Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 3, Number 1, 2017, 111–120

FIXED POINT THEOREMS FOR ĆIRIĆ TYPE CONTRACTIONS AND OTHERS IN COMPLETE METRIC SPACES

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Dedicated to the memory of Professor Ljubomir Ćirić

Abstract. Inspired by Jachymski's fixed point theorem for integral type of contractions, we prove fixed point theorems in complete metric spaces for Ćirić, Kannan and Chatterjea types of contractions.

1. Introduction

Jachymski in [7] proved splendid fixed point theorems. The following is a corollary of Theorem 9 in [7].

Theorem 1.1 (Jachymski [7]). *Let T be a mapping on a complete metric space (X, d). Assume that there exist $r \in [0, 1)$ and a function η from $[0, \infty)$ into itself satisfying*

$$\eta\big(d(Tx, Ty)\big) \le r\, \eta\big(d(x, y)\big)$$

for any $x, y \in X$ and the following (H1):

(H1) *For any sequence $\{a_n\}$ in $[0, \infty)$, $\lim_n \eta(a_n) = 0$ iff $\lim_n a_n = 0$.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.

See [3, 8, 13, 15, 16, 19] and others. Considering the case where $\eta = (t \mapsto t)$, we understand that Theorem 1.1 is one of generalizations of the Banach contraction principle [1, 4]. We note that the assumption on η is only (H1) in Theorem 1.1. So the authors think that we have completed the Banach type of this study.

Recently, in [18] we obtained the following Bogin type fixed point theorem [2]. See also [10, 11, 14].

For $\tau \in (1, \infty)$, we define a set $H(\tau)$ as follows: $\eta \in H(\tau)$ iff η is a function from $[0, \infty)$ into itself satisfying (H1) and the following:

(H2:τ) For any sequence $\{a_n\}$ in $[0, \infty)$ which converges to some $\alpha \in (0, \infty)$,

$$\eta(\alpha) < \tau \limsup_{n \to \infty} \eta(a_n)$$

holds.

2010 *Mathematics Subject Classification.* 54H25.

Key words and phrases. Fixed point, integral type of contraction, Ćirić's fixed point theorem, Kannan's fixed point theorem, Chatterjea's fixed point theorem, complete metric space.

The second author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

Theorem 1.2 ([18]). *Let T be a mapping on a complete metric space (X, d). Assume that T satisfies Condition* (B)*, that is, there exist $r \in [0, 1)$, $s, t \in (0, 1/2)$ and $\eta \in H\big((r + s + 2\,t)/t\big)$ satisfying $r + 2\,s + 2\,t = 1$ and*

$$
\begin{aligned}
\eta\big(d(Tx, Ty)\big) \le r\,\eta\big(d(x, y)\big) + s\,\eta\big(d(x, Ty)\big) + s\,\eta\big(d(Tx, y)\big) \\
+ t\,\eta\big(d(x, Tx)\big) + t\,\eta\big(d(y, Ty)\big)
\end{aligned}
\tag{1.1}
$$

for any $x, y \in X$. Assume also that there exists $u \in X$ satisfying the following:

(H3) $\big\{\eta\big(d(T^m u, T^n u)\big) : m, n \in \mathbb{N} \cup \{0\}\big\}$ *is bounded.*

Then T has a unique fixed point z. Moreover $\{T^n u\}$ converges to z.

In [17], we obtained that the following (H4) implies (H3).

(H4) For any $\beta > 0$ and $\varepsilon > 0$, there exists $M > 0$ such that $\eta(a) < (1+\varepsilon)\,\eta(a+b)$ holds for any $a > 0$ and $b \in [-\beta, +\beta]$ satisfying $\eta(a) > M$ and $a + b > 0$.

We note that we need (H1), (H2:$(r + s + 2\,t)/t$) and (H4) for Bogin type.

In this paper, strongly inspired by Theorem 1.1, we will prove fixed point theorems in complete metric spaces for Ćirić, Kannan and Chatterjea types of contractions.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. For an arbitrary set A, we denote by $\#A$ the cardinal number of A.

In this section, we give some preliminaries.

Lemma 2.1 ([18]). *Let η be a function from $[0, \infty)$ into itself satisfying* (H1)*. Let $\{x_n\}$ be a sequence in a metric space (X, d) satisfying*

$$
\lim_{n\to\infty} \limsup_{m\to\infty} \eta\big(d(x_n, x_m)\big) = 0. \tag{2.1}
$$

Then $\{x_n\}$ is Cauchy.

Lemma 2.2. *Let η be a function from $[0, \infty)$ into itself satisfying* (H1)*. Let $\{x_n\}$ be a sequence in a metric space (X, d) satisfying*

$$
\lim_{n\to\infty} \sup_{m>n} \eta\big(d(x_n, x_m)\big) = 0. \tag{2.2}
$$

Then $\{x_n\}$ is Cauchy.

Proof. Since (2.2) is stronger than (2.1), we obtain the desired result by Lemma 2.1. □

The typical example of η belonging to $H(\tau)$ is the following:

Lemma 2.3 ([17]). *Let η be a continuous function from $[0, \infty)$ into itself satisfying $\eta^{-1}(0) = \{0\}$ and $\inf\{\eta(a) : a \ge \alpha\} > 0$ for some $\alpha \in (0, \infty)$. Then $\eta \in H(\tau)$ for any $\tau \in (1, \infty)$.*

We introduce the following condition (H5). It is obvious that (H5) is strictly weaker than (H4).

(H5) There exists $\varepsilon > 0$ such that for any $\beta > 0$, there exists $M > 0$ such that

$$\eta(a) < (1+\varepsilon)\,\eta(a+b)$$

holds for any $a > 0$ and $b \in [-\beta, +\beta]$ satisfying $\eta(a) > M$ and $a + b > 0$.

Lemma 2.4. *Let η and h be functions from $[0,\infty)$ into itself. Assume that h satisfies* (H4) *and that there exist $s, t \in (0,\infty)$ satisfying*

$$s\,h(a) \le \eta(a) \le t\,h(a)$$

for any $a \in [0,\infty)$. Then η satisfies (H5).

Proof. Put $\varepsilon = 2\,t/s$. Fix $\beta > 0$. Then by (H4), there exists $M > 0$ satisfying the following:

- $h(a) < 2\,h(a+b)$ holds for any $a > 0$ and $b \in [-\beta, +\beta]$ with $h(a) > M/t$ and $a + b > 0$.

Fix $a > 0$ and $b \in [-\beta, +\beta]$ with $\eta(a) > M$ and $a + b > 0$. Then we have

$$\eta(a) \le t\,h(a) < 2\,t\,h(a+b) \le (2\,t/s)\,\eta(a+b) < (1+\varepsilon)\,\eta(a+b).$$

Therefore (H5) holds. □

Since $a \mapsto a^q$ satisfies (H4), we obtain the following; see [17].

Example 2.5. Let η be a function from $[0,\infty)$ into itself. Assume that there exist $q \in (0,\infty)$ and $s, t \in (0,\infty)$ satisfying

$$s\,a^q \le \eta(a) \le t\,a^q$$

for any $a \in [0,\infty)$. Then η satisfies (H1) and (H5).

3. Fixed Point Theorems

We begin with Ćirić type. See [6, 12].

Theorem 3.1. *Let T be a mapping on a complete metric space (X,d). Assume that there exist $r \in (0,1)$ and $\eta \in H(1/r)$ satisfying*

$$\eta\big(d(Tx,Ty)\big) \le r\,\max\big\{\eta\big(d(x,y)\big),\ \eta\big(d(x,Ty)\big),\ \eta\big(d(Tx,y)\big),\ \eta\big(d(x,Tx)\big),\ \eta\big(d(y,Ty)\big)\big\} \tag{3.1}$$

for any $x, y \in X$. Assume also that there exists $u \in X$ satisfying (H3). *Then T has a unique fixed point z. Moreover $\{T^n u\}$ converges to z.*

Remark. See Example 4.1 below. In order to prove the existence of a fixed point, we have to assume (H3) or something.

Proof. Put

$$A(m,n) = \big\{T^j u :\ j \in \mathbb{N}\cup\{0\},\ m \le j \le n\big\}, \tag{3.2}$$

$$A(m,\infty) = \big\{T^j u :\ j \in \mathbb{N}\cup\{0\},\ m \le j\big\}, \tag{3.3}$$

$$D(m,n) = \sup\big\{\eta\big(d(x,y)\big) :\ x, y \in A(m,n)\big\} \tag{3.4}$$

and

$$D(m,\infty) = \sup\big\{\eta\big(d(x,y)\big) :\ x, y \in A(m,\infty)\big\} \tag{3.5}$$

for $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq n$, where T^0 is the identity mapping on X. By (3.1), we note

$$D(m,n) \leq r\, D(m-1,n) \tag{3.6}$$

for $m, n \in \mathbb{N}$ with $m \leq n$. We also note by (3.1)

$$\max\left\{\eta\big(d(u,T^j u)\big) :\ 1 \leq j \leq n\right\} = D(0,n) \tag{3.7}$$

for $n \in \mathbb{N}$. By (H3), we note $D(0,\infty) < \infty$. By (3.6), we have

$$D(m,\infty) \leq r\, D(m-1,\infty) \leq \cdots \leq r^m\, D(0,\infty)$$

for $m \in \mathbb{N}$. So, by Lemma 2.2, $\{T^n u\}$ is Cauchy. Since X is complete, $\{T^n u\}$ converges to some $z \in X$. Arguing by contradiction, we assume $Tz \neq z$. We consider the following two cases:

- $\#\left\{n \in \mathbb{N} : \eta\big(d(T^{n+1}u, Tz)\big) \leq r\,\eta\big(d(z,Tz)\big)\right\} = \infty$
- $\#\left\{n \in \mathbb{N} : \eta\big(d(T^{n+1}u, Tz)\big) \leq r\,\eta\big(d(z,Tz)\big)\right\} < \infty$

In the first case, there exists a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ satisfying

$$\eta\big(d(T^{f(n)+1}u, Tz)\big) \leq r\,\eta\big(d(z,Tz)\big)$$

for $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} d(T^{f(n)+1}u, Tz) = d(z,Tz) > 0,$$

we have by (H2:$1/r$)

$$\begin{aligned}
\limsup_{n\to\infty} \eta\big(d(T^{f(n)+1}u, Tz)\big) &\leq r\,\eta\big(d(z,Tz)\big) \\
&< r\,\frac{1}{r}\limsup_{n\to\infty} \eta\big(d(T^{f(n)+1}u, Tz)\big) \\
&= \limsup_{n\to\infty} \eta\big(d(T^{f(n)+1}u, Tz)\big),
\end{aligned}$$

which implies a contradiction. In the second case, since

$$\lim_{n\to\infty} \max\left\{\eta\big(d(T^n u, z)\big), \eta\big(d(T^{n+1}u, z)\big), \eta\big(d(T^n u, T^{n+1}u)\big)\right\} = 0,$$

we have

$$\max\left\{\eta\big(d(T^n u, z)\big), \eta\big(d(T^{n+1}u, z)\big), \eta\big(d(T^n u, T^{n+1}u)\big)\right\} < \eta\big(d(z,Tz)\big)$$

for sufficiently large $n \in \mathbb{N}$. Thus, the following hold:

$$\begin{aligned}
&\#\left\{n \in \mathbb{N} : \eta\big(d(T^{n+1}u, Tz)\big) \leq r\,\eta\big(d(T^n u, z)\big)\right\} < \infty, \\
&\#\left\{n \in \mathbb{N} : \eta\big(d(T^{n+1}u, Tz)\big) \leq r\,\eta\big(d(T^{n+1}u, z)\big)\right\} < \infty, \\
&\#\left\{n \in \mathbb{N} : \eta\big(d(T^{n+1}u, Tz)\big) \leq r\,\eta\big(d(T^n u, T^{n+1}u)\big)\right\} < \infty.
\end{aligned}$$

Since

$$\begin{aligned}
\eta\big(d(T^{n+1}u, Tz)\big) \leq r\,\max\big\{&\eta\big(d(T^n u, z)\big), \eta\big(d(T^n u, Tz)\big), \eta\big(d(T^{n+1}u, z)\big), \\
&\eta\big(d(T^n u, T^{n+1}u)\big), \eta\big(d(z,Tz)\big)\big\}
\end{aligned}$$

for $n \in \mathbb{N}$, there exists $\nu \in \mathbb{N}$ satisfying

$$\eta\big(d(T^{n+1}u, Tz)\big) \leq r\,\eta\big(d(T^n u, Tz)\big)$$

for $n \geq \nu$. Then we have $\lim_n \eta\big(d(T^n u, Tz)\big) = 0$, thus, $Tz = \lim_n T^n u = z$ holds. This is a contradiction. We have obtained a contradiction in all cases. Therefore we have shown $Tz = z$. Let us prove that the fixed point z is unique. Let $w \in X$ be a fixed point of T. Then we have

$$\begin{aligned}
\eta\big(d(z,w)\big) &= \eta\big(d(Tz,Tw)\big) \\
&\leq r \max\big\{\eta\big(d(z,w)\big), \eta\big(d(z,Tw)\big), \eta\big(d(Tz,w)\big), \\
&\qquad\qquad \eta\big(d(z,Tz)\big), \eta\big(d(w,Tw)\big)\big\} \\
&= r \max\big\{\eta\big(d(z,w)\big), \eta\big(d(z,w)\big), \eta\big(d(z,w)\big), \\
&\qquad\qquad \eta\big(d(z,z)\big), \eta\big(d(w,w)\big)\big\} \\
&= r\, \eta\big(d(z,w)\big)
\end{aligned}$$

and hence $\eta\big(d(z,w)\big) = 0$. Hence by (H1), we obtain $z = w$. We have shown that the fixed point is unique. □

If we replace the assumption (H3) by (H5), we can also prove a fixed point theorem as Theorem 3.1.

Theorem 3.2. *Let T be a mapping on a complete metric space (X,d). Assume that there exist $r \in (0,1)$ and $\eta \in H(1/r)$ satisfying* (3.1). *Assume also* (H5). *Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

Remark. We need (H1), (H2:$1/r$) and (H5) for Ćirić type.

Proof. Fix $u \in X$. Choose $\varepsilon > 0$ appearing in (H5). We can choose $\kappa \in \mathbb{N}$ satisfying

$$(1+\varepsilon)\, r^{\kappa} < 1. \tag{3.8}$$

Put $A(m,n)$, $A(m,\infty)$, $D(m,n)$ and $D(m,\infty)$ by (3.2)–(3.5), respectively. Arguing by contradiction, we assume that (H3) does not hold. Thus,

$$D(0,\infty) = \infty \tag{3.9}$$

holds. Then we note that $T^n u$ ($n \in \mathbb{N} \cup \{0\}$) are all different. Put $\beta = d(u, T^{\kappa} u)$. Then we can choose $M > 0$ appearing in (H5). By (3.7) and (3.9), we can choose $\ell \in \mathbb{N}$ satisfying $\ell > \kappa$,

$$\eta\big(d(u, T^{\ell} u)\big) = D(0,\ell)$$

and

$$\eta\big(d(u, T^{\ell} u)\big) > M.$$

Then since

$$|d(u, T^{\ell} u) - d(T^{\kappa} u, T^{\ell} u)| \leq d(u, T^{\kappa} u) = \beta,$$

we have by (H5), (3.6) and (3.8)

$$\begin{aligned}
D(0,\ell) &= \eta\big(d(u, T^{\ell} u)\big) \\
&< (1+\varepsilon)\, \eta\big(d(T^{\kappa} u, T^{\ell} u)\big) \\
&\leq (1+\varepsilon)\, D(\kappa,\ell) \\
&\leq (1+\varepsilon)\, r\, D(\kappa-1,\ell)
\end{aligned}$$

$$\begin{aligned}&\leq (1+\varepsilon)\, r^2\, D(\kappa-2,\ell)\\ &\leq \cdots \leq (1+\varepsilon)\, r^{\kappa}\, D(0,\ell)\\ &< D(0,\ell),\end{aligned}$$

which implies a contradiction. Thus, (H3) holds. So by Theorem 3.1, we obtain the desired result. □

Next we will prove a fixed point theorem for Kannan type contractions. See [9].

Theorem 3.3. *Let T be a mapping on a complete metric space (X,d). Assume that there exist $\alpha \in (0,1/2)$ and $\eta \in H(1/\alpha)$ satisfying*

$$\eta\big(d(Tx,Ty)\big) \leq \alpha\, \eta\big(d(x,Tx)\big) + \alpha\, \eta\big(d(y,Ty)\big) \tag{3.10}$$

for any $x, y \in X$. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.

Remark. We need (H1) and (H2:$1/\alpha$) for Kannan type. We need neither (H3) nor (H5).

Proof. Fix $u \in X$. Since

$$\eta\big(d(T^n u, T^{n+1}u)\big) \leq \alpha\, \eta\big(d(T^{n-1}u, T^n u)\big) + \alpha\, \eta\big(d(T^n u, T^{n+1}u)\big)$$

for $n \in \mathbb{N}$, we have

$$\begin{aligned}\eta\big(d(T^n u, T^{n+1}u)\big) &\leq \frac{\alpha}{1-\alpha}\, \eta\big(d(T^{n-1}u, T^n u)\big)\\ &\leq \cdots \leq \left(\frac{\alpha}{1-\alpha}\right)^n \eta\big(d(u,Tu)\big)\end{aligned}$$

and hence $\lim_n \eta\big(d(T^n u, T^{n+1}u)\big) = 0$ because $\alpha/(1-\alpha) \in (0,1)$ holds. Using this, we have

$$\begin{aligned}&\lim_{n\to\infty} \sup_{m>n} \eta\big(d(T^n u, T^m u)\big)\\ &\leq \lim_{n\to\infty} \sup_{m>n} \Big(\alpha\, \eta\big(d(T^{n-1}u, T^n u)\big) + \alpha\, \eta\big(d(T^{m-1}u, T^m u)\big)\Big)\\ &\leq \lim_{n\to\infty} 2\,\alpha\, \eta\big(d(T^{n-1}u, T^n u)\big) = 0.\end{aligned}$$

So by Lemma 2.2, $\{T^n u\}$ is Cauchy. Since X is complete, $\{T^n u\}$ converges to some $z \in X$. Arguing by contradiction, we assume $Tz \neq z$. By (H2:$1/\alpha$), we have

$$\begin{aligned}\limsup_{n\to\infty} \eta\big(d(T^{n+1}u, Tz)\big) &\leq \alpha \lim_{n\to\infty} \eta\big(d(T^n u, T^{n+1}u)\big) + \alpha\, \eta\big(d(z,Tz)\big)\\ &= \alpha\, \eta\big(d(z,Tz)\big)\\ &< \alpha\, \frac{1}{\alpha} \limsup_{n\to\infty} \eta\big(d(T^{n+1}u, Tz)\big)\\ &= \limsup_{n\to\infty} \eta\big(d(T^{n+1}u, Tz)\big).\end{aligned}$$

This is a contradiction. So, we have $z = Tz$. Noting that (3.10) is stronger than (3.1), we can prove the uniqueness of z as in the proof of Theorem 1.2. □

Next we will prove a fixed point theorem for Chatterjea type contractions. See [5].

Theorem 3.4. *Let T be a mapping on a complete metric space (X,d). Assume that there exist $\alpha \in [0,1/2)$ and a function η from $[0,\infty)$ into itself satisfying* (H1) *and*

$$\eta\big(d(Tx,Ty)\big) \le \alpha\,\eta\big(d(x,Ty)\big) + \alpha\,\eta\big(d(Tx,y)\big) \tag{3.11}$$

for any $x, y \in X$. Assume also that there exists $u \in X$ satisfying (H3)*. Then T has a unique fixed point z. Moreover $\{T^n u\}$ converges to z.*

Proof. Define a function f from $\mathbb{N}$ into $[0,\infty)$ by

$$f(n) = \sup_{m>n} \eta\big(d(T^n u, T^m u)\big).$$

Then by (H3), f is well defined. For $n \in \mathbb{N}$, we have

$$\begin{aligned} f(n+1) &= \sup_{m>n} \eta\big(d(T^{n+1}u, T^{m+1}u)\big) \\ &\le \sup_{m>n} \Big(\alpha\,\eta\big(d(T^n u, T^{m+1}u)\big) + \alpha\,\eta\big(d(T^{n+1}u, T^m u)\big)\Big) \\ &\le \alpha \sup_{m>n} \eta\big(d(T^n u, T^{m+1}u)\big) + \alpha \sup_{m>n} \eta\big(d(T^{n+1}u, T^m u)\big) \\ &= \alpha \sup_{m>n} \eta\big(d(T^n u, T^{m+1}u)\big) + \alpha \sup_{m>n+1} \eta\big(d(T^{n+1}u, T^m u)\big) \\ &\le \alpha\, f(n) + \alpha\, f(n+1) \end{aligned}$$

and hence

$$f(n+1) \le \frac{\alpha}{1-\alpha}\, f(n).$$

Since $\alpha/(1-\alpha) \in [0,1)$, we obtain $\lim_n f(n) = 0$. So by Lemma 2.2, $\{T^n u\}$ is Cauchy. Since X is complete, $\{T^n u\}$ converges to some $z \in X$. Since

$$\begin{aligned} \limsup_{n\to\infty} \eta\big(d(T^{n+1}u, Tz)\big) &\le \limsup_{n\to\infty} \Big(\alpha\,\eta\big(d(T^n u, Tz)\big) + \alpha\,\eta\big(d(T^{n+1}u, z)\big)\Big) \\ &= \limsup_{n\to\infty} \alpha\,\eta\big(d(T^n u, Tz)\big), \end{aligned}$$

we can prove that $\big\{\eta\big(d(T^n u, Tz)\big)\big\}$ is bounded. So, we have $\lim_n \eta\big(d(T^n u, Tz)\big) = 0$. Therefore we have $z = Tz$ by (H1). Noting that (3.11) is stronger than (3.1), we can prove the uniqueness of z as in the proof of Theorem 1.2. □

Since (3.11) is stronger than (3.1), we obtain the following by the proof of Theorem 3.2 and Theorem 3.4.

Corollary 3.5. *Let T be a mapping on a complete metric space (X,d). Assume that there exist $\alpha \in [0,1/2)$ and a function η from $[0,\infty)$ into itself satisfying* (H1), (H5) *and* (3.11)*. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

Remark. We need (H1) and (H5) for Chatterjea type.

4. Examples

Rhoades in [12] gave the following example.

Example 4.1 (Example 3 in [12])**.** Let $X = \mathbb{N}$ and let d be as usual. Define a mapping T on X by $Tx = x + 1$. Define a function η from $[0, \infty)$ into itself by

$$\eta(a) = \exp(a) - 1.$$

Then the following hold:

(i) (X, d) is a complete metric space.
(ii) (3.11) holds for any $x, y \in X$, where $\alpha := \exp(-1) = 0.37\cdots \in (0, 1/2)$.
(iii) (1.1) holds for any $x, y \in X$, where $t := \exp(-1) \in (0, 1/2)$, $s := 1/2 - t \in (0, 1/2)$ and $r = 0$.
(iv) (3.1) holds for any $x, y \in X$, where $r := \exp(-1) \in (0, 1)$.
(v) T does not have a fixed point.
(vi) $\eta \in H(\tau)$ holds for any $\tau \in (1, \infty)$.
(vii) η does not satisfy (H5).

Remark. Example 4.1 give a negative answer to the second problem raised in [17].

Proof. (i) is obvious. Fix $x, y \in X$ with $x < y$. Then we have

$$\begin{aligned} \eta\big(d(Tx, Ty)\big) &= \exp(y - x) - 1 \\ &< \exp(y - x) - \exp(-1) \\ &= \exp(-1)\,\big(\exp(y + 1 - x) - 1\big) \\ &= \exp(-1)\,\eta\big(d(x, Ty)\big). \end{aligned}$$

Using this, we can prove (ii). (iii) and (iv) follow from (ii). (v) is obvious. By Lemma 2.3, (vi) holds. By Theorem 3.2, we can prove (vii). □

We finally give an example of a function η, which satisfies (H5) and does not satisfy (H4).

Example 4.2. Fix $q \in (0, \infty)$ and define a continuous function f from $[0, \infty)$ into $[1, 2]$ by

$$f(a) = 2 - \min\big\{|a - 2\,n| : n \in \mathbb{N} \cup \{0\}\big\}.$$

Define a continuous function η from $[0, \infty)$ into itself by

$$\eta(a) = f(a)\,a^q.$$

Then the following hold:

(i) $\eta \in H(\tau)$ holds for any $\tau \in (1, \infty)$ and η satisfies (H5).
(ii) η does not satisfy (H4).

Proof. It is obvious that f and η are continuous. It is also obvious that

$$a^q \le \eta(a) \le 2\,a^q$$

holds for any $a \in [0, \infty)$. By Lemma 2.3, $\eta \in H(\tau)$ for any $\tau \in (1, \infty)$. By Example 2.5, η satisfies (H5). We have shown (i). Let us prove (ii). We have

$$\eta(2\,n) = 2\,(2\,n)^q > 2\,(2\,n - 1)^q = (1 + 1)\,\eta(2\,n - 1)$$

for $n \in \mathbb{N}$. We note

$$\sup\{\eta(2\,n) : n \in \mathbb{N}\} = \infty \quad \text{and} \quad (2\,n) - (2\,n-1) = 1.$$

So for $\beta := 1 > 0$ and $\varepsilon := 1 > 0$, we cannot choose $M > 0$ appearing (H4). $\square$

Competing Interests

The authors declare that they have no competing interests.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[2] J. Bogin, *A generalization of a fixed point theorem of Goebel, Kirk and Shimi*, Canad. Math. Bull. **19** (1976), 7–12.

[3] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **29** (2002), 531–536.

[4] R. Caccioppoli, *Un teorema generale sull'esistenza di elementi uniti in una transformazione funzionale*, Rend. Accad. Naz. Lincei **11** (1930), 794–799.

[5] S. K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.

[6] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[7] J. Jachymski, *Remarks on contractive conditions of integral type*, Nonlinear Anal. **71** (2009), 1073–1081.

[8] J. Jachymski, J. Matkowski and T. Świątkowski, *Nonlinear contractions on semimetric spaces*, J. Appl. Anal. **1** (1995), 125–134.

[9] R. Kannan, *Some results on fixed points – II*, Amer. Math. Monthly **76** (1969), 405–408.

[10] M. Kikkawa and T. Suzuki, *Fixed point theorems for new nonlinear mappings satisfying Condition* (CC), Linear Nonlinear Anal. **1** (2015), 37–52.

[11] M. Kikkawa and T. Suzuki, *A direct proof of some resent generalization of both Ćirić's and Bogin's fixed point theorems*, Bull. Kyushu Inst. Technol. **64** (2017), 13–19.

[12] B. E. Rhoades, *Two fixed-point theorems for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **2003** (2003), 4007–4013.

[13] T. Suzuki, *Meir-Keeler contractions of integral type are still Meir-Keeler contractions*, Int. J. Math. Math. Sci. **2007** (2007), Article ID 39281, 1–6.

[14] T. Suzuki, *Fixed point theorems for a new nonlinear mapping similar to a nonspreading mapping*, Fixed Point Theory Appl. **2014**, 2014:47.

[15] T. Suzuki, *Comments on some recent generalization of the Banach contraction principle*, J. Inequal. Appl. **2016**, 2016:111.

[16] T. Suzuki, *Discussion of several contractions by Jachymski's approach*, Fixed Point Theory Appl. **2016**, 2016:91.

[17] T. Suzuki, *Some notes on generalizations of both Ćirić's and Bogin's fixed point theorems*, to appear in Pure Appl. Funct. Anal.

[18] T. Suzuki and M. Kikkawa, *Generalizations of both Ćirić's and Bogin's fixed point theorems*, J. Nonlinear Convex Anal. **17** (2016), 2183–2196.

[19] T. Suzuki and C. Vetro, *Three existence theorems for weak contractions of Matkowski type*, Int. J. Math. Stat. **6** (2010), 110–120.

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