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SUBLINEAR SCALARIZATION METHODS FOR SETS WITH RESPECT TO SET-RELATIONS

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Abstract. This paper is concerned with a certain historical background on set-relations and scalarization methods for sets. We introduce a basic idea of sublinear scalarization and its generalization as unifications of several nonlinear scalarizations for sets in a real vector space. Moreover, we show a certain possibility to use this idea to establish some kinds of set-valued inequalities. For example, we show generalized results on Fan-Takahashi minimax inequality and Gordan's alternative theorem for set-valued maps.

1. Introduction

Real numbers can be compared with each other and the relative merits of vectors in a vector space may be discussed in a similar way by using a partial ordering, but how about comparison of sets? To cite a case, when we consider evaluations on power-indices of sport teams or activity-indices of classrooms at high schools, we should deal with families whose elements are some kinds of vectors, that is, we might encounter some difficult situations on the decision-making by a certain methodology for comparisons between sets.

Besides, every real data has a margin of error and it is fuzzy; hence realized gains and losses are probabilistic but we could not obtain each correct probability in advance. Non-numerical evaluations like customer evaluation (satisfaction level) and aesthetic evaluation (that is, sense of beauty) depend upon each individual preference but we could not easily know the structure of each preference order. Of course, there are several paradigms like probability, fuzzy, game, neural network, analytical hierarchy process, analytic network process as mathematical tools to solve the problem. However, a simple mathematical approach is "set-relation," which is a methodology introduced by Kuroiwa ([9]). Based on this idea, there are several researches on set-relations and their application in the area of "set-valued optimization" and "set optimization," e.g., [7, 10, 13, 22, 21] and references cited therein.

Generally speaking, a total ordering space like the real field $\mathbb{R}$ has useful structures for preferece, evaluation, computation, or easy comparison on the values of real-valued functions. On the other hand, multiobjective programming and vector optimization are studied based on "multicriteria" evaluation like some partial orderings. For typical solution concepts, minimal and maximal notions like Pareto optimal solution or efficient solution for given outcome sets are defined with respect optimization are studied based on "multicriteria" evaluation like some partial orderings. For typical solution concepts, minimal and maximal notions like Pareto optimal solution or efficient solution for given outcome sets are defined with respect to reasonable ordering convex cones (that is, dominance cones) when the outcome space is a real vector space; see [24] and [15].

The most practical method to approach this kind of problems is "scalarization," which is used in converting a given vector optimization problem to a scalar optimization problem (or a family of such problems) such that the solutions of the latter problem are also solutions of the former problem. The notion of "weighted sum" of elements is a typical tool for the scalarization of vectors in multicriteria problems, and it is regarded as a linear functional with the weight vector. A linear functional on a vector space is a bilinear form as a function of two variables of the original vector space X and its topological dual space X^*, which is the set of all bounded linear functionals on X. Also it becomes the inner product of a given vector $\boldsymbol{x}$ and the normal vector $\boldsymbol{w}$ in case of Euclidean spaces:

$$\sum_k w_k x_k = \langle \boldsymbol{w}, \boldsymbol{x} \rangle \quad \text{for} \quad \boldsymbol{w} = (w_k) \quad \text{and} \quad \boldsymbol{x} = (x_k).$$

It is one of the most useful tools for evaluation with respect to some index of the adequacy of efficiency in multiobjective programming and vector optimization. Moreover, the notion of average of n elements is regarded as a special case of the weighted sum of a given vector in $\mathbb{R}^n$ with the weight vector $\boldsymbol{w} = (1/n, \dots, 1/n)$. This scalarization technique is a method based on the linear operation, and then it is called "linear scalarization," and the following "order-monotone" (that is, order preserving) property holds:

$$\boldsymbol{x_1} \leq_C \boldsymbol{x_2} \quad \text{implies} \quad \langle \boldsymbol{w}, \boldsymbol{x_1} \rangle \leq \langle \boldsymbol{w}, \boldsymbol{x_2} \rangle$$

if the weight vector $\boldsymbol{w}$ is chosen as an element of the dual cone C^* in X^* of the ordering convex cone C in a vector space X, where $C^* = \{\boldsymbol{y} \in X^* : \langle \boldsymbol{y}, \boldsymbol{x} \rangle \geq 0 \quad \forall \boldsymbol{x} \in C\}$, and if we write $\boldsymbol{x_1} \leq_C \boldsymbol{x_2}$ when $\boldsymbol{x_2} - \boldsymbol{x_1} \in C$. Owing to this property, any minimal or maximal element of a convex set in a vector optimization problem is characterized by optimal solutions of its scalarized problem with a certain nonzero weight vector in C^*, which is guaranteed by separation theorems for two convex sets.

From the view-point of the advanced area of vector optimization and set optimization, the linear scalarization can be regarded as a special case of the following sublinear scalarization

$$h_C(\boldsymbol{x}; \boldsymbol{k}) = \inf \{ t \in \mathbb{R} \mid \boldsymbol{x} \in t\boldsymbol{k} - C \} \tag{1.1}$$

when the ordering cone C is a half space in $\mathbb{R}^n$, that is, $C = \{\boldsymbol{x} \in X : \langle \boldsymbol{k}, \boldsymbol{x} \rangle \geq 0\}$ with normal vector $\boldsymbol{k} \in \mathbb{R}^n$ satisfying $\langle \boldsymbol{k}, \boldsymbol{k} \rangle = 1$. Indeed, we can verify that $h_C(\boldsymbol{x}; \boldsymbol{k}) = \langle \boldsymbol{k}, \boldsymbol{x} \rangle$ easily under the condition above. This scalarization method is based on the sublinearity of $h_C(\cdot; \boldsymbol{k})$ and hence it is called "sublinear scalarization." This approach is found in [2, 19, 20], and it was developed by Tammer ([3, 4]) and Luc ([14]). It is similar to the idea of Minkowski functional, which plays an important key role on seminorm connected with a topology of a locally convex topological vector space.

On the other hand, if $\boldsymbol{k} \in C$, the lower level set of $h_C(\cdot;\boldsymbol{k})$ at each height t coincides with a parallel translation of $-C$ at offset $t\boldsymbol{k}$, that is,

$$\{\boldsymbol{x} \mid h_C(\boldsymbol{x};\boldsymbol{k}) \leq t\} = t\boldsymbol{k} - C,$$

and hence $h_C(\cdot;\boldsymbol{k})$ is the smallest strictly monotonic function with respect to the ordering cone C in case that $\boldsymbol{k} \in \operatorname{int} C$, which is the topological interior of C; see page 21 in [14]. Hence, $h_C(\cdot;\boldsymbol{k})$ has the order-monotone property. Also this scalarization has a dual form as follows:

$$-h_C(-\boldsymbol{x};\boldsymbol{k}) = \sup\{t \in \mathbb{R} \mid \boldsymbol{x} \in t\boldsymbol{k} + C\}. \tag{1.2}$$

When the ordering cone C is a half space as discussed above, both sublinear functionals $h_C(\boldsymbol{x};\boldsymbol{k})$ and $-h_C(-\boldsymbol{x};\boldsymbol{k})$ of $\boldsymbol{x}$ are coincident with the value of $\langle \boldsymbol{k}, \boldsymbol{x}\rangle$. Nishizawa and Tanaka ([17]) study certain characterizations of set-valued mappings by using the inherited properties of $h_C(\cdot;\boldsymbol{k})$ and $-h_C(-\cdot;\boldsymbol{k})$ on cone-convexity and cone-semicontinuity. These observations lead us into the work to study sublinear scalarization methods for sets with respect to set-relations [9, 10] by using the ideas on the sublinear scalarization [2] and some scalarizing functions in [7]. The aim of this paper is to show a certain possibility to use sublinear scalarization methods for sets in a real vector space proposed in [11, 12] in order to establish some kinds of set-valued inequalities.

The organization of the paper is as follows. In Section 2, we introduce mathematical methodology on comparison between two sets in an ordered vector space proposed in [10] and unified nonlinear scalarizing functions for sets proposed in [11]. In Section 3, we investigate generalized results on Fan-Takahashi minimax inequality ([13, 21, 22]) and Gordan's alternative theorem for set-valued maps ([18]) as applications of sublinear scalarization methods for sets.

2. Scalarization methods for sets with respect to set-relations

Unless otherwise specified, we let X be a nonempty set, Y a real ordered topological vector space with partial order $\leq_C$ induced by a convex solid (that is, there exists nonempty interior) pointed ($C \cap (-C) = \{\theta_Y\}$) cone C, where θ_Y is the zero vector of Y, as follows: $x \leq_C y$ if $y - x \in C$ for $x, y \in Y$. In case of lack of pointedness, the binary relation induced by the cone is preorder (transitive and reflexive). We denote the algebraic sum and difference of any subsets A and B in Y by $A+B := \{a+b \mid a \in A, b \in B\}$ and $A-B := \{a-b \mid a \in A, b \in B\}$, respectively. Given $A \subset Y$, we write $tA := \{ta \mid a \in A\}$ for $t \in \mathbb{R}$ and $A+x := A+\{x\}$ for $x \in Y$. Besides, for any $A \subset Y$ we denote the topological interior, topological closure, complement of A by $\operatorname{int} A$, $\operatorname{cl} A$, A^{c}, respectively. Also, we denote the composition of two functions f and g by $g \circ f$.

Moreover, we recall some definitions of C–notions which are referred in [14]. A subset A in Y is said to be C–convex (resp., C–closed) if $A + C$ is convex (resp., closed); C–proper if $A + C \neq Y$. Moreover, A is said to be C–bounded if for each open neighborhood U of θ_Y there exists $t \geq 0$ such that $A \subset tU + C$. Furthermore, we say that F is each C–notion mentioned above if the set $F(x)$ for each $x \in E$ has the property of the corresponding C–notion.

We review the basic concepts of set-relation and nonlinear scalarization method based on sublinear-like functions with respect to each set-relation.

Definition 2.1 (Kuroiwa, Tanaka, Ha (1997), [10])**.** Let $A, B \in 2^Y \setminus \{\emptyset\}$.

(i) $A \leq_C^{(1)} B \overset{\text{def}}{\Longleftrightarrow} A \subset \bigcap_{b \in B}(b - C) \Leftrightarrow B \subset \bigcap_{a \in A}(a + C)$;
(ii) $A \leq_C^{(2)} B \overset{\text{def}}{\Longleftrightarrow} A \cap \bigcap_{b \in B}(b - C) \neq \emptyset$;
(iii) $A \leq_C^{(3)} B \overset{\text{def}}{\Longleftrightarrow} B \subset (A + C)$;
(iv) $A \leq_C^{(4)} B \overset{\text{def}}{\Longleftrightarrow} \bigcap_{a \in A}(a + C) \cap B \neq \emptyset$;
(v) $A \leq_C^{(5)} B \overset{\text{def}}{\Longleftrightarrow} A \subset (B - C)$;
(vi) $A \leq_C^{(6)} B \overset{\text{def}}{\Longleftrightarrow} A \cap (B - C) \neq \emptyset \Leftrightarrow (A + C) \cap B \neq \emptyset$.

Proposition 2.2 (Proposition 2.1 in [10])**.** *For nonempty sets $A, B \subset Y$, the following statements hold.*

$$\begin{array}{ll} A \leq_C^{(1)} B \text{ implies } A \leq_C^{(2)} B; & A \leq_C^{(1)} B \text{ implies } A \leq_C^{(4)} B; \\ A \leq_C^{(2)} B \text{ implies } A \leq_C^{(3)} B; & A \leq_C^{(4)} B \text{ implies } A \leq_C^{(5)} B; \\ A \leq_C^{(3)} B \text{ implies } A \leq_C^{(6)} B; & A \leq_C^{(5)} B \text{ implies } A \leq_C^{(6)} B. \end{array}$$

Proposition 2.3 (Proposition 2.3 in [11])**.** *For $A, B, D \in 2^Y \setminus \{\emptyset\}$, the following statements hold.*

(i) *For each $j = 1, \ldots, 6$,*
   $A \leq_C^{(j)} B$ *implies* $(A + y) \leq_C^{(j)} (B + y)$ *for* $y \in Y$, *and*
   $A \leq_C^{(j)} B$ *implies* $\alpha A \leq_C^{(j)} \alpha B$ *for* $\alpha > 0$;
(ii) *For each $j = 1, \ldots, 5$, $\leq_C^{(j)}$ is transitive, that is,*
   $A \leq_C^{(j)} B$ *and* $B \leq_C^{(j)} D$ *implies* $A \leq_C^{(j)} D$;
(iii) *For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive, that is,*
   $A \leq_C^{(j)} A$;
(iv) *For each $j = 1, \ldots, 6$,*
   $A \leq_C^{(j)} B$ *and* $y_1 \leq_C y_2$ *for* $y_1, y_2 \in Y$ *imply* $A + y_1 \leq_C^{(j)} B + y_2$;
(v) *For each $j = 1, \ldots, 6$,*
   $A \leq_C^{(j)} (te + B)$ *implies* $A \leq_C^{(j)} (se + B)$ *for any* $e \in \operatorname{int} C$ *and* $s \geq t$, *and*
   $(te + B) \leq_C^{(j)} A$ *implies* $(se + B) \leq_A^{(j)}$ *for any* $e \in \operatorname{int} C$ *and* $s \leq t$.

The six binary relations $\leq_C^{(1)}, \ldots, \leq_C^{(6)}$ are referred to as "set-relations" and they are certain generalizations of a partial ordering for vectors induced by a convex cone in a vector space. Especially, $\leq_C^{(3)}$ and $\leq_C^{(5)}$ are preorders for sets. If B is a singleton set in Definition 2.1, set-relations $\leq_C^{(2)}, \leq_C^{(3)}, \leq_C^{(6)}$ are coincident with each other, and the others $\leq_C^{(1)}, \leq_C^{(4)}, \leq_C^{(5)}$ coincide. Based on these binary relations, we introduce the following scalarizing functions for sets in a vector space, which are certain generalizations as unifications of several nonlinear scalarizations proposed in [7, 25].

Definition 2.4 (Kuwano, Tanaka, Yamada (2009), [11])**.** Let $A, B \in 2^Y \backslash \{\emptyset\}$ and $k \in \operatorname{int} C$. For each $j = 1, ..., 6$, scalarizing functions $I^{(j)}_{k,B}$ and $S^{(j)}_{k,B}$ from $2^Y \backslash \{\emptyset\}$ to $\mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I^{(j)}_{k,B}(A) := \inf \left\{ t \in \mathbb{R} \,\middle|\, A \leq^{(j)}_C (tk + B) \right\}, \tag{2.1}$$

$$S^{(j)}_{k,B}(A) := \sup \left\{ t \in \mathbb{R} \,\middle|\, (tk + B) \leq^{(j)}_C A \right\}. \tag{2.2}$$

In the above definition, k and B play key roles as a "direction" and a "reference set" as one kind of sublinear scalarization for a given set A. If B is a singleton set, $I^{(2)}_{k,B}(A) = I^{(3)}_{k,B}(A) = I^{(6)}_{k,B}(A)$, $I^{(1)}_{k,B}(A) = I^{(4)}_{k,B}(A) = I^{(5)}_{k,B}(A)$, $S^{(1)}_{k,B}(A) = S^{(2)}_{k,B}(A) = S^{(3)}_{k,B}(A)$, and $S^{(4)}_{k,B}(A) = S^{(5)}_{k,B}(A) = S^{(6)}_{k,B}(A)$. Especially, if $B = \{\theta_Y\}$ in formulas (2.1) and (2.2), we get

$$I^{(j)}_{k,\{\theta_Y\}}(A) = \begin{cases} \inf\limits_{y \in A} h_C(y; k) & \text{for each } j = 2, 3, 6; \\ \sup\limits_{y \in A} h_C(y; k) & \text{for each } j = 1, 4, 5, \end{cases} \tag{2.3}$$

and

$$S^{(j)}_{k,\{\theta_Y\}}(A) = \begin{cases} \inf\limits_{y \in A} (-h_C(-y; k)) & \text{for each } j = 1, 2, 3; \\ \sup\limits_{y \in A} (-h_C(-y; k)) & \text{for each } j = 4, 5, 6. \end{cases} \tag{2.4}$$

They are certain generalizations of four types of scalarization for sets proposed in [16]. These facts suggest a similar approach to characterize set-valued mappings in the same way used in [17] as mentioned in Section 1 and also to apply it to establish alternative theorems for set-valued maps without convexity assumptions.

For $j = 1, \ldots, 5$, scalarizing functions $I^{(j)}_{k,V}(\cdot)$ and $S^{(j)}_{k,V}(\cdot)$ with direction k and nonempty reference set V have the following monotonicity, which is referred to as "j-monotone with respect to $\leq^{(j)}_C$" in [8]:

$$A \leq^{(j)}_C B \quad \text{implies} \quad I^{(j)}_{k,V}(A) \leq I^{(j)}_{k,V}(B) \quad \text{and} \quad S^{(j)}_{k,V}(A) \leq S^{(j)}_{k,V}(B). \tag{2.5}$$

Also, if $k \in \operatorname{int} C$, the following certain properties hold, which are similar to convexity and concavity.

Proposition 2.5 (Propositions 2.14 and 2.15 in [8])**.** *For nonempty subsets A, B in Y, $\lambda \in (0, 1)$ and $k \in \operatorname{int} C$, the following statements hold:*

(i) *For each $j = 1, 2, 3$, $I^{(j)}_{k,V}(\lambda A + (1-\lambda)B) \leq \lambda I^{(j)}_{k,V}(A) + (1-\lambda) I^{(j)}_{k,V}(B)$;*

(ii) *For each $j = 4, 5, 6$, $I^{(j)}_{k,V}(\lambda A + (1-\lambda)B) \leq \lambda I^{(j)}_{k,V}(A) + (1-\lambda) I^{(j)}_{k,V}(B)$ if V is $(-C)$–convex;*

(iii) *For each $j = 1, 4, 5$, $\lambda S^{(j)}_{k,V}(A) + (1-\lambda) S^{(j)}_{k,V}(B) \leq S^{(j)}_{k,V}(\lambda A + (1-\lambda)B)$;*

(iv) *For each $j = 2, 3, 6$, $\lambda S^{(j)}_{k,V}(A) + (1-\lambda) S^{(j)}_{k,V}(B) \leq S^{(j)}_{k,V}(\lambda A + (1-\lambda)B)$ if V is C–convex.*

with the agreement that $+\infty - \infty = +\infty$ and $\alpha(+\infty) = +\infty$, $\alpha(-\infty) = -\infty$ for $\alpha > 0$.

Besides, it is easily seen that

$$I^{(j)}_{k,V}(\alpha A) = \alpha I^{(j)}_{k,\left(\frac{1}{\alpha}V\right)}(A), \quad S^{(j)}_{k,V}(\alpha A) = \alpha S^{(j)}_{k,\left(\frac{1}{\alpha}V\right)}(A) \quad \text{for all} \quad \alpha > 0 \tag{2.6}$$

and then

$$I^{(j)}_{k,V}(A+B) \;\leq\; I^{(j)}_{k,\left(\frac{1}{2}V\right)}(A) + I^{(j)}_{k,\left(\frac{1}{2}V\right)}(B), \tag{2.7}$$

$$S^{(j)}_{k,V}(A+B) \;\geq\; S^{(j)}_{k,\left(\frac{1}{2}V\right)}(A) + S^{(j)}_{k,\left(\frac{1}{2}V\right)}(B). \tag{2.8}$$

If V is a cone, then these properties (2.6)–(2.8) can be regarded as "positively homogeneous," "subadditive" and "superadditive," and hence they suggest that $I^{(j)}_{k,V}(\cdot)$ and $S^{(j)}_{k,V}(\cdot)$ have sublinear-like and superlinear-like properties, respectively.

Moreover, we would remark that these scalarizing functions have an important merit on the inherited properties on several types of cone-convexity (cone-concavity) / cone-continuity for parent set-valued map $F : X \to 2^Y$ to convexity or quasiconvexity (concavity or quasiconcavity) / semicontinuity of the composite functions

$$(I^{(j)}_{k,V} \circ F)(x) := I^{(j)}_{k,V}(F(x)) \quad \text{and} \quad (S^{(j)}_{k,V} \circ F)(x) := S^{(j)}_{k,V}(F(x))$$

for each $j = 1, \ldots, 6$ in an analogous fashion to linear scalarizing function like inner product; general results for several types of convexity and quasiconvexity are summarized in [8] and useful results and examples for cone-continuity are in [23].

Definition 2.6 (cone-convexity, [10])**.** For each $j = 1, \ldots, 6$,

(i) A map F is said to be type (j) C-convex if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq^{(j)}_C \lambda F(x_1) + (1-\lambda)F(x_2);$$

(ii) A map F is said to be type (j) properly quasi C-convex if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq^{(j)}_C F(x_1) \quad \text{or} \quad F(\lambda x_1 + (1-\lambda)x_2) \leq^{(j)}_C F(x_2);$$

(iii) A map F is said to be type (j) naturally quasi C-convex if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \leq^{(j)}_C \mu F(x_1) + (1-\mu)F(x_2).$$

Definition 2.7 (cone-concavity, [13])**.** For each $j = 1, \ldots, 6$,

(i) A map F is said to be type (j) C-concave if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \leq^{(j)}_C F(\lambda x_1 + (1-\lambda)x_2);$$

(ii) A map F is said to be type (j) properly quasi C-concave if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$F(x_1) \leq^{(j)}_C F(\lambda x_1 + (1-\lambda)x_2) \quad \text{or} \quad F(x_2) \leq^{(j)}_C F(\lambda x_1 + (1-\lambda)x_2);$$

(iii) A map F is said to be type (j) naturally quasi C-concave if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x_1) + (1-\mu)F(x_2) \leq^{(j)}_C F(\lambda x_1 + (1-\lambda)x_2).$$

TABLE 1. Inherited properties on convexity and concavity.

Assumptions ($j = 1, \ldots, 6$) on			Conclusions
ψ		F	$\psi \circ F$
j-monotone with respect to $\leq_C^{(j)}$	cv	type (j) C-convex	cv
	cv	type (j) naturally quasi C-convex	qcv
	qcv	type (j) C-convex	
	qcv	type (j) naturally quasi C-convex	
		type (j) properly quasi C-convex	
	cc	type (j) C-concave	cc
	cc	type (j) naturally quasi C-concave	qcc
	qcc	type (j) C-concave	
	qcc	type (j) naturally quasi C-concave	
		type (j) properly quasi C-concave	

Theorem 2.8 ([8]). *Let $\psi : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ and $F : X \to 2^Y$. If ψ and F satisfy each assumption in Table 1, then its correspondent conclusion holds.*

In Table 1, we denote "convex", "quasiconvex", "concave", and "quasiconcave" by "cv", "qcv", "cc", and "qcc" for short in Table 1, respectively. From property (2.5) and Proposition 2.5, Theorem 2.8 guarantees that $I_{k,V}^{(j)} \circ F$ and $S_{k,V}^{(j)} \circ F$ ($j = 1, \ldots, 5$) possibly become to be cv or qcv / cc or qcc under some suitable cone-convexity / cone-concavity assumptions on F, respectively.

On the other hand, we find useful inherited results in [23] on cone-continuity of parent set-valued map which are summarized in Table 2; each cell with symbol $(*)$ has counter examples and suitable sufficient conditions are open questions. Also cones C and $(-C)$ for cone-continuity of the set-valued map play certain roles for "l.s.c." and "u.s.c." of each composite function, respectively.

Definition 2.9 (lower continuity and upper continuity, [4]).

(i) A map F is said to be lower continuous (l.c., for short) at $\bar{x}$ if for every open set $W \subset Y$ with $F(\bar{x}) \cap W \neq \emptyset$, there exists an open neighborhood U of $\bar{x}$ such that $F(x) \cap W \neq \emptyset$ for all $x \in U$. We say that F is lower continuous on X if F is l.c. at every point $x \in X$.

(ii) A map F is said to be upper continuous (u.c., for short) at $\bar{x}$ if for every open set $W \subset Y$ with $F(\bar{x}) \subset W$, there exists an open neighborhood U of $\bar{x}$ such that $F(x) \subset W$ for all $x \in U$. We say that F is upper continuous on X if F is u.c. at every point $x \in X$.

Definition 2.10 (cone-lower continuity and cone-upper continuity, [4]).

(i) A map F is said to be C–lower continuous (C–l.c., for short) at $\bar{x}$ if for every open set $W \subset Y$ with $F(\bar{x}) \cap W \neq \emptyset$, there exists an open neighborhood U of $\bar{x}$ such that $F(x) \cap (W + C) \neq \emptyset$ for all $x \in U$. We say that F is C–lower continuous on X if F is C–l.c. at every point $x \in X$.

(ii) A map F is said to be C–upper continuous (C–u.c., for short) at $\bar{x}$ if for every open set $W \subset Y$ with $F(\bar{x}) \subset W$, there exists an open neighborhood

U of $\bar{x}$ such that $F(x) \subset (W + C)$ for all $x \in U$. We say that F is C–upper continuous on X if F is C–u.c. at every point $x \in X$.

Theorem 2.11 ([23]). *If $F : X \to 2^Y$ satisfies each assumption in Table 2, then the composite functions $I_{k,V}^{(j)} \circ F$ and $S_{k,V}^{(j)} \circ F$ $(j = 1, \ldots, 6)$ have correspondent properties.*

Table 2. Inherited properties on semicontinuity.

Assumptions on F	Conclusions on $I_{k,V}^{(j)} \circ F$		Conclusions on $S_{k,V}^{(j)} \circ F$	
	$j = 1, 4, 5$	$j = 2, 3, 6$	$j = 4, 5, 6$	$j = 1, 2, 3$
l.c.	l.s.c.	u.s.c.	l.s.c.	u.s.c.
u.c.	u.s.c.	l.s.c.	u.s.c.	l.s.c.
C-l.c.	l.s.c.	$(*)$	l.s.c.	$(*)$
C-u.c.	$(*)$	l.s.c.	$(*)$	l.s.c.
$(-C)$-l.c.	$(*)$	u.s.c.	$(*)$	u.s.c.
$(-C)$-u.c.	u.s.c.	$(*)$	u.s.c.	$(*)$

3. Applications

3.1. **Set-valued Fan-Takahashi minimax inequality.** The following theorem is equivalent to Theorem 1 in [1] of Fan-Takahashi minimax inequality; this equivalence was proved by Takahashi firstly in 1976.

Theorem 3.1 ([26]). *Let X be a nonempty compact convex subset of a topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:*

(i) *for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,*
(ii) *for each fixed $x \in X$, $f(x, \cdot)$ is quasi concave,*
(iii) *for all $x \in X$, $f(x, x) \leq 0$,*

then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for all $y \in Y$.

Based on the above theorem, we shall show four kinds of Fan-Takahashi minimax inequality for set-valued maps as applications of sublinear scalarization methods for sets by using several results in Section 2. Each proof can be referred to [13].

Theorem 3.2 ([13]). *Let X be a nonempty compact convex subset of a topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(i) *F is $(-C)$-bounded on $X \times X$,*
(ii) *for each fixed $y \in X$, $F(\cdot, y)$ is C-lower continuous,*
(iii) *for each fixed $x \in X$, $F(x, \cdot)$ is type (5) properly quasi C-concave,*
(iv) *for all $x \in X$, $F(x, x) \subset -C$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \subset -C$ for all $y \in Y$.

Theorem 3.3 ([13]). *Let X be a nonempty compact convex subset of a topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(i) *F is C-proper and C-closed on $X \times X$,*
(ii) *for each fixed $y \in X$, $F(\cdot, y)$ is C-upper continuous,*
(iii) *for each fixed $x \in X$, $F(x, \cdot)$ is type (3) properly quasi C-concave,*
(iv) *for all $x \in X$, $F(x, x) \cap (-C) \neq \emptyset$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \cap (-C) \neq \emptyset$ for all $y \in Y$.

Theorem 3.4 ([13]). *Let X be a nonempty compact convex subset of a topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(i) *F is $(-C)$-proper on $X \times X$,*
(ii) *for each fixed $y \in X$, $F(\cdot, y)$ is C-lower continuous,*
(iii) *for each fixed $x \in X$, $F(x, \cdot)$ is type (5) naturally quasi C-concave,*
(iv) *for all $x \in X$, $F(x, x) \cap \operatorname{int} C = \emptyset$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \cap \operatorname{int} C = \emptyset$ for all $y \in Y$.

Theorem 3.5 ([13]). *Let X be a nonempty compact convex subset of a topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(i) *F is compact-valued on $X \times X$,*
(ii) *for each fixed $y \in X$, $F(\cdot, y)$ is C-upper continuous,*
(iii) *for each fixed $x \in X$, $F(x, \cdot)$ is type (3) naturally quasi C-concave,*
(iv) *for all $x \in X$, $F(x, x) \not\subset \operatorname{int} C$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \not\subset \operatorname{int} C$ for all $y \in Y$.

Remark 3.6. It is easy to check that if F is a single-valued function into the real numbers then Theorems 3.2–3.5 are reduced to Theorem 3.1.

3.2. **Set-valued alternative theorems.** As another application of sublinear scalarization methods for sets, we provide 12 kinds of Gordan-type alternative theorems. If the reference set V consists of the origin, they are reduced to those in [16], which are generalizations of the following Gordan-type alternative theorem with non-negative orthant $\mathbb{R}^m_+$ as the ordering cone.

Theorem 3.7 ([5]). *Let A be an $m \times n$ matrix and then exactly one of the following systems has a solution:*

(i) *there exists $x \in \mathbb{R}^n$ such that $Ax > \theta_{\mathbb{R}^m}$,*
(ii) *there exists $y \in \mathbb{R}^m$ such that $A^{\mathrm{T}} y = \theta_{\mathbb{R}^n}$ and $y \geq \theta_{\mathbb{R}^m}, y \neq \theta_{\mathbb{R}^m}$,*

where $z_2 \geq z_1$ and $z_2 > z_1$ in $\mathbb{R}^i$ when $z_1 \leq_{\mathbb{R}^i_+} z_2$ and $z_1 \leq_{\operatorname{int} \mathbb{R}^i_+} z_2$, that is, $z_2 - z_1 \in \mathbb{R}^i_+$ and $z_2 - z_1 \in \operatorname{int} \mathbb{R}^i_+$, respectively for $i = m, n$.

This theorem focuses on geometry of finitely many vectors and the origin. In [16], Nishizawa, Onodsuka, and Tanaka gave generalized forms by using sublinear scalarizaions in formulas (2.3) and (2.4) without any convexity assumption.

Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset of Y. By using several results in Section 2, we show the following generalizations of the alternative theorems in [16]; each proof can be referred to [18].

Theorem 3.8 ([18]). *Assume that*

- *F is compact-valued on X and V is compact in case of $j = 1$;*
- *V is compact in case of $j = 2, 3$;*
- *F is compact-valued on X in case of $j = 4, 5$;*
- *no compactness assumption on F nor V in case of $j = 6$,*

then exactly one of the following two systems is consistent for $j = 1, \dots, 6$:

(i) *there exists $x \in X$ such that $F(x) \leq^{(j)}_{\operatorname{int} C} V$,*

(ii) *there exists $k \in \operatorname{int} C$ such that $(I^{(j)}_{k,V} \circ F)(x) \geq 0$ for all $x \in X$.*

Theorem 3.9 ([18]). *Assume that*

- *F is compact-valued on X and V is compact in case of $j = 1$;*
- *V is compact in case of $j = 2, 3$;*
- *F is compact-valued on X in case of $j = 4, 5$;*
- *no compactness assumption on F nor V in case of $j = 6$,*

then exactly one of the following two systems is consistent for $j = 1, \dots, 6$:

(i) *there exists $x \in X$ such that $V \leq^{(j)}_{\operatorname{int} C} F(x)$,*

(ii) *there exists $k \in \operatorname{int} C$ such that $(S^{(j)}_{k,V} \circ F)(x) \leq 0$ for all $x \in X$.*

Remark 3.10. For $j = 1$, we let $S := \bigcap_{v \in V}(v - C)$. Then $S + (-C) \subset S$ so that for all $x \in X$, $y \in F(x)$, and $k \in \operatorname{int} C$, there exists $t_y > 0$ such that $y \in \bigcap_{v \in t_y k + V}(v - C)$. Thus, there exists $\bar{t} \geq 0$ such that $(I^{(1)}_{k,V} \circ F)(x) = \inf\{t \in \mathbb{R} \mid F(x) \subset \bigcap_{v \in tk+V}(v - C)\} \leq \bar{t} < +\infty$. In a similar way, we can prove the finiteness of each composite function in the second system.

Remark 3.11. In Theorem 3.9, we set $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $F(x) = \{Ax\}$ (A is an $m \times n$ matrix), $V = \{\theta_{\mathbb{R}^m}\}$ and hence F is compact-valued on X and V is a compact set. Choose a direction vector $k \in \operatorname{int} \mathbb{R}^m_+$ and let $C = \{y \in \mathbb{R}^m \mid \langle k, y\rangle \geq 0\}$ ($\mathbb{R}^m_+ \subset C$). If we consider $\{Ax\}$ instead of A in formula (2.4), all cases in the assumption of Theorem 3.9 hold automatically and then they are reduced to Theorem 3.7. Similarly, we can check that Theorem 3.8 is another generalization of the classical one.

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