Linear and Nonlinear Analysis
Volume 3, Number 1, 2017, 19–26

Yokohama Publishers
ISSN 2188-8167

LINEAR EXTENSION MAPPINGS 2

DAVID YOST

Abstract. We study linear extension theorems in analysis and their applications. In particular, we examine the existence of large families of subspaces admitting linear Hahn-Banach extension operators, in duals of non-separable Asplund spaces. We also recall the extension theorem for 2-summing operators, and some of its applications to classical function spaces.

This paper continues our survey [41], which we will refer to as Part 1, so we number our results contiguously from there, rather than beginning afresh. We focus on particular aspects of the extension problem; for more general surveys, see [7] (which relates the Hahn-Banach Theorem to some questions in logic) or [8]. Another important topic which we do not address is the Whitney extension problem: can a function defined on a fixed subset of $\mathbb{R}^n$ be extended to a smooth function defined on all of $\mathbb{R}^n$, and can such extensions be chosen in a way that makes the resulting extension operator linear? For some idea about the vast literature on this, see for example [6, 16, 17] and the references therein.

Theorem 9 gave a partial answer to the question: when does there exist a linear selection for the Hahn-Banach Theorem? That is, given a subspace M of a Banach space X, can we find a continuous, linear mapping $L : M^* \to X^*$ such that Lf is always an extension (preferably but not necessarily norm preserving) of f? Another partial answer is given by the observation that such an extension exists for any complemented subspace. The following easy exercise gives a weak converse to that.

Lemma 10. *If $L : M^* \to X^*$ is a linear extension operator, and $R : X^* \to M^*$ is the restriction operator, then $P = LR : X^* \to X^*$ is a projection with range equal to $L(M^*)$, kernel equal to M^0 and $\|P\| = \|L\|$. In particular, M^* is isomorphic to the range of a projection on X^*.*

Clearly this holds in every Hilbert space. The existence of rich families of operators on Hilbert spaces, in particular of projections onto arbitrary closed subspaces, is well known, and their importance widely understood. In fact, every operator on a Hilbert space is a finite linear combination of projections [27]. But in general Banach spaces, the collection of operators may be much poorer; in some examples, the Calkin algebra is even finite-dimensional [39].

Unsurprisingly, the existence of a linear extension operator is impossible in general. Recall the Blaschke-Kakutani characterisation [2, Chap. 12], which asserts

2010 *Mathematics Subject Classification.* 46A22, 46B22, 46B25, 47B10.
Key words and phrases. Extension, linear selection, Hahn-Banach, Asplund space.

that if every 2-dimensional subspace of a given Banach space is the range of a contractive projection, then the Banach space is isometric to Hilbert space. This easily implies (as observed in [36]) that if such an extension operator exists for every subspace M of X, then X needs to be a Hilbert space.

One of the best known results about large families of projections on non-Hilbert spaces, more precisely long sequences of projections, is the Amir-Lindenstrauss Theorem [3] concerning weakly compactly generated spaces. These are Banach spaces containing a weakly compact subset, whose linear span is dense. We will not state their results here, but emphasize that the weak compactness of this subset was used in an essential way, via an ingenious application of Tikhonov's theorem.

Tacon [38] attempted to apply the Amir-Lindenstrauss techniques to other Banach spaces, replacing the weak compactness of a generating set by weak* compactness of the unit ball of the dual. This meant that the projections could only be constructed in the dual space, not in the original space. In view of Lemma 10, it is not surprising that his proof proceeded by finding subspaces of the original space admitting linear extension operators. This is not possible in every Banach space, and shortly we will discuss the technical obstacles to doing this, and the extra assumptions made to overcome them.

As announced in Part 1, there is always another subspace, close to the given subspace in some sense, which admits such a linear extension operator. The case $F = \{0\}$ of the following result is due to Heinrich and Mankiewicz [19], and was already stated as Theorem 9 in Part 1. A simpler proof was given later by Sims and Yost [36], and subsequently generalised [35, Proposition 2] to involve the subspace of X^*. But the first result of this sort is due to Tacon [38, Lemma 5]. Indeed the only essential difference between his result and Theorem 11 is his extraneous hypothesis that X is smooth (i.e. that for every norm one $x \in X$, there is a *unique* support functional $f_x \in X^*$ with $\|f_x\| = f_x(x) = 1$).

Theorem 11. *If M is a separable subspace of a Banach space X, and F is a separable subspace of X^*, then there is another separable subspace N of X, which contains M, and a linear mapping $L : N^* \to X^*$ such that Lf is a norm preserving extension of f for each $f \in N^*$, and $L(N^*)$ contains F.*

Abrahamsen [1] has recently offered another improvement, showing that L can also be chosen to be an almost isometric Hahn-Banach extension operator. We refer to [1] for the definition of this interesting concept, and to the references therein for its applications.

Heinrich and Mankiewicz proved Theorem 9 as a tool for the non-linear classification of Banach spaces. For example, they showed that if two Banach spaces are Lipschitz homeomorphic, then each is (linearly) isomorphic to a subspace of the bidual of the other. For recent developments in this area, see [18] and [4, Chapters 7 to 10]. Here we will pursue a different application.

But first, let us mention another extension result, due to Lindenstrauss [25], which is also relevant to the non-linear classification of Banach spaces. A simpler proof of this, based on the existence of invariant means on abelian semigroups, was given later by Pełczyński [30, pp. 61-62]. (We note that [30] also contains a comprehensive study of results related to the Borsuk-Dugundji Theorem ([5, 13];

Theorem 8 in Part 1). Another proof of the Borsuk-Dugundji Theorem is given in [20], and some interesting recent developments can be found in [9].)

Theorem 12. *Suppose that a closed subspace M of a Banach space X has the property that there exists a retract $R : X \to M$ and $\varepsilon > 0, \delta > 0$ so that for all $x, y \in X$ we have $\|Rx - Ry\| < \varepsilon$ whenever $\|x - y\| < \delta$. (In particular, this holds if M is the range of a uniformly continuous retract.) Then there is a continuous linear extension operator (not necessarily norm preserving) $L : M^* \to X^*$.*

As an example of this phenomenon, Lindenstrauss showed that c_0 is the range of a Lipschitz retract on ℓ_∞. It remains unknown whether every Banach space is the range of a uniform or Lipschitz retract on its bidual.

Returning to Theorem 11: finding one extension as just described opens the possibility of repeating this procedure again and again. It is noteworthy that Tacon was doing this with linear extension operators a full decade before the appearance of [19]. *Without loss of generality, we may suppose that N strictly contains M, and that $L(N^*)$ strictly contains F.* Denote our original subspaces by M_0 and F_0, the new subspace N given by this theorem as M_1, the extension operator as L_1, and set $F_1 = L_1(M_1^*)$. Trying to apply the theorem again gives us another separable subspace M_2 strictly containing M_1, and an extension operator $L_2 : M_2^* \to X^*$. However, we have an obstacle: we cannot ensure that the range of L_2 contains F_1, because we do not know whether F_1 is separable. We do care about the subspace F_1, because we want to apply Lemma 10 later on to obtain projections. Separability is used in an essential way in the proof of Theorem 11.

The simplest way around this difficulty is to assume that every separable subspace of X has separable dual. (Tacon made a slightly stronger assumption, which we will discuss shortly.) This is equivalent to assuming that X is an *Asplund space*, i.e. that every continuous convex function $X \to \mathbb{R}$ is Frechet differentiable on a dense G_δ set. This defines an important class of Banach spaces; for an introduction to them, see [11, §1.5], [14] or [40]. It is routine to check that any subspace of an Asplund space is also an Asplund space. It is also known that the *density character* (i.e. the minimum cardinality of a dense subset) of any Asplund space is equal to the density character of its dual.

So we assume now that X is an Asplund space, and carry on. Since M_1^* is separable, so is its continuous image F_1. Theorem 11 ensures that M_2 and L_2 also satisfy $F_2 = L_2(M_2^*) \supset F_1$. The Asplund assumption now guarantees separability of F_2, so we can do it again. We get another separable subspace M_3 strictly containing M_2, and an extension operator $L_3 : M_3^* \to X^*$, whose range F_3 is separable. Continuing, we find nice strictly increasing sequences of separable subspaces $M_n \subset X$ and $F_n \subset X^*$, and extension operators $L_n : M_n^* \to X^*$ with range F_n.

What next? It is natural to put $M_\omega = \overline{\bigcup_{n=0}^\infty M_n}$ and $F_\omega = \overline{\bigcup_{n=0}^\infty F_n}$. It is not necessary to apply Theorem 11 at this stage. Since the unit ball of $B(M_\omega^*, X^*)$ is compact in the weak* operator topology, we may take L_ω to be any limit point of the sequence $L_n R_n$.

Keep going: we get further subspaces $M_{\omega+1}, M_{\omega+2}, M_{\omega+3} \ldots$ and extension operators $L_{\omega+1}, L_{\omega+2}, L_{\omega+3} \ldots$. Applying the observation in the preceding paragraph at limit ordinals and the theorem at successor ordinals gives a long sequence of subspaces M_α and extension operators L_α, for $0 \leq \alpha \leq \omega_1$, all but the last ones being separable/having separable range.

But M_{ω_1} might not be separable; can we continue to apply the theorem? Yes, once we notice that the full strength of the separability hypothesis was not really used. The word "separable" can be replaced by "of density character at most κ" for any infinite cardinal κ. No essential changes are required for the proof. For clarity, we state explicitly this general version of Theorem 11 [36, Lemma 3].

Proposition 13. *Let κ be any infinite cardinal. If M is a subspace, with density character at most κ, of a Banach space X, and F is a subspace, with density character at most κ, of X^*, then there is another subspace N of space X, with density character at most κ, which contains M, and a linear norm preserving extension mapping $L : N^* \to X^*$ whose range $L(N^*)$ contains F.*

We have thus sketched the essential ideas of the proof of the following result [36, Theorem 4]. Some detailed, but not too onerous, book-keeping is required. It must be noted that this result is only informative for non-separable spaces. If X is separable, we can always take $M_1 = X$ and $L_1 = I$.

Proposition 14. *Let X be an Asplund space. Then there is a long sequence of subspaces M_α and linear extension mappings $L_\alpha : M_\alpha^* \to X^*$, indexed by ordinal numbers $\alpha \leq \mu$, where μ is the density character of X, so that*

(i) *$M_\alpha \subset M_\beta$ whenever $\alpha < \beta$,*
(ii) *M_α contains a dense subset no larger than α,*
(iii) *$L_\alpha(M_\alpha^*) \subset L_\beta(M_\beta^*)$ whenever $\alpha < \beta$,*
(iv) *$M_\alpha = \overline{\cup_{\beta<\alpha} M_\beta}$ whenever α is a limit ordinal,*
(v) *$M_\mu = X$ and L_μ is the identity operator.*

This result might look good, but it is actually quite unsatisfactory. The next thing we want to do is to construct projections on the dual space, and for this we consider the restriction mappings $R_\alpha : X^* \to M_\alpha^*$. From Lemma 10, $P_\alpha = L_\alpha R_\alpha$ is a contractive projection. However, serious applications of the preceding theorem require the mapping $\alpha \mapsto P_\alpha$ to be continuous, at least from the ordinal topology to the weak operator topology. This is equivalent to requiring $L_\alpha(M_\alpha^*) = \overline{\cup_{\beta<\alpha} L_\beta(M_\beta^*)}$ whenever α is a limit ordinal, and this does *not* follow routinely from previous arguments. Tacon [38] showed that this is true, when X has the property that is now referred to as *very smooth.* This means not only that X is smooth, but also that the support mapping $x \mapsto f_x$ is continuous, from the norm topology on X to the weak topology on X^*. He effectively showed that such spaces are Asplund [38, Lemma 6].

Fabian and Godefroy [15] finally proved the desired conclusion, only assuming that X is an Asplund space. Their argument depended on work of Jayne and Rogers [22, Theorem 8] and Simons [34, Lemma 2]; the latter is rather deep. We present the result with slightly informal but hopefully more suggestive notation.

Theorem 15. *Supppose Y is a dual space with the Radon-Nikodým Property, i.e. $Y = X^*$ for some Asplund space X. Then Y can be expressed as a transfinite direct sum*

$$Y = \oplus_\alpha (P_{\alpha+1} - P_\alpha)(Y),$$

where each summand $(P_{\alpha+1} - P_\alpha)(Y)$ has strictly smaller density character than Y.

This provides a powerful tool for extending results about separable spaces to non-separable spaces, by transfinite induction on the density character. Applications include renorming theorems, the existence of uncountable bases, the Mazur intersection property and the relationship between norm and Borel structures [14, Chapters 6 and 8], [11, Chapter 6]. In particular, it was shown in [15] that every dual space with the Radon-Nikodým Property has an equivalent locally uniformly convex norm.

The assumption that Y is a dual space is essential in Theorem 15. It was shown in [32] that there is a Banach space with the Radon-Nikodým Property, and a separable subspace which is not contained in any complemented separable subspace. As far as we know, the following questions are still open.

Problem 16. *Does there exist a Banach space with Radon-Nikodým Property but without any complemented infinite-dimensional separable subspaces, or without an equivalent locally uniformly convex norm?*

The strongest conclusions in this area are obtained when the assumption of weak compact generating is also made. Much work has been done to understand the minimal structure which will guarantee the existence of these long sequences of projections, leading to the concepts of projectional generator [14, Chapter 6] and projectional skeleton [23].

Stegall [37, p. 270 ff] found a more elementary proof of Theorem 15, avoiding the use of Simons' Lemma. For a modern viewpoint on this, using the idea of projectional skeleton, see [10].

p-summing operators

In this brief section, we highlight two more interesting extension theorems [28, 31]. Although they have no relationship with the preceding discussion, we believe they should be better known. They depend on the concepts of type, cotype and p-summing, which arise naturally in many situations. We then present some immediate applications to the geometry of Banach spaces.

Recall that: a sequence (x_n) in a Banach space
is called p-summable if $\sum \|x_n\|^p$ is finite; and
it is called weakly p-summable if $\sum |f(x_n)|^p$ is finite for every $f \in X^*$.
An operator is called p-summing if it sends every weakly p-summable sequence to a p-summable sequence.

This concept originated with Pietsch [31]; for a comprehensive account of their theory, see [21] or [12]. In Part 1, we were interested in replacing $\mathbb{R}$ by a Banach space in the statement of the Hahn-Banach Theorem; we needed to have some restriction on the new range space in order to have a valid theorem. The next result, the 2-Summing Extension Theorem, imposes no restriction at all on the range space; instead we need a strong restriction on the operator itself.

Theorem 17. *If $t : M \to Y$ is a 2-summing operator defined on a linear subspace M of a normed space X, then there exists a 2-summing linear extension $T : X \to Y$ of t to the whole space X, with the same 2-summing norm.*

The p-summing norm of an operator T is defined in the natural way, as

$$\inf\{\sum \|Tx_n\|^p : \sum |f(x_n)|^p \leq 1 \text{ for every } f \in X^* \text{with } \|f\| \leq 1\}.$$

Theorem 17 is not stated explicitly in [31], although it is attributed to Pietsch in [29]. It is a reasonably straightforward consequence of the arguments in [31, Satz 16]; see either [12, Theorem 4.15] or [21, Theorem 5.9] for details. The proof of this result relies (amongst other things) on the fact that subspaces of Hilbert spaces are complemented, and so does not work for p-summing operators for other values of p.

We present just two applications of this, to the relationships between classical function spaces. The first is included in [29, Proposition 3].

Corollary 18. *If a Banach space X contains $\ell_1(\Gamma)$ as a subspace, then it has $\ell_2(\Gamma)$ as a quotient space.*

Sketch Proof. There is a quotient mapping $q : \ell_1(\Gamma) \to \ell_2(\Gamma)$; send the collection of basis vectors e_γ to a dense subset of the unit ball of $\ell_2(\Gamma)$ and extend by linearity. A reformulation of Grothendieck's inequality [12, Theorem 1.13] ensures that q is 1-summing. Since $1 < 2$, we see that q is 2-summing, [31, Satz 5] or [21, 3.3], and so admits an extension $Q : X \to \ell_2(\Gamma)$, which is clearly also a quotient mapping.

The next result was first proved by different methods by Rosenthal [33, p. 203, Remark].

Corollary 19. *The Banach space ℓ_∞ has a non-separable Hilbert space as a quotient.*

Proof. There is a quotient map $\ell_1 \to C[0,1]$, hence $C[0,1]^*$ is isometric to a subspace of ℓ_∞. Considering the point evaluations on elements of $[0,1]$, we see that $C[0,1]^*$ contains $\ell_1(\Gamma)$ as a subspace, where $\Gamma = [0,1]$. So ℓ_∞ also contains $\ell_1(\Gamma)$ and the previous result is applicable.

Finally we present a result of Maurey [28], where an extension is guaranteed by imposing a condition also on the domain space.

Recall that a Banach space X has type 2 if there is a constant C such that, for any finite collection $x_1, \ldots, x_n$, we have

$$\mathrm{Ave}_\pm \left\| \sum_{i=1}^n \pm x_i \right\|^2 \leq C \sum_{i=1}^n \|x_i\|^2.$$

Here $\mathrm{Ave}_\pm$ denotes the average over all 2^n choices of sign. Likewise, a Banach space X has cotype 2 if there is a constant C such that, for any finite collection $x_1, \ldots, x_n$, we have

$$\mathrm{Ave}_\pm \left\| \sum_{i=1}^n \pm x_i \right\|^2 \geq C \sum_{i=1}^n \|x_i\|^2.$$

Clearly both properties pass to subspaces. It is of course useful to define type and cotype p for other values of p; we refer to [12, Chapter 11] for the extensive theory. Here we just recall that an L_p space has type 2 if and only if $2 \leq p < \infty$; and it has cotype 2 if and only if $1 \leq p \leq 2$. Now we can state Maurey's Extension Theorem [28].

Theorem 20. *Let X be a Banach space with type 2, and Y a Banach space with cotype 2. If $t : M \to Y$ is an operator defined on a linear subspace M of X, then there exists an extension $T : X \to Y$ of t to the whole space X, and which factors through a Hilbert space. More precisely, there is a Hilbert space H, and operators $U : X \to H$, $V : H \to Y$, with $T = VU$ and $\|U\|\|V\| \leq C\|t\|$, where C is a constant depending only on X and Y.*

This is a generalisation of Kwapień's Theorem [24], which asserts that any operator from a type 2 space to a cotype 2 space factors through a Hilbert space. An immediate consequence of that and the Lindenstrauss-Tzafriri Theorem [26] is that (isomorphically) only Hilbert spaces can have both type 2 and cotype 2.

An interesting direct consequence of Maurey's Theorem is that a Hilbert space is complemented in any superspace of type 2.

References

[1] T. A. Abrahamsen, *Linear extensions, almost isometries, and diameter two,* Extracta Math. **30** (2015), 135–151.

[2] D. Amir, Characterizations of inner product spaces. *Operator Theory: Advances and Applications,* **20**. Birkhäuser Verlag, Basel, 1986.

[3] D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. (2) **88** (1968), 35–46.

[4] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis. Vol. 1. *American Mathematical Society Colloquium Publications,* **48.** American Mathematical Society, Providence, RI, 2000.

[5] K. Borsuk, *Über Isomorphie der Funktionalräume.* Bull. Int. Acad. Polon. Sci. (1933), 1–10.

[6] A. Brudnyi and Y. Brudnyi, *Metric spaces with linear extensions preserving Lipschitz condition*, Amer. J. Math. **129** (2007), 217–314.

[7] G. Buskes, *The Hahn-Banach theorem surveyed.* Disser. Math. (Rozprawy Mat.) **327** (1993).

[8] B. L. Chalmers, M.I. Ostrovskii and B. Shekhtman, *Hahn-Banach operators: a review.* J. Comput. Anal. Appl. **5** (2003), 11–24.

[9] C. Correa and D. V. Tausk, *Extension property and complementation of isometric copies of continuous functions spaces*, Results Math. **67** (2015), 445–455.

[10] M. Cúth and M. Fabian, *Projections in duals to Asplund spaces made without Simons' lemma.* Proc. Amer. Math. Soc. **143** (2015), 301–308.

[11] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64. Longman Scientific & Technical, John Wiley & Sons, Inc., New York, 1993.

[12] J. Diestel, H. Jarchow, and A. Tonge, Absolutely summing operators. *Cambridge Studies in Advanced Mathematics*, **43**. Cambridge University Press, Cambridge, 1995.

[13] J. Dugundji, *An extension of Tietze's theorem,* Pacific J. Math. **1** (1951), 353–367.

[14] M. Fabian, Gâteaux differentiability of convex functions and topology. Weak Asplund spaces, *Canadian Mathematical Society Series of Monographs and Advanced Texts*, John Wiley & Sons, Inc., New York, 1997.

[15] M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity,* Studia Math. **91** (1988), 141–151.

[16] C. L. Fefferman, *Whitney's extension problems and interpolation of data,* Bull. Amer. Math. Soc. **46** (2009), 207–220

[17] L. Frerick, E. Jordá, and J. Wengenroth, *Whitney extension operators without loss of derivatives*, Rev. Mat. Iberoam. **32** (2016), 377-390.

[18] G. Godefroy, G. Lancien and V. Zizler, *The non-linear geometry of Banach spaces after Nigel Kalton,* Rocky Mountain J. Math. **44** (2014), 1529–1583.

[19] S. Heinrich and P. Mankiewicz, *Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces,* Studia Math. **73** (1982), 225–251.

[20] G. J. O. Jameson, *An elementary proof of the Arens and Borsuk extension theorems,* J. London Math. Soc. (2) **14** (1976), 364–368.

[21] G. J. O. Jameson, *Summing and Nuclear Norms in Banach Space Theory*, London Mathematical Society Student Texts, vol. 8, Cambridge University Press, Cambridge, 1987.

[22] J. E. Jayne and C. A. Rogers, *Borel selectors for upper semicontinuous set-valued maps,* Acta Math. **155** (1985), 41–79.

[23] W. Kubiś, *Banach spaces with projectional skeletons,* J. Math. Anal. Appl. **350** (2009), 758–776.

[24] S. Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients,* Studia Math. **44** (1972), 583–595.

[25] J. Lindenstrauss, *On nonlinear projections in Banach spaces,* Michigan Math. J. **11** (1964), 263–287.

[26] J. Lindenstrauss and L. Tzafriri, *On the complemented subspaces problem,* Israel J. Math. **9** (1971) 263–269.

[27] K. Matsumoto, *Selfadjoint operators as a real span of 5 projections,* Math. Japon. **29** (1984), 291–294.

[28] B. Maurey, *Un théorème de prolongement,* C. R. Acad. Sci. Paris Sér. A **279** (1974), 329–332.

[29] R. I. Ovsepian and A. Pełczyński, *On the existence of a fundamental total and bounded biorthogonal sequence in every separable Banach space, and related constructions of uniformly bounded orthonormal systems in L_2,* Studia Math. **54** (1975), 149–159.

[30] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions,* vol. 58 Disser. Math. Rozprawy Mat. 1968.

[31] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen,* Studia Math. **28** (1966/1967), 333–353.

[32] A.M. Plichko and D. Yost, *The Radon-Nikodm property does not imply the separable complementation property,* J. Funct. Anal. **180** (2001), 481–487.

[33] H. P. Rosenthal, *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L_p(\mu) \to L_r(\nu)$,* J. Functional Analysis **4** (1969), 176–214.

[34] S. Simons, *A convergence theorem with boundary,* Pacific J. Math. **40** (1972), 703–708.

[35] B. Sims and D. Yost, *Banach spaces with many projections*, Proc. Centre Math. Anal. Austral. Nat. Univ. **14** (1986) 335–342.

[36] B. Sims and D. Yost, *Linear Hahn-Banach extension operators,* Proc. Edinburgh Math. Soc. (2) **32** (1989), 53–57.

[37] C. Stegall, *Spaces of Lipschitz functions on Banach spaces,* in: Functional analysis (Essen, 1991), Lecture Notes in Pure and Appl. Math., **150**, Dekker, New York, 1994, pp. 265–278.

[38] D. G. Tacon, *The conjugate of a smooth Banach space,* Bull. Austral. Math. Soc. **2** (1970), 415–425.

[39] M. Tarbard, *Hereditarily indecomposable, separable $\mathcal{L}_\infty$ Banach spaces with ℓ_1 dual having few but not very few operators,* J. Lond. Math. Soc. (2) **85** (2012), 737–764.

[40] D. Yost, *Asplund spaces for beginners*, Acta. Univ. Carolinae Math.Phys. **34** (1993), 159–177.

[41] D. Yost, *Linear Extension Mappings 1*, in: Proceedings of NACA 2015, (The 9 International Conf. on Nonlinear Convex Analysis, Chiang Rai, January 21–25, 2015), S. Dhompongsa, N. Petrot, S. Plubtieng, S. Suanthai (eds),Yokohama pu blishers, Yokohama, 2016, pp. 333–340.

Manuscript received 1 April 2016
revised 21 December 2016

David Yost
Centre for Informatics & Applied Optimization, Federation University, PO Box 663, Ballarat, Vic. 3353, Australia
E-mail address: `d.yost@federation.edu.au`