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THE PROXIMAL POINT ALGORITHM IN GEODESIC SPACES WITH CURVATURE BOUNDED ABOVE

YASUNORI KIMURA AND FUMIAKI KOHSAKA

Abstract. We investigate the asymptotic behavior of sequences generated by the proximal point algorithm for convex functions in complete geodesic spaces with curvature bounded above. Using the notion of resolvents of such functions, which was recently introduced by the authors, we show the existence of minimizers of convex functions under the boundedness assumptions on such sequences as well as the convergence of such sequences to minimizers of given functions.

1. Introduction

The aim of this paper is to obtain the following two results on the asymptotic behavior of sequences generated by the proximal point algorithm for convex functions in complete CAT(1) spaces.

Theorem 1.1. *Let X be an admissible complete* CAT(1) *space, f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$, $\{\lambda_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty}\lambda_n=\infty$, and $\{x_n\}$ a sequence defined by $x_1\in X$ and*

$$x_{n+1}=\operatorname*{argmin}_{y\in X}\left\{f(y)+\frac{1}{\lambda_n}\tan d(y,x_n)\sin d(y,x_n)\right\} \tag{1.1}$$

for all $n\in\mathbb{N}$. Then the set $\operatorname{argmin}_X f$ of all minimizers of f is nonempty if and only if $\{x_n\}$ is spherically bounded and $\sup_n d(x_{n+1},x_n)<\pi/2$.

Theorem 1.2. *Let X, f, $\{\lambda_n\}$, and $\{x_n\}$ be the same as in Theorem 1.1 and suppose that $\operatorname{argmin}_X f$ is nonempty. Then the following hold.*

(i) *There exists a positive real number C such that*

$$f(x_{n+1})-\inf f(X)\le\frac{C}{\sum_{k=1}^{n}\lambda_k}\big(1-\cos d(u,x_1)\big)$$

for all $u\in\operatorname{argmin}_X f$ and $n\in\mathbb{N}$;

(ii) *$\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$.*

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It should be noted that, in this paper, we say that a CAT(1) space X is admissible if $d(v, v') < \pi/2$ for all $v, v' \in X$. We also say that a sequence $\{x_n\}$ in a CAT(1) space X is spherically bounded if

$$\inf_{y \in X} \limsup_{n \to \infty} d(y, x_n) < \frac{\pi}{2}. \tag{1.2}$$

The proximal point algorithm, introduced by Martinet [22] and Rockafellar [24], is an approximation method for finding a minimizer of a proper lower semicontinuous convex function f of a real Hilbert space X into $]-\infty, \infty]$. This algorithm generates a sequence $\{x_n\}$ by $x_1 \in X$ and

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right\} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence of positive real numbers. It is well known that the right hand side of (1.3) consists of one point $p \in X$. We identify the set $\{p\}$ with p in this case. Using the resolvent J_f of f given by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2} \|y - x\|^2 \right\} \tag{1.4}$$

for all $x \in X$, we can write the scheme (1.3) as $x_{n+1} = J_{\lambda_n f} x_n$ for all $n \in \mathbb{N}$. See [4, 25] for more details on convex analysis in Hilbert spaces.

In 1976, Rockafellar [24, Theorem 1] showed that if $\inf_n \lambda_n > 0$, then the set $\operatorname{argmin}_X f$ is nonempty if and only if $\{x_n\}$ is bounded, and that if $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is weakly convergent to an element of $\operatorname{argmin}_X f$. In 1978, Brezis and Lions [5, Théorème 9] showed the weak convergence of $\{x_n\}$ to an element of $\operatorname{argmin}_X f$ under a weaker condition that $\operatorname{argmin}_X f$ is nonempty and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Later, Güler [10, Corollary 5.1] found an example of $\{x_n\}$ in the Hilbert space ℓ^2 which does not converge strongly.

On the other hand, in 1995, Jost [11] generalized the concept of resolvent given by (1.4) in Hilbert spaces to that in more general complete CAT(0) spaces. According to [3, Section 2.2], [11, Lemma 2], and [23, Section 1.3], if f is a proper lower semicontinuous convex function of a complete CAT(0) space X into $]-\infty, \infty]$, then the resolvent J_f of f given by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2} d(y, x)^2 \right\} \tag{1.5}$$

for all $x \in X$ is a well defined single valued nonexpansive mapping of X into itself. We also know that its fixed point set $\mathcal{F}(J_f)$ is equal to $\operatorname{argmin}_X f$. See [3, 12, 13] for more details on this concept.

In 2013, Bačák [2, Theorem 1.4 and Remark 1.6] generalized the result by Brezis and Lions [5, Théorème 9] to the complete CAT(0) space setting as follows. Note that Δ-convergence is called weak convergence in [2].

Theorem 1.3 ([2, Theorem 1.4 and Remark 1.6]). *Let X be a complete* CAT(0) *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$ such that* $\operatorname{argmin}_X f$ *is nonempty, $J_{\lambda f}$ the resolvent of λf for each $\lambda > 0$, $\{\lambda_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\{x_n\}$ a sequence defined by*

$x_1 \in X$ and $x_{n+1} = J_{\lambda_n f} x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$ *and*

$$f(x_{n+1}) - \inf f(X) \leq \frac{1}{2\sum_{k=1}^{n} \lambda_k} d(u, x_1)^2$$

for all $u \in \operatorname{argmin}_X f$ and $n \in \mathbb{N}$.

Recently, the authors [15] introduced the concept of resolvents of convex functions in complete CAT(1) spaces and studied the existence and approximation of fixed points of mappings related to this concept. Considering the geometric difference between CAT(0) and CAT(1) spaces, they replaced $d(y,x)^2/2$ in (1.5) with $\tan d(y,x) \sin d(y,x)$ in the definition of resolvent below. According to [15, Theorems 4.2 and 4.6], if f is a proper lower semicontinuous convex function of an admissible complete CAT(1) space X into $]-\infty, \infty]$, then the resolvent R_f of f given by

$$R_f x = \operatorname*{argmin}_{y \in X} \big\{ f(y) + \tan d(y,x) \sin d(y,x) \big\} \tag{1.6}$$

for all $x \in X$ is a well defined single valued mapping of X into itself such that

$$\mathcal{F}(R_f) = \operatorname*{argmin}_X f \tag{1.7}$$

and

$$\begin{aligned} &\Big(C_x^2 (1 + C_y^2) C_y + C_y^2 (1 + C_x^2) C_x \Big) \cos d(R_f x, R_f y) \\ &\quad \geq C_x^2 (1 + C_y^2) \cos d(R_f x, y) + C_y^2 (1 + C_x^2) \cos d(R_f y, x) \end{aligned} \tag{1.8}$$

for all $x, y \in X$, where $C_z = \cos d(R_f z, z)$ for all $z \in X$. Using this concept, we can write the scheme (1.1) as

$$x_{n+1} = R_{\lambda_n f} x_n$$

for all $n \in \mathbb{N}$. The function $t \mapsto \tan t \sin t$ used in (1.6) is obviously a strictly increasing, continuous, and convex function on $[0, \pi/2[$ such that $\tan 0 \sin 0 = 0$ and $\tan t \sin t \to \infty$ as $t \uparrow \pi/2$. These properties are similar to those of the function $t \mapsto t^2$ on $[0, \infty[$ used in (1.5). Note that the diameters of the model spaces $\mathbb{S}^2$ and $\mathbb{R}^2$ of CAT(1) and CAT(0) spaces coincide with π and ∞, respectively and that the second order Maclaurin approximation of the function $t \mapsto \tan t \sin t$ is equal to $t \mapsto t^2$.

This paper is organized as follows. In Section 2, we recall some definitions and results needed in this paper. In Section 3, we obtain some fundamental properties of resolvents of convex functions in CAT(1) spaces. In Section 4, after obtaining Theorem 4.1, a maximization theorem in CAT(1) spaces, we give the proofs of Theorems 1.1 and 1.2. In Section 5, we obtain three corollaries of Theorems 1.1 and 1.2.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, $\mathbb{R}$ the set of all real numbers, and $\mathcal{F}(T)$ the set of all fixed points of a mapping T.

A metric space X with metric d is said to be uniquely π-geodesic if for each $x, y \in X$ with $d(x,y) < \pi$, there exists a unique mapping c of $[0,l]$ into X such that $d\big(c(t), c(t')\big) = |t - t'|$ for all $t, t' \in [0,l]$, $c(0) = x$, and $c(l) = y$, where $l = d(x,y)$. The mapping c is called the geodesic from x to y and the set $[x,y]$, which is defined as the image of c, is called the geodesic segment between x and y. We also denote by $\alpha x \oplus (1-\alpha)y$ the point $c\big((1-\alpha)d(x,y)\big)$ for each $\alpha \in [0,1]$.

Let H be a real Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ and the induced norm $\|\cdot\|$. We know that the unit sphere S_H of H is a complete metric space with the spherical metric ρ_{S_H} defined by

$$\rho_{S_H}(x,y) = \arccos \langle x, y \rangle$$

for each $x, y \in S_H$. It is also known that S_H is uniquely π-geodesic. For each distinct $x, y \in S_H$ such that $\rho_{S_H}(x,y) < \pi$, the unique geodesic c from x to y is given by

$$c(t) = (\cos t)x + (\sin t) \cdot \frac{y - \langle y, x \rangle x}{\|y - \langle y, x \rangle x\|}$$

for all $t \in [0, \rho_{S_H}(x,y)]$. We denote by $\mathbb{S}^2$ the unit sphere of the three dimensional Euclidean space $\mathbb{R}^3$.

It is known [6, Lemma 2.14 in Chapter I.2] that if X is a uniquely π-geodesic space and x_1, x_2, x_3 are points in X satisfying

$$d(x_1,x_2) + d(x_2,x_3) + d(x_3,x_1) < 2\pi, \tag{2.1}$$

then there exist $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathbb{S}^2$ such that

$$d(x_i, x_j) = \rho_{\mathbb{S}^2}(\bar{x}_i, \bar{x}_j)$$

for all $i, j \in \{1,2,3\}$. The sets Δ and $\bar{\Delta}$ given by

$$\Delta = [x_1,x_2] \cup [x_2,x_3] \cup [x_3,x_1] \quad \text{and} \quad \bar{\Delta} = [\bar{x}_1,\bar{x}_2] \cup [\bar{x}_2,\bar{x}_3] \cup [\bar{x}_3,\bar{x}_1]$$

are called the geodesic triangle with vertices x_1, x_2, x_3 and a comparison triangle for Δ, respectively. A point $\bar{p} \in \bar{\Delta}$ is called a comparison point for $p \in \Delta$ if $p \in [x_i, x_j]$, $\bar{p} \in [\bar{x}_i, \bar{x}_j]$, and $d(x_i, p) = \rho_{\mathbb{S}^2}(\bar{x}_i, \bar{p})$ for some distinct $i, j \in \{1,2,3\}$.

A uniquely π-geodesic space X is called a CAT(1) space if

$$d(p,q) \le \rho_{\mathbb{S}^2}(\bar{p}, \bar{q})$$

whenever Δ is a geodesic triangle with vertices $x_1, x_2, x_3 \in X$ satisfying (2.1), $\bar{\Delta}$ is a comparison triangle for Δ, and $\bar{p}, \bar{q} \in \bar{\Delta}$ are comparison points for $p, q \in \Delta$, respectively. We know that all nonempty closed convex subsets of a real Hilbert space H, the space (S_H, ρ_{S_H}), and all complete CAT(0) spaces are complete CAT(1) spaces. The complete CAT(1) space (S_H, ρ_{S_H}) is particularly called a Hilbert sphere. See [6] for more details on geodesic spaces.

The following lemma plays a fundamental role in the study of CAT(1) spaces.

Lemma 2.1 ([18, Corollary 2.2]). *Let X be a CAT(1) space and x_1, x_2, x_3 points of X such that (2.1) holds. If $\alpha \in [0,1]$, then*

$$\begin{aligned}
&\cos d\big(\alpha x_1 \oplus (1-\alpha)x_2, x_3\big) \sin d(x_1,x_2) \\
&\ge \cos d(x_1,x_3) \sin\big(\alpha d(x_1,x_2)\big) + \cos d(x_2,x_3) \sin\big((1-\alpha) d(x_1,x_2)\big).
\end{aligned}$$

We also know the following.

Lemma 2.2 (See the proof of [19, Lemma 4.1]). *Let X, x_1, x_2, and x_3 be the same as in Lemma 2.1. Then*

$$\cos d\Big(\frac{1}{2}x_1 \oplus \frac{1}{2}x_2, x_3\Big) \cos\Big(\frac{1}{2}d(x_1, x_2)\Big) \geq \frac{1}{2}\cos d(x_1, x_3) + \frac{1}{2}\cos d(x_2, x_3).$$

Lemma 2.3 (See, for instance, [15, Lemma 2.3]). *Let X, x_1, x_2, and x_3 be the same as in Lemma 2.1. If $d(x_1, x_3) \leq \pi/2$, $d(x_2, x_3) \leq \pi/2$, and $\alpha \in [0, 1]$, then*

$$\cos d\big(\alpha x_1 \oplus (1-\alpha)x_2, x_3\big) \geq \alpha \cos d(x_1, x_3) + (1-\alpha)\cos d(x_2, x_3).$$

Let X be a CAT(1) space and $\{x_n\}$ a sequence in X. The asymptotic center $\mathcal{A}\big(\{x_n\}\big)$ of $\{x_n\}$ is defined by

$$\mathcal{A}\big(\{x_n\}\big) = \left\{ z \in X : \limsup_{n\to\infty} d(z, x_n) = \inf_{y \in X} \limsup_{n\to\infty} d(y, x_n) \right\}.$$

The sequence $\{x_n\}$ is said to be Δ-convergent to an element $p \in X$ if

$$\mathcal{A}\big(\{x_{n_i}\}\big) = \{p\}$$

for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In this case, the point p is called the Δ-limit of $\{x_n\}$. If $\{x_n\}$ is Δ-convergent to $p \in X$, then it is bounded and each subsequence of $\{x_n\}$ is Δ-convergent to p. For a sequence $\{x_n\}$ in X, we denote by $\omega_\Delta\big(\{x_n\}\big)$ the set of all points $q \in X$ such that there exists a subsequence of $\{x_n\}$ which is Δ-convergent to q. It is known [8, Proposition 4.1 and Corollary 4.4] that if X is a complete CAT(1) space and $\{x_n\}$ is a spherically bounded sequence in X, that is, it satisfies (1.2), then $\mathcal{A}\big(\{x_n\}\big)$ is a singleton and $\{x_n\}$ has a Δ-convergent subsequence. See [8, 20] for more details on Δ-convergence. We know the following.

Lemma 2.4 ([17, Proposition 3.1]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a spherically bounded sequence in X. If $\{d(z, x_n)\}$ is convergent for all $z \in \omega_\Delta\big(\{x_n\}\big)$, then $\{x_n\}$ is Δ-convergent.*

Let X be an admissible CAT(1) space and f a function of X into $]-\infty, \infty]$. The function f is said to be convex if

$$f\big(\alpha x \oplus (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in X$ and $\alpha \in]0, 1[$. It is also said to be Δ-lower semicontinuous if

$$f(p) \leq \liminf_{n\to\infty} f(x_n)$$

whenever $\{x_n\}$ is a sequence in X which is Δ-convergent to $p \in X$. We denote by $\operatorname{argmin}_X f$ or $\operatorname{argmin}_{y\in X} f(y)$ the set of all minimizers of f. A function g of X into $[-\infty, \infty[$ is said to be concave if $-g$ is convex. We denote by $\operatorname{argmax}_X g$ the set of all maximizers of g. See [14, 26] on some examples of convex functions in CAT(1) spaces. We know the following.

Lemma 2.5 ([15, Lemma 3.1]). *Let X be an admissible complete* CAT(1) *space and f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. Then f is Δ-lower semicontinuous.*

It is clear that if A is a nonempty bounded subset of $\mathbb{R}$, I is a closed subset of $\mathbb{R}$ which contains A, and f is a continuous and nondecreasing real function on I, then $f(\sup A) = \sup f(A)$ and $f(\inf A) = \inf f(A)$. This implies the following.

Lemma 2.6. *Let I be a nonempty closed subset of $\mathbb{R}$, $\{t_n\}$ a bounded sequence in I, and f a continuous real function on I. Then the following hold.*

(i) *If f is nondecreasing, then $f(\limsup_n t_n) = \limsup_n f(t_n)$;*
(ii) *if f is nonincreasing, then $f(\limsup_n t_n) = \liminf_n f(t_n)$.*

3. Fundamental properties of resolvents in CAT(1) spaces

Let X be an admissible complete CAT(1) space and f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. It is known [15, Theorem 4.2] that for each $x \in X$, there exists a unique $\hat{x} \in X$ such that

$$f(\hat{x}) + \tan d(\hat{x}, x) \sin d(\hat{x}, x) = \inf_{y \in X} \bigl\{ f(y) + \tan d(y, x) \sin d(y, x) \bigr\}.$$

The resolvent R_f of f is defined by $R_f x = \hat{x}$ for all $x \in X$. In other words, R_f is given by (1.6) for all $x \in X$. It is known [15, Theorems 4.2 and 4.6] that R_f is a well defined single valued mapping of X into itself satisfying (1.7) and (1.8).

Using some techniques developed in the proof of [15, Theorem 4.6], we show the following fundamental result. The inequality (3.2) is a generalization of (1.8) and also a counterpart of [1, Lemma 3.1] in the CAT(1) space setting.

Lemma 3.1. *Let X be an admissible complete* CAT(1) *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$, R_η the resolvent of ηf for all $\eta > 0$, and $C_{\eta,z}$ the real number given by $C_{\eta,z} = \cos d(R_\eta z, z)$ for all $\eta > 0$ and $z \in X$. If $\lambda, \mu > 0$ and $x, y \in X$, then the inequalities*

$$\begin{aligned} &\left(\frac{1}{C_{\lambda,x}^2} + 1 \right) d(R_\lambda x, R_\mu y) \bigl(C_{\lambda,x} \cos d(R_\lambda x, R_\mu y) - \cos d(R_\mu y, x) \bigr) \\ &\geq \lambda \bigl(f(R_\lambda x) - f(R_\mu y) \bigr) \sin d(R_\lambda x, R_\mu y) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\Bigl(\lambda C_{\lambda,x}^2 (1 + C_{\mu,y}^2) C_{\mu,y} + \mu C_{\mu,y}^2 (1 + C_{\lambda,x}^2) C_{\lambda,x} \Bigr) \cos d(R_\lambda x, R_\mu y) \\ &\geq \lambda C_{\lambda,x}^2 (1 + C_{\mu,y}^2) \cos d(R_\lambda x, y) + \mu C_{\mu,y}^2 (1 + C_{\lambda,x}^2) \cos d(R_\mu y, x) \end{aligned} \tag{3.2}$$

hold.

Proof. Let $\lambda, \mu > 0$ and $x, y \in X$ be given. Set $D = d(R_\lambda x, R_\mu y)$ and

$$z_t = t R_\mu y \oplus (1 - t) R_\lambda x$$

for all $t \in]0, 1[$. By the definition of R_λ and the convexity of f, we have

$$\begin{aligned} &\lambda f(R_\lambda x) + \tan d(R_\lambda x, x) \sin d(R_\lambda x, x) \\ &\quad \leq \lambda f(z_t) + \tan d(z_t, x) \sin d(z_t, x) \\ &\quad \leq t \lambda f(R_\mu y) + (1 - t) \lambda f(R_\lambda x) + \tan d(z_t, x) \sin d(z_t, x) \end{aligned}$$

and hence we have

$$\begin{aligned} & t\lambda\big(f(R_\lambda x) - f(R_\mu y)\big) \\ & \le \tan d(z_t, x) \sin d(z_t, x) - \tan d(R_\lambda x, x) \sin d(R_\lambda x, x) \\ & = \left(\frac{1}{\cos d(z_t, x) \cos d(R_\lambda x, x)} + 1\right) \big(\cos d(R_\lambda x, x) - \cos d(z_t, x)\big). \end{aligned} \tag{3.3}$$

On the other hand, Lemma 2.1 implies that

$$\begin{aligned} & \cos d(z_t, x) \sin d(R_\mu y, R_\lambda x) \\ & \ge \cos d(R_\mu y, x) \sin\big(t d(R_\mu y, R_\lambda x)\big) + \cos d(R_\lambda x, x) \sin\big((1-t) d(R_\mu y, R_\lambda x)\big). \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we have

$$\begin{aligned} & t\lambda\big(f(R_\lambda x) - f(R_\mu y)\big) \sin D \\ & \le \left(\frac{1}{\cos d(z_t, x) \cos d(R_\lambda x, x)} + 1\right) \\ & \quad \times \Big[\cos d(R_\lambda x, x)\Big(\sin D - \sin\big((1-t)D\big)\Big) - \cos d(R_\mu y, x) \sin(tD)\Big] \\ & = \left(\frac{1}{\cos d(z_t, x) \cos d(R_\lambda x, x)} + 1\right) \cdot 2 \sin\left(\frac{t}{2} D\right) \\ & \quad \times \left[\cos d(R_\lambda x, x) \cos\left(\left(1 - \frac{t}{2}\right) D\right) - \cos d(R_\mu y, x) \cos\left(\frac{t}{2} D\right)\right] \end{aligned}$$

and hence

$$\begin{aligned} & \lambda\big(f(R_\lambda x) - f(R_\mu y)\big) \sin D \\ & \le \left(\frac{1}{\cos d(z_t, x) \cos d(R_\lambda x, x)} + 1\right) \cdot \frac{2}{t} \sin\left(\frac{t}{2} D\right) \\ & \quad \times \left[\cos d(R_\lambda x, x) \cos\left(\left(1 - \frac{t}{2}\right) D\right) - \cos d(R_\mu y, x) \cos\left(\frac{t}{2} D\right)\right]. \end{aligned}$$

Letting $t \downarrow 0$, we obtain

$$\lambda\big(f(R_\lambda x) - f(R_\mu y)\big) \sin D \le \left(\frac{1}{C_{\lambda,x}^2} + 1\right) D\big(C_{\lambda,x} \cos D - \cos d(R_\mu y, x)\big).$$

Thus (3.1) holds.

If $D > 0$, then (3.1) implies that

$$\begin{aligned} & \mu C_{\mu,y}^2 \big(1 + C_{\lambda,x}^2\big)\big(C_{\lambda,x} \cos D - \cos d(R_\mu y, x)\big) \\ & \ge \frac{\lambda \mu C_{\lambda,x}^2 C_{\mu,y}^2}{D} \big(f(R_\lambda x) - f(R_\mu y)\big) \sin D \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & \lambda C_{\lambda,x}^2 \big(1 + C_{\mu,y}^2\big)\big(C_{\mu,y} \cos D - \cos d(R_\lambda x, y)\big) \\ & \ge \frac{\lambda \mu C_{\lambda,x}^2 C_{\mu,y}^2}{D} \big(f(R_\mu y) - f(R_\lambda x)\big) \sin D. \end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), we obtain (3.2). It is obvious that the equality in (3.2) holds in the case when $D = 0$. □

As a direct consequence of Lemma 3.1, we obtain the following.

Corollary 3.2. *Let X, f, $\{R_\eta\}$, and $\{C_{\eta,z}\}$ be the same as in Lemma 3.1. If $\lambda > 0$, $x \in X$, and $y \in \operatorname{argmin}_X f$, then the inequalities*

$$\frac{\pi}{2}\left(\frac{1}{C_{\lambda,x}^2}+1\right)\bigl(C_{\lambda,x}\cos d(y,R_\lambda x)-\cos d(y,x)\bigr) \geq \lambda\bigl(f(R_\lambda x)-f(y)\bigr) \tag{3.7}$$

and

$$C_{\lambda,x}\cos d(y,R_\lambda x) \geq \cos d(y,x) \tag{3.8}$$

hold.

Proof. Let $\lambda > 0$, $x \in X$, and $y \in \operatorname{argmin}_X f$ be given. Since $f(R_\lambda x) - f(y) \geq 0$ and $\sin t \geq 2t/\pi$ for all $t \in [0, \pi/2]$, it follows from (1.7) and (3.1) that

$$\begin{aligned}&\left(\frac{1}{C_{\lambda,x}^2}+1\right) d(y,R_\lambda x)\bigl(C_{\lambda,x}\cos d(y,R_\lambda x)-\cos d(y,x)\bigr)\\&\quad \geq \lambda\bigl(f(R_\lambda x)-f(y)\bigr)\cdot\frac{2d(y,R_\lambda x)}{\pi}.\end{aligned}$$

This implies that (3.7) holds when $d(y, R_\lambda x) > 0$. Note that the equality in (3.7) clearly holds when $d(y, R_\lambda x) = 0$. It then follows from (3.7) that

$$\frac{\pi}{2}\left(\frac{1}{C_{\lambda,x}^2}+1\right)\bigl(C_{\lambda,x}\cos d(y,R_\lambda x)-\cos d(y,x)\bigr) \geq 0$$

and hence (3.8) holds. □

4. The proximal point algorithm in CAT(1) spaces

We need the following maximization theorem in the proof of Theorem 1.1.

Theorem 4.1. *Let X be an admissible complete* CAT(1) *space, $\{z_n\}$ a spherically bounded sequence in X, $\{\beta_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty}\beta_n = \infty$, and g the real function on X defined by*

$$g(y) = \liminf_{n\to\infty}\frac{1}{\sum_{l=1}^{n}\beta_l}\sum_{k=1}^{n}\beta_k\cos d(y,z_k) \tag{4.1}$$

for all $y \in X$. Then g is a concave and nonexpansive function of X into $[0, 1]$ and $\operatorname{argmax}_X g$ is a singleton.

Proof. Set $\sigma_n = \sum_{l=1}^{n}\beta_l$ for all $n \in \mathbb{N}$. Since X is admissible, we know that

$$\frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k\cos d(y,z_k) \in\ ]0,1]$$

and hence $g(y) \in [0, 1]$ for all $y \in X$.

We next show that g is concave and nonexpansive. If $y_1, y_2 \in X$ and $\alpha \in]0,1[$, then it follows from Lemma 2.3 that

$$\cos d\big(\alpha y_1 \oplus (1-\alpha)y_2, z_k\big) \geq \alpha \cos d(y_1, z_k) + (1-\alpha)\cos d(y_2, z_k)$$

for all $k \in \mathbb{N}$. This implies that

$$\begin{aligned}&\frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k \cos d\big(\alpha y_1 \oplus (1-\alpha)y_2, z_k\big)\\ &\geq \frac{\alpha}{\sigma_n}\sum_{k=1}^{n}\beta_k\cos d(y_1, z_k) + \frac{1-\alpha}{\sigma_n}\sum_{k=1}^{n}\beta_k\cos d(y_2, z_k)\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the lower limit, we obtain

$$g\big(\alpha y_1 \oplus (1-\alpha)y_2\big) \geq \alpha g(y_1) + (1-\alpha)g(y_2)$$

and hence g is concave. The nonexpansiveness of $t \mapsto \cos t$ and the triangle inequality imply that

$$\cos d(y_1, z_k) \leq d(y_1, y_2) + \cos d(y_2, z_k)$$

for all $k \in \mathbb{N}$ and hence we have

$$g(y_1) - g(y_2) \leq d(y_1, y_2)$$

Similarly, we can see that $g(y_2) - g(y_1) \leq d(y_1, y_2)$. Thus g is nonexpansive.

We next show that $\operatorname{argmax}_X f$ is nonempty. The spherical boundedness of $\{z_n\}$ implies that

$$0 \leq \inf_{y\in X}\limsup_{n\to\infty} d(y, z_n) < \frac{\pi}{2}.$$

Since $t \mapsto \cos t$ is continuous and decreasing on $[0, \pi/2]$, Lemma 2.6 implies that

$$\begin{aligned}0 &< \cos\left(\inf_{y\in X}\limsup_{n\to\infty} d(y, z_n)\right)\\ &= \sup_{y\in X}\cos\left(\limsup_{n\to\infty} d(y, z_n)\right) = \sup_{y\in X}\liminf_{n\to\infty}\cos d(y, z_n).\end{aligned} \tag{4.2}$$

On the other hand, we can see that

$$\liminf_{n\to\infty}\cos d(y, z_n) \leq \liminf_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k\cos d(y, z_k) \tag{4.3}$$

for all $y \in X$. In fact, setting $\gamma_n = \cos d(y, z_n)$ for all $n \in \mathbb{N}$, we know that, for each $\gamma < \liminf_n \gamma_n$, there exists $n_0 \in \mathbb{N}$ such that $\gamma < \gamma_k$ for all $k \geq n_0$. Thus, if $p \in \mathbb{N}$, then we have

$$\begin{aligned}\frac{1}{\sigma_{n_0+p}}\sum_{k=1}^{n_0+p}\beta_k\gamma_k &= \frac{1}{\sigma_{n_0+p}}\left(\sum_{k=1}^{n_0}\beta_k\gamma_k + \sum_{k=n_0+1}^{n_0+p}\beta_k\gamma_k\right)\\ &> \frac{1}{\sigma_{n_0+p}}\left(\sum_{k=1}^{n_0}\beta_k\gamma_k + \sum_{k=n_0+1}^{n_0+p}\beta_k\gamma\right)\end{aligned}$$

$$= \frac{1}{\sigma_{n_0+p}} \sum_{k=1}^{n_0} \beta_k \gamma_k + \left(1 - \frac{\sigma_{n_0}}{\sigma_{n_0+p}}\right) \gamma.$$

Since $\sigma_n \to \infty$ as $n \to \infty$, we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \gamma_k &= \liminf_{p\to\infty} \frac{1}{\sigma_{n_0+p}} \sum_{k=1}^{n_0+p} \beta_k \gamma_k \\ &\geq \liminf_{p\to\infty} \left(\frac{1}{\sigma_{n_0+p}} \sum_{k=1}^{n_0} \beta_k \gamma_k + \left(1 - \frac{\sigma_{n_0}}{\sigma_{n_0+p}}\right) \gamma \right) = \gamma. \end{aligned}$$

Since $\gamma < \liminf_n \gamma_n$ is arbitrary, we know that (4.3) holds.

By (4.2) and (4.3), we have

$$0 < \sup_{y\in X} \liminf_{n\to\infty} \cos d(y, z_n) \leq \sup_{y\in X} g(y) =: l. \tag{4.4}$$

By the definition of l, there exists a sequence $\{y_n\}$ in X such that $g(y_n) \leq g(y_{n+1})$ for all $n \in \mathbb{N}$ and $g(y_n) \to l$ as $n \to \infty$. If $m \geq n$, then Lemma 2.2 implies that

$$\cos d\Big(\frac{1}{2} y_n \oplus \frac{1}{2} y_m, z_k\Big) \cos\Big(\frac{1}{2} d(y_n, y_m)\Big) \geq \frac{1}{2} \cos d(y_n, z_k) + \frac{1}{2} \cos d(y_m, z_k)$$

for all $k \in \mathbb{N}$. This gives us that

$$g\Big(\frac{1}{2} y_n \oplus \frac{1}{2} y_m\Big) \cos\Big(\frac{1}{2} d(y_n, y_m)\Big) \geq \frac{1}{2} g(y_n) + \frac{1}{2} g(y_m).$$

Since $l = \sup g(X)$ and $g(y_n) \leq g(y_m)$, we then obtain

$$l \cos\Big(\frac{1}{2} d(y_n, y_m)\Big) \geq \frac{1}{2} g(y_n) + \frac{1}{2} g(y_m) \geq g(y_n). \tag{4.5}$$

Noting that (4.4) implies that $0 < l \leq 1$, we have

$$d(y_n, y_m) \leq 2 \arccos \frac{g(y_n)}{l} \tag{4.6}$$

whenever $m \geq n$. Since $g(y_n)/l \to 1$ as $n \to \infty$, the right hand side of (4.6) converges to 0. Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, the sequence $\{y_n\}$ converges to some $p \in X$. By the continuity of g and the choice of $\{y_n\}$, we obtain

$$g(p) = \lim_{n\to\infty} g(y_n) = l.$$

Thus p is an element of $\operatorname{argmax}_X g$.

We finally show that $\operatorname{argmax}_X g$ consists of one point. Suppose that p and q are elements of $\operatorname{argmax}_X g$. As in the proof of (4.5), we can see that

$$l \cos\Big(\frac{1}{2} d(p, q)\Big) \geq \frac{1}{2} g(p) + \frac{1}{2} g(q) = l.$$

Since $l > 0$, we then obtain $\cos\big(d(p, q)/2\big) = 1$. Consequently, we have $p = q$. $\square$

Now, we are ready to give the proofs of Theorems 1.1 and 1.2. In these proofs, we denote by R_η and $C_{\eta,z}$ the resolvent of ηf for all $\eta > 0$ and the real number given by $C_{\eta,z} = \cos d(R_\eta z, z)$ for all $\eta > 0$ and $z \in X$, respectively.

The proof of Theorem 1.1. We first show the if part. Suppose that $\{x_n\}$ is spherically bounded and

$$\sup_n d(x_{n+1}, x_n) < \frac{\pi}{2}. \tag{4.7}$$

Set

$$\beta_n = \frac{\lambda_n C^2_{\lambda_n, x_n}}{1 + C^2_{\lambda_n, x_n}} \quad \text{and} \quad \sigma_n = \sum_{k=1}^{n} \beta_k$$

for all $n \in \mathbb{N}$. It is obvious that $\beta_n > 0$ for all $n \in \mathbb{N}$. It also follows from (4.7) that

$$0 < \cos\left(\sup_n d(x_{n+1}, x_n)\right) = \inf_n \cos d(x_{n+1}, x_n) = \inf_n C_{\lambda_n, x_n}$$

and hence it follows from

$$\beta_n \geq \frac{\lambda_n C^2_{\lambda_n, x_n}}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty$$

that $\sum_{n=1}^{\infty} \beta_n = \infty$. Thus Theorem 4.1 ensures that the real function g on X, which is defined by

$$g(y) = \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cos d(y, x_{k+1})$$

for all $y \in X$, has a unique maximizer p on X.

Let μ be a positive real number. By (3.2), we have

$$\begin{aligned} &\left(\lambda_k C^2_{\lambda_k, x_k}(1 + C^2_{\mu,p}) + \mu C^2_{\mu,p}(1 + C^2_{\lambda_k, x_k})\right) \cos d(x_{k+1}, R_\mu p) \\ &\geq \lambda_k C^2_{\lambda_k, x_k}(1 + C^2_{\mu,p}) \cos d(x_{k+1}, p) + \mu C^2_{\mu,p}(1 + C^2_{\lambda_k, x_k}) \cos d(R_\mu p, x_k) \end{aligned}$$

and hence

$$\begin{aligned} &\frac{\lambda_k C^2_{\lambda_k, x_k}}{1 + C^2_{\lambda_k, x_k}} \cos d(x_{k+1}, R_\mu p) \\ &\geq \frac{\lambda_k C^2_{\lambda_k, x_k}}{1 + C^2_{\lambda_k, x_k}} \cos d(x_{k+1}, p) + \frac{\mu C^2_{\mu,p}}{1 + C^2_{\mu,p}} \big(\cos d(R_\mu p, x_k) - \cos d(R_\mu p, x_{k+1})\big) \end{aligned} \tag{4.8}$$

for all $k \in \mathbb{N}$. Summing up (4.8) with respect to $k \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} &\frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cos d(x_{k+1}, R_\mu p) \\ &\geq \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cos d(x_{k+1}, p) + \frac{\mu C^2_{\mu,p}}{1 + C^2_{\mu,p}} \cdot \frac{\cos d(R_\mu p, x_1) - \cos d(R_\mu p, x_{n+1})}{\sigma_n} \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\sigma_n \to \infty$ as $n \to \infty$, we obtain

$$g(R_\mu p) = \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cos d(x_{k+1}, R_\mu p)$$

$$\geq \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cos d(x_{k+1}, p) = g(p).$$

Then it follows from $\operatorname{argmax}_X g = \{p\}$ that $R_\mu p = p$. By (1.7), we know that

$$\mathcal{F}(R_\mu) = \operatorname*{argmin}_X \mu f = \operatorname*{argmin}_X f$$

and hence we conclude that p is an element of $\operatorname{argmin}_X f$.

We next show the only if part. Suppose that $\operatorname{argmin}_X f$ is nonempty and let u be an element of $\operatorname{argmin}_X f$. It follows from (3.8) that

$$\begin{aligned} \min\bigl\{\cos d(x_{n+1}, x_n), \cos d(u, x_{n+1})\bigr\} &\geq \cos d(x_{n+1}, x_n) \cos d(u, x_{n+1}) \\ &\geq \cos d(u, x_n). \end{aligned} \tag{4.9}$$

The admissibility of X and (4.9) imply that

$$\max\bigl\{d(x_{n+1}, x_n), d(u, x_{n+1})\bigr\} \leq d(u, x_n) \leq d(u, x_1) < \frac{\pi}{2} \tag{4.10}$$

and hence $\{x_n\}$ is spherically bounded and $\sup_n d(x_{n+1}, x_n) < \pi/2$. □

The proof of Theorem 1.2. We first show (i). Set $l = \sup_n d(x_{n+1}, x_n)$. By Theorem 1.1, we know that $\{x_n\}$ is spherically bounded and $l < \pi/2$. Letting

$$K = \frac{1}{\cos^2 l} + 1,$$

we have

$$\frac{1}{\cos^2 d(x_{n+1}, x_n)} + 1 \leq K \tag{4.11}$$

for all $n \in \mathbb{N}$. Let u be an element of $\operatorname{argmin}_X f$. By the definitions of R_{λ_n} and $\{x_n\}$, we know that

$$\begin{aligned} f(u) &\leq f(x_{n+1}) \\ &\leq f(x_{n+1}) + \frac{1}{\lambda_n} \tan d(x_{n+1}, x_n) \sin d(x_{n+1}, x_n) \\ &\leq f(x_n) \end{aligned} \tag{4.12}$$

for all $n \in \mathbb{N}$. On the other hand, it follows from (3.7) that

$$\begin{aligned} &\lambda_n\bigl(f(x_{n+1}) - f(u)\bigr) \\ &\leq \frac{\pi}{2} \left(\frac{1}{\cos^2 d(x_{n+1}, x_n)} + 1 \right) \bigl(\cos d(u, x_{n+1}) - \cos d(u, x_n)\bigr) \end{aligned} \tag{4.13}$$

for all $n \in \mathbb{N}$. If $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, n\}$, then it follows from (4.11), (4.12), and (4.13) that

$$\begin{aligned} \lambda_k\bigl(f(x_{n+1}) - f(u)\bigr) &\leq \lambda_k\bigl(f(x_{k+1}) - f(u)\bigr) \\ &\leq \frac{K\pi}{2} \bigl(\cos d(u, x_{k+1}) - \cos d(u, x_k)\bigr). \end{aligned}$$

Hence we obtain

$$\begin{aligned}\big(f(x_{n+1}) - \inf f(X)\big) \sum_{k=1}^{n} \lambda_k &\leq \frac{K\pi}{2}\big(\cos d(u, x_{n+1}) - \cos d(u, x_1)\big) \\ &\leq \frac{K\pi}{2}\big(1 - \cos d(u, x_1)\big).\end{aligned}$$

Letting $C = K\pi/2$, we obtain the desired inequality.

We finally show (ii). Since $\sum_{n=1}^{\infty} \lambda_n = \infty$, it follows from (i) that

$$\lim_{n\to\infty} f(x_n) = \inf f(X). \tag{4.14}$$

We then show that $\{d(z, x_n)\}$ is convergent for all $z \in \omega_\Delta\big(\{x_n\}\big)$. If z is an element of $\omega_\Delta\big(\{x_n\}\big)$, then we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is Δ-convergent to z. By Lemma 2.5 and (4.14), we obtain

$$f(z) \leq \liminf_{i\to\infty} f(x_{n_i}) = \lim_{n\to\infty} f(x_n) = \inf f(X)$$

and hence z is an element of $\operatorname{argmin}_X f$. Thus $\omega_\Delta\big(\{x_n\}\big)$ is a subset of $\operatorname{argmin}_X f$. It also follows from (4.10) that $\{d(z, x_n)\}$ is convergent. Thus, Lemma 2.4 implies that $\{x_n\}$ is Δ-convergent to some $x_\infty \in X$. This gives us that

$$\{x_\infty\} = \omega_\Delta\big(\{x_n\}\big) \subset \operatorname*{argmin}_X f.$$

Consequently, $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$. □

5. Three corollaries of Theorems 1.1 and 1.2

Using Theorems 1.1 and 1.2, we obtain the following three corollaries.

Corollary 5.1. *Let X be an admissible complete* CAT(1) *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$, R_f the resolvent of f, and x an element of X. Then the following hold.*

(i) *The set $\operatorname{argmin}_X f$ is nonempty if and only if $\{R_f^n x\}$ is spherically bounded and $\sup_n d(R_f^{n+1}x, R_f^n x) < \pi/2$;*

(ii) *if $\operatorname{argmin}_X f$ is nonempty, then $\{R_f^n x\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$ and there exists a positive real number C such that*

$$f(R_f^n x) - \inf f(X) \leq \frac{C}{n}\big(1 - \cos d(u, x_1)\big)$$

for all $u \in \operatorname{argmin}_X f$ and $n \in \mathbb{N}$.

Proof. Set $\lambda_n = 1$ for all $n \in \mathbb{N}$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and $x_{n+1} = R_{\lambda_n f} x_n$ for all $n \in \mathbb{N}$. Then we have $x_n = R_f^{n-1} x$ and $\sum_{k=1}^{n} \lambda_k = n$ for all $n \in \mathbb{N}$. Thus Theorems 1.1 and 1.2 imply the conclusion. □

Remark 5.2. The part (i) of Corollary 5.1 is related to [15, (i) of Theorem 7.1], where it is shown that $\operatorname{argmin}_X f$ is nonempty if and only if there exists $w \in X$ such that

$$\limsup_{n\to\infty} d(R_f y, R_f^n w) < \frac{\pi}{2}$$

for all $y \in X$. On the other hand, the former part of (ii) is a refinement of [15, (ii) of Theorem 7.1], where it is additionally assumed that

$$\limsup_{n\to\infty} d(R_f y, y_n) < \frac{\pi}{2}$$

whenever $\{y_n\}$ is a sequence in X which is Δ-convergent to $y \in X$.

Corollary 5.3. *Let X be a nonempty closed convex admissible subset of a Hilbert sphere (S_H, ρ_{S_H}), both f and $\{\lambda_n\}$ the same as in Theorem 1.1, and $\{x_n\}$ a sequence defined by $x_1 \in X$ and*

$$x_{n+1} = \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{\lambda_n} \tan \rho_{S_H}(y, x_n) \sin \rho_{S_H}(y, x_n) \right\}$$

for all $n \in \mathbb{N}$. Then the following hold.

(i) *The set $\operatorname{argmin}_X f$ is nonempty if and only if $\{x_n\}$ is spherically bounded and $\sup_n \rho_{S_H}(x_{n+1}, x_n) < \pi/2$;*

(ii) *if $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$ and there exists a positive real number C such that*

$$f(x_{n+1}) - \inf f(X) \le \frac{C}{\sum_{k=1}^{n} \lambda_k} \bigl(1 - \cos \rho_{S_H}(u, x_1)\bigr)$$

for all $u \in \operatorname{argmin}_X f$ and $n \in \mathbb{N}$.

Proof. Since $(X, \rho_{S_H}|_{X\times X})$ is an admissible complete CAT(1) space, Theorems 1.1 and 1.2 imply the conclusion. □

Corollary 5.4. *Let κ be a positive real number, X a complete $\mathrm{CAT}(\kappa)$ space such that $d(v, v') < \pi/(2\sqrt{\kappa})$ for all $v, v' \in X$, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$, $\{\lambda_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\{x_n\}$ a sequence defined by $x_1 \in X$ and*

$$x_{n+1} = \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{\lambda_n} \tan\bigl(\sqrt{\kappa} d(y, x_n)\bigr) \sin\bigl(\sqrt{\kappa} d(y, x_n)\bigr) \right\}$$

for all $n \in \mathbb{N}$. Then the following hold.

(i) *The set $\operatorname{argmin}_X f$ is nonempty if and only if*

$$\inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) < \frac{\pi}{2\sqrt{\kappa}} \quad \text{and} \quad \sup_n d(x_{n+1}, x_n) < \frac{\pi}{2\sqrt{\kappa}};$$

(ii) *if $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$ and there exists a positive real number C such that*

$$f(x_{n+1}) - \inf f(X) \le \frac{C}{\sum_{k=1}^{n} \lambda_k} \Bigl(1 - \cos\bigl(\sqrt{\kappa} d(u, x_1)\bigr)\Bigr)$$

for all $u \in \operatorname{argmin}_X f$ and $n \in \mathbb{N}$.

Proof. Since (X, d) is a complete $\mathrm{CAT}(\kappa)$ space if and only if $(X, \sqrt{\kappa} d)$ is a complete CAT(1) space, Theorems 1.1 and 1.2 imply the conclusion. □

Note added in proof

We finally note that Theorem 1.1 and the part (ii) of Theorem 1.2 were announced in the talk [21] based on [15,16] and the present paper. On the other hand, Espínola and Nicolae [9] studied the proximal point algorithm and the splitting proximal point algorithm for convex functions in CAT(κ) spaces with positive κ. The Δ-convergence result in the part (ii) of Corollary 5.4 was also found in [9]. In [7], the authors in this paper and the authors in [9] confirmed that there is the overlapping stated above and that these two papers were independently written.

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Y. Kimura
Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: yasunori@is.sci.toho-u.ac.jp

F. Kohsaka
Department of Mathematical Sciences, Tokai University, Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan
E-mail address: f-kohsaka@tsc.u-tokai.ac.jp