Linear and Nonlinear Analysis
Volume 3, Number 1, 2017, 1–18
Yokohama Publishers
ISSN 2188-8167

MANN AND HALPERN ITERATIONS FOR THE SPLIT COMMON FIXED POINT PROBLEM IN BANACH SPACES

WATARU TAKAHASHI

Abstract. In this paper, we consider the split common fixed point problem in Banach spaces. Then using the idea of Mann's iteration, we first prove a weak convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Furthermore, using the idea of Halpern's iteration, we obtain a strong convergence theorem for finding a solution of the problem in Banach spaces. It seems that these results are first in Banach spaces.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\kappa > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \kappa\|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [6] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1},\ 1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2},\ 1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2,\ 1 \leq j \leq n$, the *split common null point problem* [6] is to find a point $z \in H_1$ such that

$$z \in \big(\cap_{i=1}^m A_i^{-1}0\big) \cap \big(\cap_{j=1}^n T_j^{-1}(B_j^{-1}0)\big),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

2010 *Mathematics Subject Classification.* 47H05, 47H09.
Key words and phrases. Split common fixed point problem, metric projection, metric resolvent, fixed point, iteration procedure, duality mapping.
The author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility peoblem and generalized split feasibility peoblems including the split common null point problem in Hilbert spaces; see, for instance, [6, 8, 14, 30]. However, it is difficult to solve such results outside Hilbert spaces. Recently, by using the hybrid methods of [15, 16, 18], Takahashi [22, 23, 24] proved strong convergence theorems for finding solutions of the feasibility problem and the split common null point problem in Banach spaces. Furthermore, by using the shrinking projection method [27], Takahashi [26] proved a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. On the other hand, in 1953, Mann [12] introduced the following iteration process. Let C be a nonempty, closed and convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. In 1967, Halpern [9] also gave an iteration process as follows: Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations of iterative processes for finding fixed points of nonexpansive mappings.

In this paper, motivated by these problems and methods, we consider the split common fixed point problem in Banach spaces. Then using the idea of Mann's iteration, we first prove a weak convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Furthermore, using the idea of Halpern's iteration, we obtain a strong convergence theorem for finding a solution of the problem in Banach spaces. It seems that these results are first in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [21] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [19]. The following result was proved by Takahashi and Toyoda [28].

Lemma 2.1 ([28]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $n \in \mathbb{N}$ and $u \in C$, then $\{P_C x_n\}$ converges strongly to some $z \in C$, where P_C is the metric projection on H onto C.*

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [19] and [20]. We know the following result.

Lemma 2.2 ([19]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.3 ([19]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of of E into 2^{E^*}. A multi-valued mapping A on E is said to be monotone if $\langle x-y, u^*-v^*\rangle \geq 0$ for all $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [4]; see also [20, Theorem 3.5.4].

Theorem 2.4 ([4])**.** *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [20].

Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [26] if, for any $x \in E$ and $q \in F(U)$,

$$\langle x-q, J(x-Ux)\rangle \geq \frac{1-\eta}{2}\|x-Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Examples We know examples of η-demimetric mappings from [26, 25].

(1) Let H be a Hilbert space and let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x-y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is k-demimetric; see [26].

(2) Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $U : C \to H$ is called generalized hybrid [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x-y\|^2$$

for all $x, y \in C$. Such a mapping U is called (α, β)-generalized hybrid. A (1,0)-generalized hybrid mapping is nonexpansive. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 0-demimetric; see [25].

(3) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric; see [26].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric; see [26].

Lemma 2.5 ([26])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.*

We also know the following lemmas:

Lemma 2.6 ([2], [32])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 ([11])**.** *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ satisfies $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$*;*
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Weak convergence theorem

In this section, we prove a weak convergence theorem of Mann's type iteration for the split common fixed point problem in Banach spaces. Let E be a Banach space and let D be a nonempty, closed and convex subset of E. A mapping $U : D \to E$ is called demiclosed if for a sequence $\{x_n\}$ in D such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds.

Theorem 3.1. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be a nonexpansive mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. For any $x_1 = x \in H$, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T\big(I - rA^*J_F(A - UA)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ and $r \in (0,\infty)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1 \quad \text{and} \quad 0 < r\|AA^*\| < 1 - \eta$$

for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges weakly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = \lim_{n\to\infty} P_{F(T)\cap A^{-1}F(U)}x_n$.

Proof. Since T is nonexpansive, $F(T)$ is closed and convex [21]. We also have from Lemma 2.5 that $F(U)$ is closed and convex. Then $F(T) \cap A^{-1}F(U)$ is closed and convex. Since $F(T) \cap A^{-1}F(U)$ is nonempty, the metric projection $P_{F(T)\cap A^{-1}F(U)}$ of H onto $F(T) \cap A^{-1}F(U)$ is well-defined. Let $z \in F(T) \cap A^{-1}F(U)$. Then $z = Tz$ and $Az - UAz = 0$. Put $y_n = T\big(x_n - rA^*J_F(Ax_n - UAx_n)\big)$ for all $n \in \mathbb{N}$. Since T is nonexpansive, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \big\|T\big(x_n - rA^*J_F(Ax_n - UAx_n)\big) - Tz\big\|^2 \\
&\le \|x_n - rA^*J_F(Ax_n - UAx_n) - z\|^2 \\
&= \|x_n - z - rA^*J_F(Ax_n - UAx_n)\|^2 \\
&= \|x_n - z\|^2 - 2\langle x_n - z, rA^*J_F(Ax_n - UAx_n)\rangle \\
&\qquad + \|rA^*J_F(Ax_n - UAx_n)\|^2 \\
&\le \|x_n - z\|^2 - 2r\langle Ax_n - Az, J_F(Ax_n - UAx_n)\rangle \\
&\qquad + r^2\|AA^*\|\|J_F(Ax_n - UAx_n)\|^2 \\
&\le \|x_n - z\|^2 - r(1-\eta)\|Ax_n - UAx_n\|^2 \\
&\qquad + r^2\|AA^*\|\|Ax_n - UAx_n\|^2 \\
&= \|x_n - z\|^2 + r(r\|AA^*\| - (1-\eta))\|Ax_n - UAx_n\|^2.
\end{aligned} \tag{3.1}$$

From $0 < r\|AA^*\| < 1 - \eta$ we have that $\|y_n - z\| \le \|x_n - z\|$ for all $n \in \mathbb{N}$ and hence

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n x_n + (1-\beta_n)y_n - z\| \\
&\le \beta_n \|x_n - z\| + (1-\beta_n)\|y_n - z\| \\
&\le \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\
&\le \|x_n - z\|.
\end{aligned}$$

Then $\lim_{n\to\infty} \|x_n - z\|$ exists. Thus $\{x_n\}$, $\{Ax_n\}$ and $\{y_n\}$ are bounded. Using the equality (2.2), we have that for $n \in \mathbb{N}$ and $z \in F(T) \cap A^{-1}F(U)$

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n x_n + (1-\beta_n)y_n - z\|^2 \\
&= \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|x_n - z\|^2 \\
&\quad + (1-\beta_n)r(r\|AA^*\| - (1-\eta))\|Ax_n - UAx_n\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\
&= \|x_n - z\|^2 + (1-\beta_n)r(r\|AA^*\| - (1-\eta))\|Ax_n - UAx_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - y_n\|^2.
\end{aligned}$$

Therefore, we have that $\beta_n(1-\beta_n)\|x_n - y_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2$ and

$$(1-\beta_n)r(1-\eta-r\|AA^*\|)\|Ax_n - UAx_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Thus we have from $0 < a \le \beta_n \le b < 1$ that

$$\lim_{n\to\infty} \|x_n - y_n\|^2 = 0 \text{ and } \lim_{n\to\infty} \|Ax_n - UAx_n\|^2 = 0. \tag{3.2}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using $\lim_{n\to\infty}\|Ax_n - UAx_n\| = 0$ and the demiclosedness of U, we have that $Aw = UAw$ and hence $w \in A^{-1}F(U)$. We also have that

$$\begin{aligned}\|x_n - Tx_n\| &= \|x_n - y_n + y_n - Tx_n\| \\ &= \|x_n - y_n + T\big(x_n - rA^*J_F(Ax_n - UAx_n)\big) - Tx_n\| \\ &\leq \|x_n - y_n\| + \|x_n - rA^*J_F(Ax_n - UAx_n) - x_n\| \\ &= \|x_n - y_n\| + \|rA^*J_F(Ax_n - UAx_n)\| \to 0.\end{aligned}$$

Since $x_{n_i} \rightharpoonup w$ and a nonexpansive T is demiclosed [19], we have $w = Tw$. This implies that $w \in F(T) \cap A^{-1}F(U)$.

We next show that if $x_{n_i} \rightharpoonup x^*$ and $x_{n_j} \rightharpoonup y^*$, then $x^* = y^*$. We know $x^*, y^* \in F(T)\cap A^{-1}F(U)$ and hence $\lim_{n\to\infty}\|x_n - x^*\|$ and $\lim_{n\to\infty}\|x_n - y^*\|$ exist. Suppose $x^* \neq y^*$. Since H satisfies Opial's condition, we have that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x^*\| &= \lim_{i\to\infty}\|x_{n_i} - x^*\| < \lim_{i\to\infty}\|x_{n_i} - y^*\| \\ &= \lim_{n\to\infty}\|x_n - y^*\| = \lim_{j\to\infty}\|x_{n_j} - y^*\| \\ &< \lim_{j\to\infty}\|x_{n_j} - x^*\| = \lim_{n\to\infty}\|x_n - x^*\|.\end{aligned}$$

This is a contradiction. Then we have $x^* = y^*$. Therefore, $x_n \rightharpoonup x^* \in F(T) \cap A^{-1}F(U)$. Moreover, since for any $z \in F(T) \cap A^{-1}F(U)$

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad \forall n \in \mathbb{N},$$

we have from Lemma 2.1 that $P_{F(T)\cap A^{-1}F(U)}x_n \to z_0$ for some $z_0 \in F(T)\cap A^{-1}F(U)$. The property of metric projection implies that

$$\langle x^* - P_{F(T)\cap A^{-1}F(U)}x_n, x_n - P_{F(T)\cap A^{-1}F(U)}x_n\rangle \leq 0.$$

Therefore, we have

$$\|x^* - z_0\|^2 = \langle x^* - z_0, x^* - z_0\rangle \leq 0.$$

This means that $x^* = z_0$, i.e., $x_n \rightharpoonup z_0$. □

4. Strong convergence theorem

In this section, we prove a strong convergence theorem of Halpern's type iteration for the split common fixed point problem in Banach spaces.

Theorem 4.1. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be a nonexpansive mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u_n + (1-\alpha_n)T(x_n - rA^*J_F(I-U)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $r \in (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < r\|AA^*\| < 1-\eta, \quad \lim_{n\to\infty} \alpha_n = 0,$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad 0 < a \le \beta_n \le b < 1$$

where $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)}u$.

Proof. As in the proof of Theorem 3.1, $F(T) \cap A^{-1}F(U)$ is nonempty, closed and convex and hence the metric projection $P_{F(T)\cap A^{-1}F(U)}$ of H onto $F(T) \cap A^{-1}F(U)$ is well-defined. Put $z_n = T(I - rA^*J_F(I-U)A)x_n$ for all $n \in \mathbb{N}$. Let $z \in F(T) \cap A^{-1}F(U)$. We have that $z = Tz$ and $Az - UAz = 0$. As in the proof of Theorem 3.1, we have that

$$\begin{aligned}
\|z_n - z\|^2 &= \|T(I - rA^*J_F(I-U)A)x_n - Tz\|^2 \\
&\le \|x_n - rA^*J_F(I-U)Ax_n - z\|^2 \\
&\le \|x_n - z\|^2 - 2r\langle Ax_n - Az, J_F(I-U)Ax_n\rangle \\
&\quad + r^2\|AA^*\|\,\|(I-U)Ax_n\|^2 \\
&\le \|x_n - z\|^2 - r(1-\eta)\|Ax_n - UAx_n\|^2 + r^2\|AA^*\|\,\|(I-U)Ax_n\|^2 \\
&= \|x_n - z\|^2 + r(r\|AA^*\| - (1-\eta))\,\|(I-U)Ax_n\|^2 .
\end{aligned} \tag{4.1}$$

From $0 < r\|AA^*\| < (1-\eta)$ we have that $\|z_n - z\| \le \|x_n - z\|$ for all $n \in \mathbb{N}$. Put $y_n = \alpha_n u_n + (1-\alpha_n)T(x_n - rA^*J_F(I-U)Ax_n)$. We have that

$$\begin{aligned}
\|y_n - z\| &= \|\alpha_n(u_n - z) + (1-\alpha_n)(z_n - z)\| \\
&\le \alpha_n\|u_n - z\| + (1-\alpha_n)\|z_n - z\| \\
&\le \alpha_n\,\|u_n - z\| + (1-\alpha_n)\,\|x_n - z\| .
\end{aligned}$$

Using this, we get that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(y_n - z)\| \\
&\le \beta_n\,\|x_n - z\| + (1-\beta_n)\,\|y_n - z\| \\
&\le \beta_n\,\|x_n - z\| + (1-\beta_n)(\alpha_n\,\|u_n - z\| + (1-\alpha_n)\,\|x_n - z\|) \\
&= (1 - \alpha_n(1-\beta_n))\|x_n - z\| + \alpha_n(1-\beta_n)\|u_n - z\|.
\end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}}\|u_n - z\| \le M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_k - z\| \le K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}
\|x_{k+1} - z\| &\le (1 - \alpha_k(1-\beta_k))\|x_k - z\| + \alpha_k(1-\beta_k)\|u_k - z\| \\
&\le (1 - \alpha_k(1-\beta_k))K + \alpha_k(1-\beta_k)K = K.
\end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. Take $z_0 = P_{F(T)\cap A^{-1}F(U)}u$. Since

$z_n = T(I - rA^* J_F(I-U)A)x_n$, we have that

$$x_{n+1} - x_n = \beta_n x_n + (1-\beta_n)\{\alpha_n u_n + (1-\alpha_n)z_n\} - x_n$$

and hence

$$\begin{aligned} x_{n+1} - x_n - &(1-\beta_n)\alpha_n u_n \\ &= \beta_n x_n + (1-\beta_n)(1-\alpha_n)z_n - x_n \\ &= (1-\beta_n)\{(1-\alpha_n)z_n - x_n\} \\ &= (1-\beta_n)\{z_n - x_n - \alpha_n z_n\}. \end{aligned}$$

Thus we have that

$$\begin{aligned} &\langle x_{n+1} - x_n - (1-\beta_n)\alpha_n u_n, x_n - z_0\rangle \\ &\quad = (1-\beta_n)\langle z_n - x_n, x_n - z_0\rangle - (1-\beta_n)\langle \alpha_n z_n, x_n - z_0\rangle \\ &\quad = -(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - (1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{4.2}$$

From (2.3) and (4.1), we have that

$$\begin{aligned} 2\langle x_n - &z_n, x_n - z_0\rangle \\ &= \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|z_n - z_0\|^2 \\ &\geq \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|x_n - z_0\|^2 \\ &= \|z_n - x_n\|^2. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we have that

$$\begin{aligned} 2\langle x_{n+1} - &x_n, x_n - z_0\rangle \\ &= 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - 2(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \\ &\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{4.4}$$

Furthermore, using (2.3) and (4.4), we have that

$$\begin{aligned} \|x_{n+1} - z_0\|^2 - \|x_n - x_{n+1}\|^2 - &\|x_n - z_0\|^2 \\ &\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned}$$

Setting $\Gamma_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n - &\|x_n - x_{n+1}\|^2 \\ &\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{4.5}$$

Noting that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\beta_n x_n + (1-\beta_n)\{\alpha_n u_n + (1-\alpha_n)z_n\} - x_n\| \\ &= \|(1-\beta_n)\alpha_n(u_n - z_n) + (1-\beta_n)(z_n - x_n)\| \\ &\leq (1-\beta_n)\big(\|z_n - x_n\| + \alpha_n\|u_n - z_n\|\big), \end{aligned} \tag{4.6}$$

we have that

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &\le (1-\beta_n)^2\big(\|z_n-x_n\|+\alpha_n\|u_n-z_n\|\big)^2\\ &= (1-\beta_n)^2\|z_n-x_n\|^2\\ &\quad+(1-\beta_n)^2\big(2\alpha_n\|z_n-x_n\|\|u_n-z_n\|+\alpha_n^2\|u_n-z_n\|^2\big).\end{aligned} \tag{4.7}$$

Thus we have from (4.5) and (4.7) that

$$\begin{aligned}\mathit{\Gamma}_{n+1}-\mathit{\Gamma}_n &\le \|x_n-x_{n+1}\|^2+2(1-\beta_n)\alpha_n\langle u_n,x_n-z_0\rangle\\ &\quad-(1-\beta_n)\|z_n-x_n\|^2-2(1-\beta_n)\alpha_n\langle z_n,x_n-z_0\rangle\\ &\le (1-\beta_n)^2\|z_n-x_n\|^2\\ &\quad+(1-\beta_n)^2\big(2\alpha_n\|z_n-x_n\|\|u_n-z_n\|+\alpha_n^2\|u_n-z_n\|^2\big)\\ &\quad+2(1-\beta_n)\alpha_n\langle u_n,x_n-z_0\rangle-(1-\beta_n)\|z_n-x_n\|^2\\ &\quad-2(1-\beta_n)\alpha_n\langle z_n,x_n-z_0\rangle\end{aligned}$$

and hence

$$\begin{aligned}&\mathit{\Gamma}_{n+1}-\mathit{\Gamma}_n+\beta_n(1-\beta_n)\|z_n-x_n\|^2\\ &\quad\le(1-\beta_n)^2\big(2\alpha_n\|z_n-x_n\|\|u_n-z_n\|+\alpha_n^2\|u_n-z_n\|^2\big)\\ &\qquad+2(1-\beta_n)\alpha_n\langle u_n,x_n-z_0\rangle-2(1-\beta_n)\alpha_n\langle z_n,x_n-z_0\rangle.\end{aligned} \tag{4.8}$$

We will divide the proof into two cases.

Case 1: Suppose that there exists a natural number N such that $\mathit{\Gamma}_{n+1} \le \mathit{\Gamma}_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty}\mathit{\Gamma}_n$ exists and then $\lim_{n\to\infty}(\mathit{\Gamma}_{n+1}-\mathit{\Gamma}_n)=0$. Using $\lim_{n\to\infty}\alpha_n=0$ and $0<a\le\beta_n\le b<1$, we have from (4.8) that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{4.9}$$

From (4.6) we have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.10}$$

We also have that

$$\begin{aligned}\|y_n-z_n\| &= \|\alpha_n u_n+(1-\alpha_n)z_n-z_n\|\\ &= \alpha_n\|u_n-z_n\|\to 0.\end{aligned} \tag{4.11}$$

Furthermore, from $\|y_n-x_n\|\le\|y_n-z_n\|+\|z_n-x_n\|$, we have that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{4.12}$$

We show that $\limsup_{n\to\infty}\langle u-z_0,y_n-z_0\rangle\le 0$, where $z_0=P_{F(T)\cap A^{-1}F(U)}u$. Put $l=\limsup_{n\to\infty}\langle u-z_0,y_n-z_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l=\lim_{i\to\infty}\langle u-z_0,y_{n_i}-z_0\rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w\in H$. From $\|x_n-y_n\|\to 0$, $\{x_{n_i}\}$ converges weakly to $w\in H$. Since $\|z_n-x_n\|\to 0$, we also have that $\{z_{n_i}\}$ converges weakly to $w\in H$. On the other hand, from (4.1) we have that

$$\begin{aligned}r(1-\eta-r\|AA^*\|)\,\|(I-U)Ax_n\|^2 &\le \|x_n-z\|^2-\|z_n-z\|^2\\ &= (\|x_n-z\|-\|z_n-z\|)(\|x_n-z\|+\|z_n-z\|)\end{aligned} \tag{4.13}$$

$$\leq \|x_n - z_n\| \left(\|x_n - z\| + \|z_n - z\|\right).$$

Then we get from $\|x_n - z_n\| \to 0$ that

$$\lim_{n\to\infty} \|Ax_n - UAx_n\| = 0. \tag{4.14}$$

Since $\{x_{n_i}\}$ converges weakly to $w \in H$ and A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using $\lim_{n\to\infty} \|Ax_n - UAx_n\| = 0$ and the demiclosedness of U, we have that $Aw = UAw$. We also have that

$$\begin{aligned}
\|x_n - Tx_n\| &= \|x_n - z_n + z_n - Tx_n\| \\
&= \|x_n - z_n + T\big(x_n - rA^*J_F(Ax_n - UAx_n)\big) - Tx_n\| \\
&\leq \|x_n - z_n\| + \|x_n - rA^*J_F(Ax_n - UAx_n) - x_n\| \\
&= \|x_n - z_n\| + \|rA^*J_F(Ax_n - UAx_n)\| \to 0.
\end{aligned}$$

Since $x_{n_i} \rightharpoonup w$ and a nonexpansive T is demiclosed [19], we have $w = Tw$. This implies that $w \in F(T) \cap A^{-1}F(U)$. Since $\{y_{n_i}\}$ converges weakly to $w \in F(T) \cap A^{-1}F(U)$, we have that

$$l = \lim_{i\to\infty} \langle u - z_0, y_{n_i} - z_0\rangle = \langle u - z_0, w - z_0\rangle \leq 0.$$

Since $y_n - z_0 = \alpha_n(u_n - z_0) + (1-\alpha_n)(T(x_n - rA^*J_F(I-U)Ax_n) - z_0)$, we have from (2.1) that

$$\begin{aligned}
\|y_n - z_0\|^2 \leq{}& (1-\alpha_n)^2 \|T(x_n - rA^*J_F(I-U)Ax_n) - z_0\|^2 \\
&+ 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle.
\end{aligned}$$

From (4.1), we have

$$\|y_n - z_0\|^2 \leq (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle.$$

This implies that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \beta_n \|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2 \\
&\leq \beta_n \|x_n - z_0\|^2 \\
&\quad + (1-\beta_n)\Big((1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle\Big) \\
&= \big(\beta_n + (1-\beta_n)(1-\alpha_n)^2\big)\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\
&\leq (\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\
&= (1 - (1-\beta_n)\alpha_n)\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\
&= (1 - (1-\beta_n)\alpha_n)\|x_n - z_0\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n(\langle u_n - u, y_n - z_0\rangle + \langle u - z_0, y_n - z_0\rangle).
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$, by Lemma 2.6 we obtain that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.7 that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. Thus we have from (4.8) that for all $n \in \mathbb{N}$,

$$\begin{aligned}\beta_{\tau(n)}(1-\beta_{\tau(n)})\|z_{\tau(n)} - x_{\tau(n)}\|^2 \\ &\leq (1-\beta_{\tau(n)})^2 2\alpha_{\tau(n)}\|z_{\tau(n)} - x_{\tau(n)}\|\|u_{\tau(n)} - z_{\tau(n)}\| \\ &\quad + (1-\beta_{\tau(n)})^2\alpha_{\tau(n)}^2\|u_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)}, x_{\tau(n)} - z_0\rangle \\ &\quad - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle z_{\tau(n)}, x_{\tau(n)} - z_0\rangle.\end{aligned} \tag{4.15}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < a \leq \beta_n \leq b < 1$, we have from (4.15) that

$$\lim_{n\to\infty}\|z_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{4.16}$$

As in the proof of Case 1 we have that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{4.17}$$

and

$$\lim_{n\to\infty}\|y_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{4.18}$$

Since $\|y_{\tau(n)} - x_{\tau(n)}\| \leq \|y_{\tau(n)} - z_{\tau(n)}\| + \|z_{\tau(n)} - x_{\tau(n)}\|$, we have that

$$\lim_{n\to\infty}\|y_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{4.19}$$

For $z_0 = P_{F(T)\cap A^{-1}F(U)}u$, let us show that $\limsup_{n\to\infty}\langle z_0 - u, y_{\tau(n)} - z_0\rangle \geq 0$. Put

$$l = \limsup_{n\to\infty}\langle z_0 - u, y_{\tau(n)} - z_0\rangle.$$

Without loss of generality, there exists a subsequence $\{y_{\tau(n_i)}\}$ of $\{y_{\tau(n)}\}$ such that $l = \lim_{i\to\infty}\langle z_0 - u, y_{\tau(n_i)} - z_0\rangle$ and $\{y_{\tau(n_i)}\}$ converges weakly to some point $w \in H$. From $\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$, $\{x_{\tau(n_i)}\}$ converges weakly to $w \in H$. Furthermore, since $\|z_{\tau(n)} - x_{\tau(n)}\| \to 0$, we also have that $\{z_{\tau(n_i)}\}$ converges weakly to $w \in H$. As in the proof of Case 1 we have that $w \in F(T)\cap A^{-1}F(U)$. Then we have

$$l = \lim_{i\to\infty}\langle z_0 - u, y_{\tau(n_i)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \geq 0.$$

As in the proof of Case 1, we also have that

$$\left\|y_{\tau(n)} - z_0\right\|^2 \leq (1-\alpha_{\tau(n)})^2\left\|x_{\tau(n)} - z_0\right\|^2 + 2\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle$$

and then

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 &\leq \beta_{\tau(n)}\left\|x_{\tau(n)} - z_0\right\|^2 + (1-\beta_{\tau(n)})\left\|y_{\tau(n)} - z_0\right\|^2 \\ &\leq \left(1 - (1-\beta_{\tau(n)})\alpha_{\tau(n)}\right)\left\|x_{\tau(n)} - z_0\right\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have that

$$(1-\beta_{\tau(n)})\alpha_{\tau(n)}\left\|x_{\tau(n)} - z_0\right\|^2 \leq 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle.$$

Since $(1-\beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\left\|x_{\tau(n)} - z_0\right\|^2 \leq 2\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle$$

$$= 2\langle u_{\tau(n)} - u, y_{\tau(n)} - z_0\rangle + 2\langle u - z_0, y_{\tau(n)} - z_0\rangle.$$

Thus we have that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - z_0\|^2 \le 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$. From (4.17), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.7 again, we obtain that

$$\|x_n - z_0\| \le \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

5. Applications

In this section, using Theorem 3.1, we first get well-known and new weak convergence theorems which are connected with the split common fixed point problems in Banach spaces. We know the following result obtained by Marino and Xu [13]; see also [29].

Lemma 5.1 ([13, 29])**.** *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \le k < 1$. Let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [10]; see also [31].

Lemma 5.2 ([10, 31])**.** *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 5.3. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $F(T) \ne \emptyset$ and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction with $F(U) \ne \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \ne 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \ne \emptyset$. For any $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T\big(I - rA^*(A - UA)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0, 1]$ and $r \in (0, \infty)$ satisfy the following:

$$0 < a \le \beta_n \le b < 1 \quad \text{and} \quad 0 < r\|AA^*\| < 1 - k$$

for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges weakly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = \lim_{n\to\infty} P_{F(T)\cap A^{-1}F(U)}x_n$.

Proof. Since U is a k-strict pseudo-contraction of H_2 into H_2 such that $F(U) \ne \emptyset$, from (1) in Examples, U is k-demimetric. Furthermore, from Lemma 5.1, U is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Theorem 5.4. *Let H_1 and H_2 be Hilbert spaces. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $F(T) \ne \emptyset$ and let $U : H_2 \to H_2$ be a generalized hybrid mapping with $F(U) \ne \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \ne 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \ne \emptyset$. For any $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T\big(I - rA^*(A - UA)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ *and* $r \in (0,\infty)$ *satisfy the following:*

$$0 < a \le \beta_n \le b < 1 \quad \text{and} \quad 0 < r\|AA^*\| < 1$$

for some $a, b \in \mathbb{R}$. *Then* $\{x_n\}$ *converges weakly to a point* $z_0 \in F(T) \cap A^{-1}F(U)$, *where* $z_0 = \lim_{n\to\infty} P_{F(T)\cap A^{-1}F(U)}x_n$.

Proof. Since U is a generalized hybrid mapping of H_2 into H_2 such that $F(U) \neq \emptyset$, from (2) in Examples, U is 0-demimetric. Furthermore, from Lemma 5.2, U is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Theorem 5.5. *Let* H *be a Hilbert space and let* F *be a smooth, strictly convex and reflexive Banach space. Let* J_F *be the duality mapping on* F. *Let* C *and* D *be nonempty, closed and convex subsets of* H *and* F, *respectively. Let* P_C *and* P_D *be the metric projections of* H *onto* C *and* F *onto* D, *respectively. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $C \cap A^{-1}D \neq \emptyset$. *For any* $x_1 = x \in H$, *define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C\big(I - rA^*J_F(A - P_D A)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ *and* $r \in (0,\infty)$ *satisfy the following:*

$$0 < a \le \beta_n \le b < 1 \quad \text{and} \quad 0 < r\|AA^*\| < 2$$

for some $a, b \in \mathbb{R}$. *Then* $\{x_n\}$ *converges weakly to a point* $z_0 \in C \cap A^{-1}D$, *where* $z_0 = \lim_{n\to\infty} P_{C\cap A^{-1}D}x_n$.

Proof. Since P_C is the metric projection of H onto C, P_C is nonexpansive. Furthermore, since P_D is the metric projection of F onto D, from (3) in Examples, P_D is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$, then $p = P_D p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$. It is clear that $P_D x_n \rightharpoonup p$ and $\|J_F(x_n - P_D x_n)\| = \|x_n - P_D x_n\| \to 0$. Since P_D is the metric projection of F onto D, we have that

$$\langle P_D x_n - P_D p, J_F(x_n - P_D x_n) - J_F(p - P_D p)\rangle \ge 0.$$

Therefore, $-\|p - P_D p\|^2 = \langle p - P_D p, -J_F(p - P_D p)\rangle \ge 0$ and hence $p = P_D p$. This implies that P_D is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Theorem 5.6. *Let* H *be a Hilbert space and let* F *be a uniformly convex and smooth Banach space. Let* J_F *be the duality mapping on* F. *Let* T *and* B *be maximal monotone operators of* H *into* H *and* F *into* F^*, *respectively. Let* Q_μ *be the resolvent of* T *for* $\mu > 0$ *and let* J_λ *be the metric resolvent of* B *for* $\lambda > 0$, *respectively. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $T^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. *For any* $x_1 = x \in H$, *define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Q_\mu\big(I - rA^*J_F(A - J_\lambda A)\big)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset [0,1]$ *and* $r \in (0,\infty)$ *satisfy the following:*

$$0 < a \le \beta_n \le b < 1 \quad \text{and} \quad 0 < r\|AA^*\| < 2$$

for some $a, b \in \mathbb{R}$. *Then* $\{x_n\}$ *converges weakly to a point* $z_0 \in T^{-1}0 \cap A^{-1}(B^{-1}0)$, *where* $z_0 = \lim_{n\to\infty} P_{T^{-1}0\cap A^{-1}(B^{-1}0)}x_n$.

Proof. Since Q_μ is the resolvent of T on H, Q_μ is nonexpansive. Furthermore, since J_λ is the metric resolvent of B for $\lambda > 0$, from (4) in Examples, J_λ is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - J_\lambda x_n \to 0$, then $p = J_\lambda p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - J_\lambda x_n \to 0$. It is clear that $J_\lambda x_n \rightharpoonup p$ and $\|J_F(x_n - J_\lambda x_n)\| = \|x_n - J_\lambda x_n\| \to 0$. Since J_λ is the metric resolvent of B, we have from [3] that

$$\langle J_\lambda x_n - J_\lambda p, J_F(x_n - J_\lambda x_n) - J_F(p - J_\lambda p)\rangle \geq 0.$$

Therefore, $-\|p - J_\lambda p\|^2 = \langle p - J_\lambda p, -J_F(p - J_\lambda p)\rangle \geq 0$ and hence $p = J_\lambda p$. This implies that J_λ is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Similarly, using Theorem 4.1, we have the following strong convergence theorems.

Theorem 5.7. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0,1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction with $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. For $x_1 = x \in H_1$, let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u_n + (1-\alpha_n)T(x_n - rA^*(I-U)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $r \in (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < r\|AA^*\| < 1-k, \quad \lim_{n\to\infty} \alpha_n = 0,$$
$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

where $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)}u$.

Theorem 5.8. *Let H_1 and H_2 be Hilbert spaces. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $U : H_2 \to H_2$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. For $x_1 = x \in H_1$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u_n + (1-\alpha_n)T(x_n - rA^*(I-U)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $r \in (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < r\|AA^*\| < 1, \quad \lim_{n\to\infty} \alpha_n = 0,$$
$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

where $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)}u$.

Theorem 5.9. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\big(\alpha_n u_n + (1 - \alpha_n) P_C(x_n - rA^* J_F(I - P_D)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $r \in (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy

$$0 < r\|AA^*\| < 2, \quad \lim_{n\to\infty} \alpha_n = 0,$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

where $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D}u$.

Theorem 5.10. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let T and B be maximal monotone operators of H into H and F into F^*, respectively. Let Q_μ be the resolvent of T for $\mu > 0$ and let J_λ be the metric resolvent of B for $\lambda > 0$, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $T^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\big(\alpha_n u_n + (1 - \alpha_n) Q_\mu(x_n - rA^* J_F(I - J_\lambda)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $r \in (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy

$$0 < r\|AA^*\| < 2, \quad \lim_{n\to\infty} \alpha_n = 0,$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1$$

where $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in T^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{T^{-1}0\cap A^{-1}(B^{-1}0)}u$.

References

[1] S. M. Alsulami and W. Takahashi, *The split common null point problem for maximal monotone mappings in Hilbert spaces and applications*, J. Nonlinear Convex Anal. **15** (2014), 793–808.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuous properties*, J. Nonlinear Convex Anal. **10** (2009), 131–147.

[4] F. E. Browder, *Nonlinear maximal monotone operators in Banach spaces*, Math. Ann. **175** (1968), 89–113.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[6] C. Byrne, Y. Censor, A. Gibali and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759–775.

[7] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.

[8] Y. Censor and A. Segal, *The split common fixed-point problem for directed operators*, J. Convex Anal. **16** (2009), 587–600.

[9] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957-961.

[10] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert space*, Taiwanese J. Math. **14** (2010), 2497–2511.

[11] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.

[12] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[13] G. Marino and H.-K. Xu, *Weak and strong convergence theorems for strich pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[14] A. Moudafi, *The split common fixed point problem for demicontractive mappings*, Inverse Problems **26** (2010), 055007, 6 pp.

[15] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[16] S. Ohsawa and W. Takahashi, *Strong convergence theorems for resolvents of maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **81** (2003), 439–445.

[17] Z. Opial, *Weak covergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[18] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Programming Ser. A **87** (2000), 189–202.

[19] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[20] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.

[21] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[22] W. Takahashi, *The split feasibility problem in Banach spaces*, J. Nonlinear Convex Anal. **15** (2014), 1349–1355.

[23] W. Takahashi, *The split common null point problem in Banach spaces*, Arch. Math. (Basel) **104** (2015), 357–365.

[24] W. Takahashi, *The split common null point problem in two Banach spaces*, J. Nonlinear Convex Anal. **16** (2015), 2343–2350.

[25] W. Takahashi, *The split common fixed point problem and strong convergence theorems by hybrid methods in two Banach spaces*, J. Nonlinear Convex Anal. **17** (2016), 1051–1067.

[26] W. Takahashi, *The split common fixed point problem and the shrinking projection method in Banach spaces*, J. Convex Anal. **24** (2017), to appear.

[27] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[28] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[29] W. Takahashi, N.-C. Wong and J.-C. Yao, *Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 553–575.

[30] W. Takahashi, H.-K. Xu and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal. **23** (2015), 205–221.

[31] W. Takahashi, J.-C. Yao and P. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 567–586.

[32] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.

Manuscript received 22 January 2016
revised 6 April 2016

Wataru Takahashi
Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: `wataru@is.titech.ac.jp; wataru@a00.itscom.net`