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PECULIAR HOMOMORPHISMS ON ALGEBRAS OF VECTOR-VALUED CONTINUOUSLY DIFFERENTIABLE MAPS

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ABSTRACT. We study unital homomorphisms between Banach algebras of vector-valued continuously differentiable maps. We provide sufficient conditions under which every unital homomorphism between these Banach algebras are given as composition operators of some special form.

1. INTRODUCTION

Every homomorphism which preserves identity (unital homomorphism) between unital semisimple commutative Banach algebras is represented as a composition operator induced by the associated continuous map between the maximal ideal spaces. It is not always the case that the converse holds; a continuous map between the maximal ideal spaces need not always define a composition operator between underlying algebras; some restriction on the map between the maximal ideal spaces is required. Suppose that $\psi : \mathrm{Lip}(K_1, E_1) \to \mathrm{Lip}(K_2, E_2)$ is a unital homomorphism between the algebras of all Lipschitz maps on a compact metric space K_j into a unital semisimple commutative Banach algebra E_j with the maximal ideal space $M(E_j)$. It is well known that the maximal ideal space of $\mathrm{Lip}(K_j, E_j)$ is homeomorphic to $K_j \times M(E_j)$ and we may suppose that $\mathrm{Lip}(K_j, E_j)$ is a subalgebra $C(K_j \times M(E_j))$ of the algebra of all complex-valued continuous functions on $K_j \times M(E_j)$ through the Gelfand transform. Botelho and Jamison [1] proved that if K_2 is connected and E_j is the algebra of convergent sequences or the algebra of bounded sequences, then there exist continuous maps $\varphi_1 : K_2 \times M(E_2) \to K_1$ and $\varphi_2 : M(E_2) \to M(E_1)$ such that $\psi(F)(x, \phi) = F(\varphi_1(x, \phi), \varphi_2(\phi))$ for every $(x, \phi) \in K_2 \times M(E_2)$; φ_2 depends only on $M(E_2)$, not on K_2. Oi [7] generalized this result by proving that it is the case where E_j is a unital commutative C^*-algebra. We may say that such a homomorphism is peculiar since the corresponding continuous map between the maximal ideal spaces has a special form such that φ_2 depends only on $M(E_2)$, not on K_2. We call such a homomorphism of type BJ.

In [3] we study homomorphisms of type BJ for admissible quadruples and algebras of vector-valued Lipschitz maps. One may imagine that the results for admissible quadruples in [3] are applicable for $C^1([0, 1], E)$ of the algebra of vector-valued

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continuously differentiable maps defined on the unit interval $[0,1]$ in a similar way to the case of $\mathrm{Lip}(K,E)$. In this paper we exhibit several sufficient conditions on the Banach algebra E which ensure that every unital homomorphism on $C^1([0,1],E)$ is of type BJ. These results are similar in some part but different in the other to results for algebras of vector-valued Lipschitz maps in [3]. We make use of results in [3] for proofs of the main results in this paper. On the other hand some new idea is needed for the case of $C^1([0,1],E)$, it is worth publishing results for $C^1([0,1],E)$ in this paper.

2. Preliminary

Let E be a unital commutative Banach algebra. The maximal ideal space of E is the set $M(E)$ of all non-zero multiplicative linear functional with the Gelfand topology; the relative topology of the weak-$*$ topology of E^*. The maximal ideal space is a compact Hausdorff space. The Gelfand transform of $a \in E$ is denoted by $\Gamma_E(a)$; $\Gamma_E(a) : M(E) \to \mathbb{C}$, $\Gamma_E(a)(\phi) = \phi(a)$ for $\phi \in M(E)$. For simplicity of notation, we sometimes denote the Gelfand transform of a by $\hat{a}$. The Gelfand topology on $M(E)$ is the weakest topology that $\hat{a}$ is continuous for every $a \in E$. For a subset S of E, the set $\{\Gamma_E(a) : a \in S\}$ is denoted by $\Gamma_E(S)$ or $\widehat{S}$. The set $\widehat{S}$ is also called the Gelfand transform of S. We denote the spectrum of a by $\sigma(a)$, the spectral radius by $r(a)$, the group of all invertible elements by E^{-1} and the unit element by 1_E. The Jacobson radical, the intersection of all maximal ideals, of E is denoted by $\mathrm{rad}(E)$. For $a \in E$, we have $a \in \mathrm{rad}(E)$ if and only if $r(a) = 0$ if and only if $\sigma(a) = \{0\}$ (cf. [5, Proposition 3.5.1, Theorem 3.5.1]). We say that E is semisimple if $\mathrm{rad}(E) = \{0\}$. Hence E is semisimple if and only if the Gelfand map $\Gamma_E : E \to \widehat{E}$ is an isomorphism. For the theory of commutative Banach algebras, see e.g., [2, 4, 5, 8].

The algebra of all continuous maps from a compact Hausdorff space X into a unital commutative Banach algebra E is denoted by $C(X,E)$. For $S \subset X$ the supremum norm on S for $f \in C(X,E)$ is defined as

$$\|f\|_{\infty(S)} = \sup_{x \in S} \|f(x)\|_E.$$

With the supremum norm $\|\cdot\|_{\infty(X)}$, the algebra $C(X,E)$ is a unital commutative Banach algebra.

Let E be a unital commutative Banach algebra. An E-valued function algebra *in the strong sense* is as follows.

Definition 1. We say that A is an E-valued function algebra on a compact Hausdorff space X in the strong sense if A is a subalgebra of $C(X,E)$ for a unital commutative Banach algebra E such that the following conditions are satisfied.

(1.1) A is a Banach algebra under some norm $\|\cdot\|_A$,
(1.2) A contains the constant maps,
(1.3) A separates the points of X, that is, for every pair x and y of different points in X, there exists f in A such that $f(x) \neq f(y)$,
(1.4) for every $x \in X$ the evaluation map $e_x : A \to E$ defined by $f \mapsto f(x)$ is continuous.

The algebra $C(X,E)$ is an E-valued function algebra on X in the strong sense with the norm $\|\cdot\|_{\infty(X)}$. Suppose that a subalgebra A of $C(X,E)$ is a Banach algebra under some norm. If E is semisimple, then $e_x : A \to E$ is automatically continuous for every $x \in X$ by a theorem of Šilov (cf. [8, Theorem 3.1.11]).

Nikou and O'Farrell defined E-valued function algebra [6, Definition 1.1]. We make a replace (1) of Definition 1.1 in [6] by (1.2) of Definition 1 in this paper, and we define an E-valued function algebra *in the strong sense*. Note that a $\mathbb{C}$-valued function algebra in the sense of Nikou and O'Farrell is a $\mathbb{C}$-valued function algebra in the strong sense.

Let $A_{\mathbb{C}}$ be a $\mathbb{C}$-valued function algebra on a compact Hausdorff space X in the strong sense and E a unital commutative Banach algebra. For a pair $f \in A_{\mathbb{C}}$ and $b \in E$, the map $f \otimes b \in C(X,E)$ is defined as $(f \otimes b)(x) = f(x)b$ for $x \in X$. We denote

$$A_{\mathbb{C}} \otimes E = \left\{ \sum_{j=1}^{n} f_j \otimes b_j : n \in \mathbb{N},\ f_j \in A_{\mathbb{C}},\ b_j \in E\ (j = 1, 2, \ldots, n) \right\},$$

where $\mathbb{N}$ is the set of all positive integers.

Let $A_{\mathbb{C}}$ be a $\mathbb{C}$-valued function algebra on X in the strong sense. Then we have $\{e_x : x \in X\} \subset M(A_{\mathbb{C}})$, and the map $x \mapsto e_x$ from X into $M(A_{\mathbb{C}})$ is a continuous injection. Hence X is embedded in $M(A_{\mathbb{C}})$ as a compact subset. We call $A_{\mathbb{C}}$ *natural* if the map $x \mapsto e_x$ is a surjection, that is, if X is homeomorphic to $\{e_x : x \in X\} = M(A_{\mathbb{C}})$ through the map $x \mapsto e_x$.

An admissible quadruple was defined by Nikou and O'Farrell in [6].

Definition 2 (cf. [6])**.** Let X be a compact Hausdorff space and E a commutative Banach algebra with unit. By an admissible quadruple we mean a quadruple $(X, E, B, \widetilde{B})$, where

(2.1) $B \subset C(X)$ is a natural $\mathbb{C}$-valued function algebra on X,
(2.2) $\widetilde{B} \subset C(X,E)$ is an E-valued function algebra on X in the strong sense,
(2.3) $B \otimes E \subset \widetilde{B}$ and
(2.4) $\{\lambda \circ f : f \in \widetilde{B}, \lambda \in M(E)\} \subset B$.

Note that two definitions of an admissible quadruple by Definition 2.1 in [6] and Definition 2 are formally different since $\widetilde{B} \subset C(X,E)$ is an E-valued function algebra on X in the strong sense in the above definition while $\widetilde{B} \subset C(X,E)$ is an E-valued function algebra in the sense of Nikou and O'Farrell in [6]. But due to the condition (5) in Definition 2.1 in [6], $\widetilde{B}$ in Definition 2.1 in [6] automatically satisfies (1.2) of Definition 1. Therefore an admissible quadruple defined by Definition 2.1 in [6] and one defined by Definition 2 in this paper are equivalent. Let X be a compact Hausdorff space and E a unital commutative Banach algebra. Then $(X, E, C(X), C(X,E))$ is an admissible quadruple.

Let K be a compact metric space with metric $d(\cdot,\cdot)$. Let E be a unital commutative Banach algebra. We say that a function $F : K \to E$ is a Lipschitz map from K into E if the Lipschitz constant $L(F) = \sup_{x \neq y} \frac{\|F(x)-F(y)\|_E}{d(x,y)}$ is finite. The algebra of all Lipschitz maps from K into E is denoted by $\mathrm{Lip}(K,E)$. Then $\mathrm{Lip}(K,E)$ is a unital commutative Banach algebra with the norm $\|\cdot\|_L = L(\cdot) + \|\cdot\|_{\infty(K)}$. Then $(K, E, \mathrm{Lip}(K, \mathbb{C}), \mathrm{Lip}(K, E))$ is an admissible quadruple. Conditions which ensure that every unital homomorphism on $\mathrm{Lip}(K, E)$ is of type BJ were studied in [3].

Let $C^1([0,1])$ be the algebra of all continuously differentiable complex-valued functions on the unit interval $[0,1]$. It is well known that $C^1([0,1])$ is a Banach algebra with respect to the norm $\|f\|_{C^1} = \|f'\|_{\infty([0,1])} + \|f\|_{\infty([0,1])}$ for $f \in C^1([0,1])$. Let E be a unital commutative Banach algebra. The algebra of all continuously differentiable maps from $[0,1]$ to E is denoted by $C^1([0,1], E)$. With the norm $\|f\|_{C^1} = \|f'\|_{\infty([0,1])} + \|f\|_{\infty([0,1])}$ for $f \in C^1([0,1], E)$ $C^1([0,1], E)$ is a unital commutative Banach algebra. Then $([0,1], E, C^1([0,1]), C^1([0,1], E))$ is an admissible quadruple.

Due to Nikou and O'Farrell [6] we define as follows.

Definition 3. Let $(X, E, B, \widetilde{B})$ be an admissible quadruple. Let $\pi : X \times M(E) \to M(\widetilde{B})$ be given by $\pi(x, \phi) = \phi \circ e_x$, where $\phi \circ e_x(F) = \phi(F(x))$ for every $F \in \widetilde{B}$. Then by a routine argument π is a continuous injection. We say that an admissible quadruple $(X, E, B, \widetilde{B})$ is natural if the associated map π is bijective.

If an admissible quadruple $(X, E, B, \widetilde{B})$ is natural, then the map π is a homeomorphism since $X \times M(E)$ is compact and $M(\widetilde{B})$ is Hausdorff. In this case the maximal ideal space of $\widetilde{B}$ coincides with $\{\phi \circ e_x : x \in X,\ \phi \in M(E)\}$, and it is homeomorphic to $X \times M(E)$. Hence we may suppose that

$$\widehat{\widetilde{B}} \subset C(X \times M(E)) \tag{2.1}$$

by identifying (x, ϕ) and $\phi \circ e_x$ through π. The following proposition is proved in [3].

Proposition 4. *Let $(X, E, B, \widetilde{B})$ be an admissible quadruple. Suppose that B is dense in $C(X)$. Suppose also that $\widetilde{B}$ is inverse-closed; $F \in \widetilde{B}$ with $\Gamma_{\widetilde{B}}(F)(\phi \circ e_x) \neq 0$ for every pair $x \in X$ and $\phi \in M(E)$ implies $F^{-1} \in \widetilde{B}$. Then $(X, E, B, \widetilde{B})$ is natural.*

Let E be a unital commutative Banach algebra. By the Stone-Weierstrass theorem $C^1([0,1])$ is dense in $C([0,1])$, and $C^1([0,1], E)$ is inverse-closed by the definition of a vector-valued continuously differentiable maps. Hence by Proposition 4 the maximal ideal space of $C^1([0,1], E)$ is homeomorphic to $[0,1] \times M(E)$. Hence the admissible quadruple $([0,1], E, C^1([0,1]), C^1([0,1], E))$ is natural and

$$C^1\widehat{([0,1], E)} \subset C([0,1] \times M(E)). \tag{2.2}$$

We say that $(X, E, B, \widetilde{B})$ is semisimple if so is $\widetilde{B}$.

Proposition 5. *An admissible quadruple $(X, E, B, \widetilde{B})$ is semisimple if and only if E is semisimple.*

A proof is given in [3] and is omitted. Suppose that E is semisimple and $(X, E, B, \widetilde{B})$ is natural. Then $\widetilde{B}$ is semisimple by Proposition 5; we may identify $\widetilde{B}$ and $\widehat{\widetilde{B}}$. Hence we may suppose that

$$\widetilde{B} \subset C(X \times M(E)) \tag{2.3}$$

by (2.1). In particular, we may suppose that

$$C^1([0,1],E) = C^1(\widehat{[0,1]},E).$$

In the following we write $C^1([0,1],E)$ instead of $C^1(\widehat{[0,1]},E)$. By (2.2) we may suppose that

$$C^1([0,1],E) \subset C([0,1] \times M(E)) \tag{2.4}$$

if E is semisimple.

Since the maximal ideal space of $C^1([0,1],E)$ is homeomorphic to $[0,1] \times M(E)$ and $\{\phi \circ e_x : x \in [0,1],\ \phi \in M(E)\}$ respectively we may also suppose that

$$\Gamma_{C^1([0,1],E)}(C^1([0,1]) \otimes E) \subset C^1([0,1],E) \subset C([0,1] \times M(E)).$$

3. Algebra homomorphisms and isomorphisms

In this section we study sufficient conditions on a Banach algebra of vector-valued continuously differentiable maps which ensure that every unital homomorphism or isomorphism on it is of type BJ. The following three lemmata are versions of [3, Lemmata 12, 13 and 14].

Lemma 6. *Suppose that $G_1, \ldots, G_n$ are open sets with $\cup_{j=1}^n G_j = [0,1]$. Then there exist $f_1, \ldots, f_n \in C^1([0,1])$ such that $0 \leq f_j \leq 1$ on $[0,1]$ and $f_j = 0$ on $[0,1] \setminus G_j$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n f_j = 1$ on $[0,1]$.*

Lemma 6 is well known and we omit a proof.

Lemma 7. *Let E be a unital semisimple commutative Banach algebra. Then we have*

$$C^1([0,1],E) \subset \overline{\Gamma_{C^1([0,1],E)}(C^1([0,1]) \otimes E)}$$

where $\bar{\cdot}$ denotes the uniform closure on $[0,1] \times M(E)$.

Proof. Let $F \in C^1([0,1],E)$. Let $\varepsilon > 0$ be arbitrary. Then there exists a finite number of points $x_1, \ldots, x_n \in [0,1]$ and open neighborhoods $x_1 \in G_1, \ldots, x_n \in G_n$ such that $\cup_{j=1}^n G_j = [0,1]$ and

$$\|F(x) - F(x_j)\|_E \leq \varepsilon, x \in G_j$$

for every $j = 1, 2, \ldots, n$. Then we have by Lemma 6 that there exist $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in C^1([0,1])$ such that $0 \leq \Lambda_j \leq 1$ on $[0,1]$, $\Lambda_j = 0$ on $[0,1] \setminus G_j$ for $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n \Lambda_j = 1$ on $[0,1]$. Put $F_\varepsilon = \sum_{j=1}^n \Lambda_j F(x_j) \in C^1([0,1]) \otimes E$. By some calculation we obtain that $\|F - F_\varepsilon\|_{\infty([0,1] \times M(E))} \leq \varepsilon$. As $F \in C^1([0,1],E)$ and ε are arbitrary, we have the conclusion. □

Lemma 8. *The usual topology on $[0,1]$, the Gelfand topology induced by $C^1([0,1])$, and the relative topology induced by the operator norm topology on the dual space of $C^1([0,1])$ all coincide with each other.*

Proof. It is well known that the maximal ideal space $M(C^1([0,1]))$ with the Gelfand topology is homeomorphic to $[0,1]$ with the usual topology. In fact, $x \mapsto e_x$ defines a homeomorphism from $[0,1]$ onto $M(C^1([0,1]))$. We prove that the Gelfand topology of $M(C^1([0,1]))$, which is the topology induced by the weak-$*$ topology inherited from the dual space of $C^1([0,1])$, is homeomorphic to the topology induced by the metric inherited from the dual space of $C^1([0,1])$. Just for the simplicity we denote $M(C^1([0,1]))$ with the Gelfand topology by M_w and $M(C^1([0,1]))$ with the topology induced by the metric inherited from the dual space of $C^1([0,1])$ by M_s. Let $\mathrm{Id} : M_s \to M_w$ be the identity map. Since the topology induced by the metric inherited from the dual space of $C^1([0,1])$ is stronger than the Gelfand topology, the map Id is continuous. For $x \in [0,1]$, e_x denotes the point evaluation on $C^1([0,1])$ at x. We denote the norm of the dual space of $C^1([0,1])$ by $\|\cdot\|_*$. Let $f \in C^1([0,1])$. Recall that the Lipschitz constant of f is $L(f) = \sup_{t\neq s} \frac{|f(t)-f(s)|}{|t-s|}$. It is easy to see that $\|f'\|_{\infty([0,1])} \leq L(f)$. By the mean value theorem we have

$$\begin{aligned}\frac{|f(s)-f(t)|}{|s-t|} &\leq \frac{|\operatorname{Re} f(s) - \operatorname{Re} f(t)|}{|s-t|} + \frac{|\operatorname{Im} f(s) - \operatorname{Im} f(t)|}{|s-t|} \\ &\leq \|\operatorname{Re} f'\|_{\infty([0,1])} + \|\operatorname{Im} f'\|_{\infty([0,1])} \leq 2\|f'\|_{\infty([0,1])}\end{aligned}$$

for every $s, t \in [0,1]$ with $s \neq t$. Thus $L(f) \leq 2\|f'\|_{\infty([0,1])}$. Hence we have

$$\|e_x - e_y\|_* = \sup_{\|f\|_{C^1}\leq 1} |f(x)-f(y)| \leq \sup_{\|f'\|_{\infty([0,1])}\leq 1} |f(x)-f(y)| \leq 2|x-y|.$$

Since the usual topology and the Gelfand topology on $[0,1]$ coincide we infer that Id^{-1} is continuous. We conclude that Id is a homeomorphism. $\square$

We proved the following two theorems in [3].

Theorem 9. *Let E_j be a unital commutative Banach algebra and $(X_j, E_j, B_j, \widetilde{B_j})$ an admissible quadruple for $j = 1, 2$. Suppose that $\widehat{\widetilde{B_1}} \subset \overline{\Gamma_{\widetilde{B_1}}(B_1 \otimes E_1)}$, where $\bar{\cdot}$ denotes the uniform closure on $M(\widetilde{B_1})$. Suppose that X_2 is connected with respect to the relative topology induced by the metric inherited from the dual space of B_2 and that $M(E_1)$ is totally disconnected with respect to the relative topology induced by the metric inherited from the dual space of E_1. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be a unital homomorphism. Then there exist a continuous map $\tau : M(E_2) \to M(E_1)$ and a continuous map $\varphi : X_2 \times M(E_2) \to X_1$ which satisfy that*

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times M(E_2)$$

for every $F \in \widetilde{B_1}$.

Theorem 10. *Let E_j be a unital commutative Banach algebra and $(X_j, E_j, B_j, \widetilde{B_j})$ an admissible quadruple for $j = 1, 2$. Suppose that $\widehat{\widetilde{B_1}} \subset \overline{\Gamma_{\widetilde{B_1}}(B_1 \otimes E_1)}$, where $\bar{\cdot}$ denotes the uniform closure on $M(\widetilde{B_1})$. Suppose that X_2 is connected. Suppose that $M(E_1)$ is totally disconnected with respect to the Gelfand topology. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be a unital homomorphism. Then there exist a continuous map $\tau : M(E_2) \to M(E_1)$ and a continuous map $\varphi : X_2 \times M(E_2) \to X_1$ which satisfy that*

$$\Gamma_{\widetilde{B_2}}(\psi(F))(\phi \circ e_x) = \Gamma_{\widetilde{B_1}}(F)(\tau(\phi) \circ e_{\varphi(x,\phi)}), \quad (x, \phi) \in X_2 \times M(E_2)$$

for every $F \in \widetilde{B_1}$.

Applying Theorems 9 or 10 for $([0,1], E_j, C^1([0,1]), C^1([0,1], E_j))$ we obtain the following.

Corollary 11. *Suppose that E_j is a unital semisimple commutative Banach algebra for $j = 1, 2$. Let $\psi : C^1([0,1], E_1) \to C^1([0,1], E_2)$ be a unital homomorphism. Suppose that $M(E_1)$ is totally disconnected with respect to either the Gelfand topology or the relative topology induced by the metric inherited from the dual space of E_1. Then there exist a continuous map $\tau : M(E_2) \to M(E_1)$ and a continuous map $\varphi : [0,1] \times M(E_2) \to [0,1]$ such that for each $\phi \in M(E_2)$ the map $\varphi(\cdot, \phi) : [0,1] \to [0,1]$ is continuously differentiable, which satisfy that*

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in [0,1] \times M(E_2)$$

for every $F \in C^1([0,1], E_1)$.

Proof. The maximal ideal space of $C^1([0,1], E_j)$ is homeomorphic to $[0,1] \times M(E_j)$ by Proposition 4. We have the inclusion

$$C^1([0,1], E_1) \subset \overline{\Gamma_{C^1([0,1],E)}(C^1([0,1]) \otimes E)}$$

by Lemma 7. Since $[0,1]$ is connected we have that $[0,1]$ is also connected with respect to the relative topology induced by the metric inherited from the dual space of $C^1([0,1])$ by Lemma 8. By Theorems 9 or 10 there exist a continuous map $\tau : M(E_2) \to M(E_1)$ and a continuous map $\varphi : [0,1] \times M(E_2) \to [0,1]$ which satisfy that

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in [0,1] \times M(E_2) \tag{3.1}$$

for every $F \in C^1([0,1], E_1)$. To prove that the map $\varphi(\cdot, \phi) : [0,1] \to [0,1]$ is continuously differentiable for each $\phi \in M(E_2)$, define $\widetilde{\psi^\phi} : C^1([0,1]) \to C^1([0,1])$ by $\widetilde{\psi^\phi}(f)(x) = \phi(\psi(f \otimes 1_{E_1})(x))$, $f \in C^1([0,1])$. Then $\widetilde{\psi^\phi}$ is a unital homomorphism from $C^1([0,1])$ into $C^1([0,1])$. Then $\widetilde{\psi^\phi}$ is continuous by a theorem of Šilov (cf. [8, Theorem 3.1.11]). On the other hand we have by (3.1) that $\widetilde{\psi^\phi}(f) = f(\varphi(\cdot, \phi))$, $f \in C^1([0,1])$. Letting f the identity function we have that $\varphi(\cdot, \phi) : [0,1] \to [0,1]$ is continuously differentiable. □

The following theorem is proved in [3, Theorem 19].

Theorem 12. *Let A_j be a uniform algebra and $(X_j, A_j, B_j, \widetilde{B_j})$ an admissible quadruple for $j = 1, 2$. Suppose that $\widetilde{B_j}$ is natural for $j = 1, 2$. Suppose that* $\mathrm{Ch}(B_2)$ *is connected with respect to the relative topology induced by the metric inherited from the dual space of B_2. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be an isomorphism. Then there exist a homeomorphism $\tau : M(A_2) \to M(A_1)$ and a continuous map $\varphi : X_2 \times M(A_2) \to X_1$ such that the map $\varphi(\cdot, \phi) : X_2 \to X_1$ is a homeomorphism for each $\phi \in M(A_2)$ which satisfy that*

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times M(A_2)$$

for every $F \in \widetilde{B_1}$. In particular, A_1 is isomorphic to A_2 and B_1 is isomorphic to B_2.

Applying Theorem 12 we obtain the following.

Corollary 13. *Let A_j be a uniform algebra for $j = 1, 2$. Suppose that $\psi : C^1([0,1], A_1) \to C^1([0,1], A_2)$ is an algebra isomorphism. Then there exist a homeomorphism $\tau : M(A_2) \to M(A_1)$ and a continuous map $\varphi : [0,1] \times M(A_2) \to [0,1]$ such that for each $\phi \in M(A_2)$, the map $\varphi(\cdot, \phi) : [0,1] \to [0,1]$ is a C^1-diffeomorphism which satisfy that*

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in [0,1] \times M(A_2)$$

for every $F \in C^1([0,1], A_1)$. In particular, A_1 is algebraically isomorphic to A_2.

Proof. The Choquet boundary for $C^1([0,1])$ is $[0,1]$. By Lemma 8, $[0,1]$ is connected with respect to the relative topology induced by the metric inherited from the dual space of $C^1([0,1])$. Applying Theorem 12 we can prove Corollary 13 as in the proof of Corollary 11 except that the map $\varphi(\cdot, \phi)$ is a C^1-diffeomorphism for every $\phi \in M(A_2)$. Let $\phi \in M(A_2)$ be fixed. By Theorem 12 and Corollary 11 we see that $\varphi(\cdot, \phi)$ is continuouly differentiable, and is a homeomorphsim. Applying the equation (12) in the proof of [3, Theorem 19] we see that $\varphi(\cdot, \phi)^{-1}$ is continuouly differentiable. It follows that $\varphi(\cdot, \phi)$ is C^1-diffeomorphism. $\square$

Note that an algebraic isomorphism between uniform algebra is automatically isometric.

Several examples of unital semisimple commutative Banach algebras E such that the maximal ideal space are discrete with respect to the relative topology induced by the metric inherited from the dual space of E are exhibited in [3]. Examples of unital semisimple commutative Banach algebras whose maximal ideal spaces are totally disconnected is also exhibited in [3].

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