \overline{\mathbb{Q}}, \\ (0, 0, 0), t = +\infty, \end{cases} \quad \tilde{h}(t) = \begin{cases} (0, 1, 1), t \in \mathbb{Q}, \\ (0, 0, \frac{1}{t^2}), t \in \overline{\mathbb{Q}}, \\ (0, 0, 0), t = +\infty, \end{cases}$$

$\mho_n(t) = (1 + \frac{1}{n})\mho(t)$, where (u, v, w) denotes triangular fuzzy number, $\mathbb{Q}$ is rational, $\overline{\mathbb{Q}}$ is irrational. According to Definition 2.4, $\{\mho_n \in FH[2, +\infty\}$, and $\mho_n(t) \to \mho(t), t \in [c, +\infty)$. In addition, $\tilde{h}(t)$ and $\tilde{g}(t)$ is (FH) integrable on $[2, +\infty)$ (refer to [7]), obviously, $\tilde{g}(t) \preceq \mho_n(t) \preceq \tilde{h}(t)$.

In fact,

$$f_\lambda^-(t) = \begin{cases} \frac{1}{2}\lambda, t \in \mathbb{Q}, \\ 0, t \in \overline{\mathbb{Q}}, \\ 0, t = +\infty, \end{cases} \quad f_\lambda^+(t) = \begin{cases} \frac{1}{2}, t \in \mathbb{Q}, \\ \frac{(1-\lambda)}{e^{t^2}}, t \in \overline{\mathbb{Q}}, \\ 0, t = +\infty. \end{cases}$$

Since $(f_n)_\lambda^+(t) \leq (1 + \frac{1}{n})\frac{1}{e^{t^2}}$, $(f_n)_\lambda^-(t), (f_n)_\lambda^+(t)$ are Henstock integrable uniformly for $\lambda \in [0, 1]$ and

$$\begin{aligned} \int_2^{+\infty} (f_n)_\lambda^-(t)\mathrm{d}t &= 0, \\ \int_2^{+\infty} (f_n)_\lambda^+(t)\mathrm{d}t &= \lim_{d \to +\infty} \int_2^d (f_n)_\lambda^+(t)\mathrm{d}t \\ &= \left(1 + \frac{1}{n}\right)(1 - \lambda)\frac{\sqrt{\pi}}{2e^2}. \end{aligned}$$

Therefore, $\mho(t) \in FH[2, +\infty)$ with

$$\lim_{n \to \infty} \int_2^{+\infty} \mho_n(t)\mathrm{d}t = \int_2^{+\infty} \mho(t)\mathrm{d}t = \tilde{H}$$

where $[\tilde{H}]_\lambda = [0, (1 - \lambda)\frac{\sqrt{\pi}}{2e^2}]$.

4. Global existence of solution for generalized FDES

Considering the-FDE as follows

$$\begin{cases} x' = \mho(x, t), \quad t \in [c, d], \\ x(c) = x_0, \quad x_0 \in E^n, \end{cases} \tag{4.1}$$

where $\mho : E^n \times [c, +\infty] \to E^n$ and $\mho(x, t)$ is fuzzy Henstock integrable. Next, we discuss the global existence for generalized solution of (4.1).

Definition 4.1. $x : [c, +\infty) \longrightarrow E^n$ is a generalized solution of (4.1) iff for $t > c$,

$$x(t) = x(c) + \int_c^t \mho(s, x(s))\mathrm{d}s$$

or

$$x(t) = x(c) + (-1) \cdot \int_c^t \mho(s, x(s))\mathrm{d}s.$$

The main conclusions of this article are as follow.

Theorem 4.2. *Let* $\mho : E^n \times [c, +\infty] \longrightarrow E^n$. *if*

(H_1) $\mho(\cdot, t)$ *is continuous for a.e.* $t \in [c, +\infty)$;

(H_2) $\int_c^{+\infty} \mho(s, x(s))\mathrm{d}s$ *exist for* $x \in E^n$;

(H_3) *there exists* $\tilde{g}, \tilde{h} : [c, +\infty) \longrightarrow E^n$, *s.t.*

$$\tilde{g}(t) \preceq \mho_n(t) \preceq \tilde{h}(t), \quad t \in [c, +\infty)$$

then for any $x_0 \in E^n$, (4.1) *exists a global solution.*

Lemma 4.3. *Let* $\mho : E^n \times [c, +\infty] \longrightarrow E^n$ *satisfy* $(H_1), (H_2), (H_3)$, *if* $\varphi : [c, +\infty] \longrightarrow E^n$ *is continuous, then* $\int_c^{+\infty} \mho(s, \varphi(s))\mathrm{d}s$ *exists.*

Proof. Since $\varphi : [c, +\infty] \longrightarrow E^n$ is continuous, then $\varphi(s)$ can be represented the limit of the fuzzy-number-valued function sequence $\{\varphi_n\}$, according to Condition (H_2), for each n, $\int_c^{+\infty} \mho(s, \varphi_n(s))\mathrm{d}s$ exists, by Condition (H_3) and Theorem 3.15, the conclusion is true. □

Lemma 4.4. *Let* $\mho : E^n \times [c, +\infty] \longrightarrow E^n$ *satisfy* $(H_1), (H_2), (H_3)$, *for* $\forall x_0 \in E^n$ *and* $d > c$, *set*

$$\varphi_n(t) = \begin{cases} x_0, & t \in [c, c + (d-c)/n], \\ x_0 + \int_c^{t-(d-c)/n} \mho(s, \varphi_{n-1}(s))\mathrm{d}s, & t \in [c + (d-c)/n, d]. \end{cases} \tag{4.2}$$

then $\{\varphi_n(t)\}$ *has a subsequence of pointwise convergence.*

Proof of Theorem 4.2. For given $d > c$ and $x_0 \in E^n$, according to Lemma 4.3, suppose $\{\varphi_n(t)\}, n = 1, 2, \ldots$ has a subsequence of pointwise convergence on $[c, d]$, the subsequence is also denoted as φ_n, let $\lim\limits_{n\to\infty} \varphi_n(t) = \varphi(t), t \in [c, d]$. According to (H_1), for a.e. $t \in [c, d]$,

$$\lim_{n\to\infty} \mho(t, \varphi_n(t)) = \mho(t, \varphi(t)).$$

Therefore, φ_n is continuous, according to Lemma 4.3, $\int_c^{+\infty} \mho(s, \varphi(s))\mathrm{d}s$ exists. By Condition (H_3) and Convergence Theorem 3.15

$$\lim_{n\to\infty} \int_c^{t-(d-c)/n} \mho(s, \varphi_n(s))\mathrm{d}s = \lim_{n\to\infty} \int_c^t \mho(s, \varphi(s))\mathrm{d}s -_H \int_{t-(d-c)/n}^t \mho(s, \varphi_n(s))\mathrm{d}s.$$

According to Condition (H_3), we can proof easily $\int_{t-(d-c)/n}^t \mho(s, \varphi_n(s))\mathrm{d}s \longrightarrow \tilde{0}$. In (4.2), we put the limit of convergent subsequence $\{\varphi_n\}$ and gain

$$\varphi(t) = x(0) + \int_c^t \mho(s, \varphi(s))\mathrm{d}s, \quad t \in [c, d].$$

□

According to arbitrariness of $d > c$, the generalized global solution of (4.1) exits.

Theorem 4.5. *Let $\mho : E^n \times [c, +\infty] \longrightarrow E^n$ satisfy two conditions (H_1), (H_2) and (H_4), for $\{x(t), t \in [c, +\infty)\}$ which is continuous*

$$\{F_x(t) = \int_c^{+\infty} \mho(s, x(s))\mathrm{d}s, \quad x \in \{x(t)\}, \quad t \in [c, +\infty)\} \tag{4.3}$$

about x is uniformly ACG^* and $\{F_x(t)\}$ is equi-continuous. Then the initial value (4.1) *has a generalized global solution under the meaning of (FH) integrals.*

Proof. Since $\{F_x(t)\}$ is equi-continuous, and $F_x(c) = \tilde{0}$ for $x \in \{x(t)\}$, therefore, $\{F_x(t)\}$ is bounded with $[c, +\infty) < M$ uniformly. Suppose B is a space, the values of acquisition are defined on $[c, +\infty)$ and constituted of E^n all continuous functions, for $\varphi(t), \psi(t) \in B$, we define distance:

$$H(\psi(t), \varphi(t)) = \sup_{t \in [c, +\infty)} D(\psi(t), \varphi(t))$$

for $x_0 \in E^n$, considering the subset $K = \{x, D(x, x_0) \leq M, x \in B\}$ and

$$(Tx)(t) = x_0 + \int_c^{+\infty} \mho(s, x(s))\mathrm{d}s.$$

Therefore, K is closed, bounded and convex. Since,

$$H(Tx, x_0) = \sup_{t \in [c, +\infty)} D(\int_c^{+\infty} \mho(s, x(s))\mathrm{d}s, \tilde{0}) = \sup_{t \in [c, +\infty)} D(F_x(t), \tilde{0}) \leq M.$$

Thus, $T(K) \subset K$, that is, T is the operator in K.

Next, we will proof that T is continuous. Suppose $x_n(t) \longrightarrow x(t)$ where $x(t), x_n(t) \in K$. Since $\mho$ is continuous

$$\lim_{n \to \infty} \mho(t, x_n(t)) = \mho(t, x(t)), \quad t \in [c, +\infty).$$

According to condition (H_4), $\{F_n(t) = \int_c^{+\infty} \mho(s, x_n(s))\mathrm{d}s, x_n \in K\}$ is uniform ACG^* on $[c, +\infty)$, and $\{F_n(t)\}$ is equi-continuous, thus, it is uniformly bounded. $\{F_n(t)\}$ is convergent on $[c, +\infty)$ uniformly by Ascoli-Arzela Theorem. From Theorem 3.2,

$$\lim_{n \to \infty} \int_c^{+\infty} \mho(s, x_n(s))\mathrm{d}s = \int_c^{+\infty} \mho(s, x(s))\mathrm{d}s, \quad t \in [c, +\infty).$$

Therefore, we obtain

$$\begin{aligned} H(Tx_n, Tx) &= \sup_{t \in [c, +\infty)} D(F_n(t), F_x(t)) \\ &\quad + \sup_{t \in [c, +\infty)} D\left(\int_c^{+\infty} \mho(s, x_n(s))\mathrm{d}s, \int_c^{+\infty} \mho(s, x(s))\mathrm{d}s\right) \longrightarrow 0. \end{aligned}$$

That is, T is continuous.

Finally, we proof T is compact. For any bounded subset N in K, $T(N) = \{F_x(t), x \in N\}$ is a continuous function family on $[c, +\infty)$, by codition $(H_4), T(N)$ is uniform bounded and equi-continuous. So, $T(N)$ is compact relatively in K by Arzel*á*-Ascoli Theorem. That is, T is compact.

According to Schauder fixed point theorem, T has at least one $x(t) \in K$ s.t. $Tx(\cdot) = x(\cdot)$, therefore, $x(t) = x_0 + \int_c^{+\infty} \mho(s, x(s))\mathrm{d}s$ is the generalized global solution of (4.1). $\square$

Example 4.6. Let's discuss the following fuzzy systems:

$$\begin{cases} \tilde{x}'(t) = \mho(\tilde{x}, t) + \tilde{\varphi}(t), t \in [-1, +\infty), \\ \tilde{x}(-1) = \tilde{x}_0 \end{cases} \tag{4.4}$$

where $\mho(\tilde{x}, t)$ is (FH)-integrable, $\tilde{\varphi}(t) = \chi_{\{h(t)\}} + \tilde{W}$ and

$$\tilde{W}(s) = \begin{cases} 1 - s, 0 < s < 1, \\ s + 1, -1 \le s \le 0, \\ 0, \qquad others. \end{cases}$$

and

$$h(t) = \begin{cases} 0, & t = 0, \\ 2t \sin \frac{1}{t^2} - \frac{2}{t} \cos \frac{1}{t^2}, t \neq 0, \end{cases}$$

A generalized solution of (4.6) is $\tilde{x}(t)$ iff

$$\tilde{x}(t) = \tilde{x}_0 + (FH) \int_{-1}^{+\infty} (\mho(\tilde{x}(s), s) + \tilde{\varphi}(s))\mathrm{d}s$$

hold true. Therefore,

$$\tilde{x}(t) = \tilde{\psi}(t) + (FH) \int_{-1}^{+\infty} \mho(\tilde{x}(s), s)\mathrm{d}s$$

for $\tilde{\psi} = \tilde{x}_0 + \chi_{\{H(t)\}} + \tilde{W} \cdot t$ with

$$H(t) = \begin{cases} 0, \qquad t = 0, \\ t^2 \sin \frac{1}{t^2}, t \neq 0. \end{cases}$$

Let $\mho(\tilde{x}, s)$ satisfies Eq. (3) and $\tilde{\psi}_x$ is uniformly ACG* and equi-continuous on $[-1, +\infty)$. Therefore, (4.6) has generalized global solution under the meaning of (FH) integrals.

5. Conclusions

In this article, we put forward the weakly uniformly fuzzy Henstock integrability and obtain the necessary and sufficient condition about weakly uniformly Henstock integrability. In addition, we give and prove some important lemmas and convergence theorems of generalized fuzzy Henstock integral. Finally, we derive the existence of global solutions for generalized FDES as the application. In future studies, we will apply the convergence theorems to solve the existence of solutions to generalized fuzzy integral equations. In addition, these theorems can also be used to solve numerical integration and numerical differentiation problems for fuzzy Henstock integrals on infinite intervals.

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