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POINTWISE MULTIPLIERS ON SEVERAL FUNCTION SPACES – A SURVEY –

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Dedicated to Professor Mikio Kato on his 68th birthday

Abstract. This is a survey for pointwise multipliers on function spaces. We present a description on basic properties of pointwise multipliers and give the characterization of pointwise multipliers on Lorentz, Orlicz, Musielak-Orlicz, Morrey, BMO and Campanato spaces together with several examples. Moreover, we give some applications of pointwise multipliers on BMO.

1. Introduction

The pointwise multiplication (product) of two functions is a basic operation. In this survey we focus the characterization of functions g which are operators from a function space to another function space as maps $f \mapsto fg$. More precisely, let $\Omega = (\Omega, \mu)$ be a complete σ-finite measure space. We denote by $L^0(\Omega)$ the set of all measurable functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. Then $L^0(\Omega)$ is a linear space under the usual sum and scalar multiplication. Let $E_1, E_2 \subset L^0(\Omega)$ be subspaces. We say that a function $g \in L^0(\Omega)$ is a pointwise multiplier from E_1 to E_2, if the pointwise multiplication fg is in E_2 for any $f \in E_1$. We denote by $\mathrm{PWM}(E_1, E_2)$ the set of all pointwise multipliers from E_1 to E_2. We abbreviate $\mathrm{PWM}(E, E)$ to $\mathrm{PWM}(E)$. For example,

$$\mathrm{PWM}(L^0(\Omega)) = L^0(\Omega).$$

The pointwise multipliers are basic operators on function spaces and thus the characterization of pointwise multipliers is not only interesting itself but also sometimes very useful to other study.

The space of pointwise multipliers between function spaces is natural to consider between Banach or quasi-Banach ideal spaces (i.e. complete quasi-normed spaces with the lattice property, see (2.8) for its definition), but there are natural spaces like BMO spaces and Campanato spaces which are not Banach ideal spaces. In our considerations we want to cover also these spaces. To do this we give basic properties on pointwise multipliers in the second section in which we do not always assume the lattice property.

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For $p \in (0,\infty]$, $L^p(\Omega)$ denotes the usual Lebesgue space equipped with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, d\mu(x)\right)^{1/p}, \text{ if } p \neq \infty,$$
$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

Then $L^p(\Omega)$ is a complete quasi-normed space (quasi-Banach space). If $p \in [1,\infty]$, then it is a Banach space. It is well known as Hölder's inequality that

$$\|fg\|_{L^{p_2}(\Omega)} \le \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_3}(\Omega)},$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0,\infty]$, $i = 1,2,3$. This shows that

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \supset L^{p_3}(\Omega).$$

Conversely, we can show the reverse inclusion by using the uniform boundedness theorem or the closed graph theorem. That is,

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) = L^{p_3}(\Omega). \tag{1.1}$$

If $p_1 = p_2 = p$, then

$$\mathrm{PWM}(L^{p}(\Omega)) = L^{\infty}(\Omega). \tag{1.2}$$

Proofs of (1.1) and (1.2) are in Maligranda and Persson [35, Proposition 3 and Theorem 1]. See Theorem 3.1 in Section 3 for (1.2).

In this paper we give proofs of (1.1), (1.2) and their generalization, see Section 3. By the generalization we have, for example,

$$\mathrm{PWM}(L^{p_1,\infty}(\Omega), L^{p_2,\infty}(\Omega)) = L^{p_3,\infty}(\Omega),$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0,\infty]$, $i = 1,2,3$, where $L^{p_i,\infty}(\Omega)$ are the weak Lebesgue spaces.

On the other hand, the results in Section 3 cannot be applied to $\mathrm{BMO}(\Omega)$. In 1976 Stegenga [64] and Janson [15] gave the characterization of $\mathrm{PWM}(\mathrm{BMO}(\Omega))$ for $\Omega = \mathbb{T}$ and $\Omega = \mathbb{T}^n$, respectively. After then the history is the following:

- Nakai and Yabuta [55] (1985) for $\Omega = \mathbb{R}^n$.
- Nakai and Yabuta [56] (1997) and Nakai [41] (1997) for spaces of homogeneous type (Ω, d, μ).
- Liu and Da. Yang [29] (2014) for $(\mathbb{R}^n, \mu)$ with the Gauss measure.
- Nakai and Sadasue [51] (2014) for probability spaces $(\Omega, \mathcal{F}, P)$.
- Li, Nakai and Do. Yang [26] (preprint) for $(\mathbb{R}^n, \mu)$ with non-doubling measures.

The result of $\mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n))$ was used by Lerner [23] to show the boundedness of the Hardy-Littlewood maximal operator on generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent.

To characterize the pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ in [55] we introduced the function space $\mathrm{BMO}_\phi(\mathbb{R}^n)$ with

$$\phi(x,r) = \frac{1}{\log(r + 1/r + |x|)}, \quad x \in \mathbb{R}^n, \ r \in (0,\infty).$$

This function space was extended to generalized Morrey-Campanato spaces with variable growth condition, see [37, 38, 41–44, 46, 47, 56], etc.

The paper is organized as follows. In Section 2 we give basic properties of pointwise multipliers in which we do not always assume the lattice property. In Section 3 we give the characterization of pointwise multipliers on function spaces with the lattice property. We state the results on pointwise multipliers on Lorentz, Orlicz, Musielak-Orlicz and Morrey spaces in Sections 4 and BMO and Campanato spaces in Section 5. Then we give some applications of pointwise multipliers on BMO in Section 6. Finally, we give some results for Besov and Triebel-Lizorkin spaces in Section 7 with the definition of pointwise multiplication fg for tempered distributions f and g.

2. Basic properties of pointwise multipliers

In this paper we always assume that the function spaces $E \subset L^0(\Omega)$ have the following property, see Kantorovich and Akilov's book [18, pages 94] in which this property is referred to as $\operatorname{supp} E = \Omega$:

(2.1) If a measurable subset $\Omega_1 \subset \Omega$ satisfies that

$$\mu(\{x \in \Omega : f(x) \neq 0\} \setminus \Omega_1) = 0 \text{ for every } f \in E, \qquad \text{then } \mu(\Omega \setminus \Omega_1) = 0.$$

Recall that a subspace $E \subset L^0(\Omega)$ (which is not necessary to be equipped with a norm or quasi-norm) is an ideal space if

$$f \in E,\ h \in L^0(\Omega),\ |h| \le |f| \text{ a.e.} \implies h \in E. \tag{2.2}$$

The following is a basic property of ideal spaces.

Proposition 2.1 ([18, Corollary 2 on page 95])**.** *If E is an ideal space, then from the assumption* (2.1) *it follows that there exists a partition $\{\Omega_m\}$ of Ω such that each Ω_m is a measurable set with finite measure and that the characteristic function of Ω_m is in E.*

For pointwise multipliers on ideal spaces we have the following simple propositions.

Proposition 2.2. *Let $E \subset L^0(\Omega)$ be a subspace. Then E is an ideal space if and only if $L^\infty(\Omega) \subset \mathrm{PWM}(E)$.*

Proof. Let $L^\infty(\Omega) \subset \mathrm{PWM}(E)$. If $f \in E$, $h \in L^0(\Omega)$, $|h| \le |f|$ a.e., then

$$g(x) = \begin{cases} h(x)/f(x), & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0 \end{cases}$$

is in $L^\infty(\Omega)$ and $h = fg \in E$. Conversely, let E be an ideal space. If $g \in L^\infty(\Omega)$ and $\alpha = \|g\|_{L^\infty(\Omega)}$, then, for all $f \in E$, $|fg| \le |\alpha f|$ a.e. and $\alpha f \in E$. Hence $fg \in E$. That is, $L^\infty(\Omega) \subset \mathrm{PWM}(E)$. □

Proposition 2.3. *Let $E \subset L^0(\Omega)$ be a subspace. If E is an ideal space, then $\mathrm{PWM}(L^\infty(\Omega), E) = E$.*

Proof. Let $g \in \mathrm{PWM}(L^\infty(\Omega), E)$. Then $g = 1g \in E$, since the constant function 1 is in $L^\infty(\Omega)$. Conversely, let $g \in E$. Then, for all $f \in L^\infty(\Omega)$ with $\alpha = \|f\|_{L^\infty(\Omega)}$, $|fg| \le |\alpha g|$ a.e. and $\alpha g \in E$. Hence $fg \in E$ by the ideal property. That is, $g \in \mathrm{PWM}(L^\infty(\Omega), E)$. $\square$

Next, recall that $\|\cdot\|$ is a quasi-norm on a linear space E if there exists $\kappa \in [1, \infty)$ such that, for all $f, g \in E$ and scalars α,

(i) $\|f\| \ge 0$, $\|f\| = 0$ if and only if $f = 0$,
(ii) $\|\alpha f\| = |\alpha|\|f\|$,
(iii) $\|f + g\| \le \kappa(\|f\| + \|g\|)$.

If $\kappa = 1$, then $\|\cdot\|$ is a norm. For any quasi-norm $\|\cdot\|$, there exists a metric $d(f, g)$ depending only on $f - g$ such that

$$d(f, g) \le \|f - g\|^p \le 2d(f, g), \tag{2.3}$$

where $0 < p \le 1$, $\kappa = 2^{(1/p)-1}$. Actually, letting

$$d_0(f) = \inf\left\{\sum_j \|f_j\|^p : f = \sum_j f_j(\text{finite sum})\right\} \tag{2.4}$$

and $d(f, g) = d_0(f - g)$, we have that

$$d_0(f) \le \|f\|^p \le 2d_0(f) \tag{2.5}$$

and (2.3). See for example [21, Theorem 1.12 on page 12] and [17, Theorem 1.2 on page 5]. If we take

$$\|f\|_1 = \inf\left\{\left(\sum_j \|f_j\|^p\right)^{1/p} : f = \sum_j f_j(\text{finite sum})\right\},$$

then $\|f\|_1$ is p-subadditive and this is just the Aoki-Rolewicz theorem (see [59, pages 92–93], [17, page 7], [60, pages 95-96], [32, page 86] and [33, pages 6–8]. In the last two publications it is even written history of this theorem). We also note the following properties on d_0 in (2.4):

(i) $d_0(f) \ge 0$, $d_0(f) = 0 \Leftrightarrow f = 0$.
(ii) $d_0(f) = d_0(-f)$.
(iii) $d_0(f + g) \le d_0(f) + d_0(g)$.
(iv) $\lim_{\alpha_n \to 0} d_0(\alpha_n f) = 0$, $\lim_{\|f_n\| \to 0} d_0(\alpha f_n) = 0$.

Let $E_1, E_2 \subset L^0(\Omega)$ be quasi-normed spaces. Then we say that $g \in \mathrm{PWM}(E_1, E_2)$ is a bounded operator if there exists a positive constant β such that

$$\|fg\|_{E_2} \le \beta\|f\|_{E_1} \text{ for all } f \in E_1.$$

In this case, we define the operator norm of $g \in \mathrm{PWM}(E_1, E_2)$ as

$$\|g\|_{\mathrm{Op}} = \inf\{\beta > 0 : \|fg\|_{E_2} \le \beta\|f\|_{E_1} \text{ for all } f \in E_1\}.$$

Note that $g \in \mathrm{PWM}(E_1, E_2)$ is a bounded operator if and only if g is a continuous operator from E_1 to E_2.

Let $E \subset L^0(\Omega)$ be a quasi-normed space, which is not necessary to be an ideal space. In this paper we say that E has the subsequence property if

$$(2.6)\quad f_j \to f \text{ in } E\ (j\to\infty) \quad \Longrightarrow \quad \exists \{f_{j(k)}\} \text{ (subsequence) s.t. } f_{j(k)} \to f \text{ a.e. } (k \to \infty).$$

The following is a basic property of the pointwise multipliers.

Theorem 2.4. *Let $E_1, E_2 \subset L^0(\Omega)$ be complete quasi-normed spaces, which are not necessary to be ideal spaces. If both E_1 and E_2 have the subsequence property* (2.6)*, then each $g \in \mathrm{PWM}(E_1, E_2)$ is a bounded operator.*

Proof. Let $g \in \mathrm{PWM}(E_1, E_2)$, and let $f_j \to f$ in E_1 and $f_j g \to h$ in E_2. Then there exists a subsequence $\{f_{j(k)}\}$ such that $f_{j(k)} \to f$ a.e. Moreover, since $f_{j(k)} g \to h$ in E_2, there exists a subsequence $\{f_{j(k(\ell))} g\}$ such that $f_{j(k(\ell))} g \to h$ a.e. On the other hand, $f_{j(k)} g \to fg$ a.e. Then $h = fg$ a.e. That is, g has a closed graph. Hence it is a bounded operator by the closed graph theorem. See for example [72, Theorem 1 on page 79] for the closed graph theorem. □

Let $A \subset \Omega$ be a measurable set. Recall that $f_j \to f$ in measure on A if, for all $\epsilon > 0$,

$$\mu(\{x \in A : |f_j(x) - f(x)| > \epsilon\}) \to 0 \ (j \to \infty).$$

Let $\Omega = \bigcup_m \Omega_m$ with $\mu(\Omega_m) < \infty$, $m = 1, 2, \dots$, and let $f_j \in L^0(\Omega)$, $j = 1, 2, \dots$. It is known from the measure theory that, if $f_j \to f$ in measure on Ω_m for each m, then there exists a subsequence $f_{j(k)}$ such that $f_{j(k)} \to f$ a.e. Hence, we have the following theorem which doesn't use the ideal property.

Theorem 2.5. *Let $E \subset L^0(\Omega)$ be a quasi-normed space. Assume that there exists a sequence of subsets $\Omega_m \subset \Omega$ with $\Omega = \bigcup_m \Omega_m$ and $\mu(\Omega_m) < \infty$, $m = 1, 2, \dots$, such that, for any sequence of functions $f_j \in E$, $j = 1, 2, \dots$,*

$$(2.7)\qquad f_j \to 0 \text{ in } E \quad \Longrightarrow \quad f_j \to 0 \text{ in measure on } \Omega_m \text{ for each } m.$$

Then E has the subsequence property (2.6)*.*

Corollary 2.6. *Let $E_i \subset L^0(\Omega)$, $i = 1, 2$, be complete quasi-normed spaces. Assume that there exists a sequence of subsets $\Omega_m \subset \Omega$ with $\Omega = \bigcup_m \Omega_m$ and $\mu(\Omega_m) < \infty$, $m = 1, 2, \dots$, such that both E_1 and E_2 have the property* (2.7)*. Then all $g \in \mathrm{PWM}(E_1, E_2)$ are bounded operators.*

We say that a quasi-normed space E has the lattice property if the following holds:

$$(2.8)\qquad f \in E,\ h \in L^0(\Omega),\ |h| \le |f| \text{ a.e.} \quad \Longrightarrow \quad h \in E,\ \|h\|_E \le \|f\|_E.$$

We don't use the lattice property in Theorem 2.5 or Corollary 2.6. On the other hand, using the lattice property, we have the following theorem which is an extension of [18, Theorem 1 on page 96] which is for normed spaces, see at the end of this section for the proof.

Theorem 2.7. *Let a quasi-normed space $E \subset L^0(\Omega)$ have the lattice property* (2.8)*. For any sequence of functions $f_j \in E$, $j = 1, 2, \dots$, if $f_j \to 0$ in E, then $f_j \to 0$ in measure on every measurable set with finite measure.*

Corollary 2.8. *If E_1 and E_2 are quasi-Banach ideal spaces (that is, complete quasi-normed spaces with the lattice property), then all $g \in \mathrm{PWM}(E_1, E_2)$ are bounded operators.*

Remark 2.1. Theorems 2.5 and 2.7 and Corollaries 2.6 and 2.8 are all applicable to Lebesgue, Orlicz, Musielak-Orlicz, Lorentz and Morrey spaces, etc. However, for BMO and Campanato spaces, we can use only Theorem 2.5 and Corollary 2.6. Actually, BMO and Campanato spaces don't have the lattice property (2.8), while they have the property (2.7), see Proposition 5.5 in which we can take balls with radius 2^m as Ω_m.

For a quasi-normed space $E \subset L^0(\Omega)$ with the lattice property and for a positive constant θ, let $E^\theta = \{f \in L^0(\Omega) : |f|^\theta \in E\}$ and $\|f\|_{E^\theta} = (\||f|^\theta\|_E)^{1/\theta}$. Then E^θ is a quasi-normed space with the lattice property. If E is a normed space and $\theta \ge 1$, then E^θ is also a normed space. It is easy to show that the following proposition holds.

Proposition 2.9 ([35, (g) on page 326])**.** *Let $E_i \subset L^0(\Omega)$ $(i = 1, 2, 3)$ be quasi-normed spaces with the lattice property. If $\mathrm{PWM}(E_1, E_2) = E_3$ and $0 < \theta < \infty$, then $\mathrm{PWM}(E_1^\theta, E_2^\theta) = E_3^\theta$.*

At the end of this section we prove Theorem 2.7. The proof is almost same as Kantorovich and Akilov's book [18, page 96].

Proof of Theorem 2.7. By Proposition 2.1 there exists a partition $\{\Omega_m\}_m$ of Ω such that $\mu(\Omega_m) < \infty$ and $\chi_{\Omega_m} \in E$. If $f_j \to f$ in measure on Ω_m for each m, then, for any measurable set A with finite measure, $f_j \to f$ in measure on A. Actually, $\mu(A) = \sum_m \mu(A \cap \Omega_m) < \infty$ implies that, for each $\epsilon > 0$ and $\delta > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{m>m_0} \mu(A \cap \Omega_m) < \delta,$$

and there exists $j_0 \in \mathbb{N}$ such that, if $j \ge j_0$ then

$$\mu(\{x \in \Omega_m : |f_j(x) - f(x)| > \epsilon\}) < \delta/m_0 \quad m = 1, 2, \dots, m_0.$$

Therefore,

$$\begin{aligned}\mu(\{x \in A &: |f_j(x) - f(x)| > \epsilon\}) \\ &\le \sum_{m\le m_0} \mu(\{x \in \Omega_m : |f_j(x) - f(x)| > \epsilon\}) + \sum_{m>m_0} \mu(A \cap \Omega_m) < 2\delta.\end{aligned}$$

Moreover, since $f_j \to f$ in E implies $f_j\chi_{\Omega_m} \to f\chi_{\Omega_m}$ in E, we may assume that $\mu(\Omega) < \infty$ and the function 1 is in E.

In the following, we prove that $f_j \to f$ in E and $f_j \not\to f$ in measure on Ω yield a contradiction. By passing to a subsequence if necessary, we may also assume that there exist numbers $\epsilon, \delta > 0$ such that the following conditions are satisfied for all $j \in \mathbb{N}$:

$$\mu(\{x \in \Omega : |f_j(x) - f(x)| > \epsilon\}) \ge \delta, \tag{2.9}$$

$$\|f_j - f\| < \epsilon/2^j. \tag{2.10}$$

Write

$$B_j = \{x \in \Omega : |f_j(x) - f(x)| > \epsilon\}, \quad B = \bigcap_{j=1}^{\infty} \bigcup_{k=j+1}^{\infty} B_k. \tag{2.11}$$

By (2.9) we have

$$\mu(B_j) \geq \delta \ (j \in \mathbb{N}), \quad \mu(B) \geq \delta. \tag{2.12}$$

By (2.10), bearing in mind that $\epsilon\chi_{B_j} \leq |f_j - f|$, we have

$$\|\chi_{B_j}\| < 1/2^j. \tag{2.13}$$

We now introduce the sets

$$C_{j,s} = \bigcup_{k=j+1}^{j+s} (B_k \cap B) \subset B.$$

Then for every $j \in \mathbb{N}$ the sequence $\{C_{j,s}\}_s$ is non-decreasing and, by (2.11),

$$B = \bigcup_{s=1}^{\infty} C_{j,s}.$$

Hence for each $j \in \mathbb{N}$ there exists a suffix s_j such that

$$\mu(B \setminus C_{j,s_j}) < 1/2^{j+1}. \tag{2.14}$$

Write

$$D_j = \bigcap_{k=j+1}^{\infty} C_{k,s_k}.$$

Then $D_j \subset B$ and the sequence $\{D_j\}_j$ is clearly non-decreasing. Since

$$B \setminus D_j = B \setminus \bigcap_{k=j+1}^{\infty} C_{k,s_k} = \bigcup_{k=j+1}^{\infty} (B \setminus C_{k,s_k}),$$

we see from (2.14) that

$$\mu(B \setminus D_j) = \sum_{k=j+1}^{\infty} \mu(B \setminus C_{k,s_k}) < 1/2^j.$$

Therefore $\mu(B \setminus (\bigcup_{j=1}^{\infty} D_j)) = 0$. Let d_0 and p be as in (2.4) and (2.5). By (2.13), when $k > j$ we have

$$\begin{aligned} \left\|\chi_{D_j}\right\|^p \leq \left\|\chi_{C_{k,s_k}}\right\|^p \leq \left\| \sum_{i=k+1}^{k+s_k} \chi_{B_i} \right\|^p \leq 2\, d_0 \left(\sum_{i=k+1}^{k+s_k} \chi_{B_i} \right) \\ \leq 2 \sum_{i=k+1}^{k+s_k} d_0\,(\chi_{B_i}) \leq 2 \sum_{i=k+1}^{k+s_k} \|\chi_{B_i}\|^p < \frac{2}{1-1/2^p} \frac{1}{2^{p(k+1)}}, \end{aligned}$$

so that $\chi_{D_j} = 0$, that is, $\mu(D_j) = 0$. Therefore $\mu(B) = 0$, contradicting (2.12). $\square$

3. Pointwise multipliers on function spaces with the lattice property

Recall that E is a Banach (or quasi-Banach) ideal space if E is a complete normed (or quasi-normed) space with the lattice property (2.8).

General properties of the space of pointwise multipliers $\mathrm{PWM}(E_1, E_2)$ for Banach ideal spaces E_1, E_2 were studied by Maligranda and Persson [35, pp. 326–330], and Kolwicz, Leśnik and Maligranda [19, pp. 879–880]. For symmetric Banach spaces (that is, rearrangement invariant spaces) E_1, E_2 they were proved by Kolwicz, Leśnik and Maligranda [19, pp. 881–887]. For ideal Banach spaces, the following result is known and it was proved by Maligranda and Persson [35].

Theorem 3.1 ([35])**.** *If $E \subset L^0(\Omega)$ is a Banach ideal space, then*

$$\mathrm{PWM}(E) = L^\infty(\Omega).$$

Remark 3.1. (i) In [35] they also proved that $\mathrm{PWM}(E, [E]) = L^\infty(\Omega)$, where $[E]$ is the maximal normed extension of E in the sense of Abramovič. That is,

$$[E] = \Big\{ f \in L^0(\Omega) : \|f\|_{[E]} = \sup\{\|g\|_E : g \in E, 0 \le g \le |f|\} < \infty \Big\}.$$

(ii) The proof of Theorem 3.1 adapts to ideal quasi-Banach spaces.

As mentioned in Section 1, Hölder's inequality

$$\|fg\|_{L^{p_2}(\Omega)} \le \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_3}(\Omega)}$$

implies

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \supset L^{p_3}(\Omega),$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0, \infty]$, $i = 1, 2, 3$. The reverse inclusion can be shown by using the uniform boundedness theorem or the closed graph theorem. The following theorem is an extension of

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) = L^{p_3}(\Omega)$$

for general quasi-normed spaces by using the uniform boundedness theorem.

Theorem 3.2 (cf. [39])**.** *Let $E_i \subset L^0(\Omega)$, $i = 1, 2, 3$. Suppose that E_1 is a complete quasi-normed space with the lattice property, that E_2 is a quasi-normed space with the lattice property, that E_3 is a quasi-normed space with the monotone completeness property;*

$$f_j \in E \ (j = 1, 2 \dots),\ f_j \ge 0,\ f_j \nearrow f \ \text{a.e. and} \ \sup_j \|f_j\|_E < \infty, \quad \Longrightarrow \quad f \in E_3. \tag{3.1}$$

Suppose also that there exists $\Omega_m \subset \Omega$, $m = 1, 2, \dots$, such that

$$\mu(\Omega_m) < \infty,\ \Omega_1 \subset \Omega_2, \subset \dots,\ \Omega = \bigcup_m \Omega_m, \tag{3.2}$$

and that

$$\{f \in L^\infty(\Omega) : \{f \ne 0\} \subset \Omega_m \ \text{for some} \ m\} \subset E_3. \tag{3.3}$$

If

$$\mathrm{PWM}(E_1, E_2) \supset E_3,$$

and if all $g \in E_3$ *are bounded operators as elements in* $\mathrm{PWM}(E_1, E_2)$ *and*

$$C^{-1}\|g\|_{E_3} \le \|g\|_{\mathrm{Op}} \le C\|g\|_{E_3}$$

holds for all $g \in E_3$ *and some positive constant* C *independent of* g*, then*

$$\mathrm{PWM}(E_1, E_2) = E_3.$$

Remark 3.2. (i) If E_3 has the lattice property, then by Proposition 2.1 we can take Ω_m, $m = 1, 2, \dots$, which satisfy (3.2) and (3.3).
(ii) From the conclusion of Theorem 3.2 it follows that, for every $g \in \mathrm{PWM}(E_1, E_2)$, its operator norm is equivalent to $\|g\|_{E_3}$.

Proof of Theorem 3.2. Let $g \in \mathrm{PWM}(E_1, E_2)$. By the lattice property of E_2, the real and imaginary parts of g and their positive and negative parts are also in $\mathrm{PWM}(E_1, E_2)$. Then it is enough to prove the case $g \ge 0$, since E_3 is a linear space. Let

$$g_j(x) = \begin{cases} 0 & x \in \Omega \setminus \Omega_j, \\ g(x) & x \in \Omega_j \text{ and } |g(x)| \le j, \\ j & x \in \Omega_j \text{ and } |g(x)| > j, \end{cases} \quad \text{for } j = 1, 2, \dots.$$

Then, for any $f \in E_1$, we have $fg \in E_2$ and $|fg_j| \le |fg|$. It follows from the lattice property that

$$\|fg_j\|_{E_2} \le \|fg\|_{E_2} \quad (j = 1, 2, \dots).$$

By the uniform boundedness theorem and the assumption, we have

$$\sup_j \|g_j\|_{\mathrm{Op}} < \infty \quad \text{and} \quad \sup_j \|g_j\|_{E_3} < \infty.$$

Therefore the monotone completeness property implies that g is in E_3. Note that a complete quasi-normed space is a complete quasi metric space and it cannot express as a countable union of closed subsets each of which does not contain non-empty open set. Thus we can apply the uniform boundedness theorem to our case, see [72, Theorem 1 in page 68]. ☐

Corollary 3.3. *Let* $E \subset L^0(\Omega)$ *be a quasi-Banach space. Then*

$$\mathrm{PWM}(E) = L^\infty(\Omega) \quad \text{and} \quad \|g\|_{\mathrm{Op}} = \|g\|_{L^\infty(\Omega)}, \tag{3.4}$$

if and only if E *has the lattice property* (2.8)*.*

The first part of the following proof is the same as in [22, 35, 39].

Proof of Corollary 3.3. Assume that E is a quasi-Banach space with the lattice property. Note that $L^\infty(\Omega)$ has the monotone completeness property and contains all finitely simple functions. Let $g \in L^\infty(\Omega)$. Then by the lattice property of E we have that $g \in \mathrm{PWM}(E)$ and $\|g\|_{\mathrm{Op}} \le \|g\|_{L^\infty(\Omega)}$. If $g = 0$, then $\|g\|_{\mathrm{Op}} = \|g\|_{L^\infty(\Omega)} = 0$. If $g \in L^\infty(\Omega)$ and $g \ne 0$, for any η such that $0 < \eta < \|g\|_{L^\infty(\Omega)}$, choose Ω_m in Proposition 2.1 such that $\chi_{\Omega_m} \in E$ and that

$$0 < \mu(A_\eta \cap \Omega_m) < \infty, \quad A_\eta = \{x \in \Omega : |g(x)| > \eta\},$$

and let h_η be the characteristic function of $A_\eta \cap \Omega_m$. Then $h_\eta \in E$ by the lattice property. From the inequality $\eta|h_\eta| \le |h_\eta g|$ a.e. it follows that

$$\eta \le \frac{\|h_\eta g\|_E}{\|h_\eta\|_E} \le \|g\|_{\mathrm{Op}}.$$

This shows that $\|g\|_{\mathrm{Op}} = \|g\|_{L^\infty(\Omega)}$ for all $g \in L^\infty(\Omega)$. Then $\mathrm{PWM}(E) = L^\infty(\Omega)$ by Theorem 3.2.

Conversely, assume (3.4). Let $f \in E$, $h \in L^0(\Omega)$, $|h| \le |f|$ a.e. Then, letting

$$g(x) = \begin{cases} h(x)/f(x), & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0, \end{cases}$$

we have $\|g\|_{\mathrm{Op}} = \|g\|_{L^\infty(\Omega)} \le 1$ and

$$\|h\|_E = \|fg\|_E \le \|g\|_{\mathrm{Op}}\|f\|_E \le \|f\|_E.$$

Hence, E has the lattice property. □

A quasi-normed space E has the Fatou property if

$$(3.5) \quad f_j \in E\ (j = 1, 2 \dots),\ f_j \ge 0,\ f_j \nearrow f \text{ a.e. and } \sup_j \|f_j\|_E < \infty, \\ \implies \quad f \in E \text{ and } \|f\|_E \le \sup_j \|f_j\|_E.$$

Theorem 3.4. *Let $E_i \subset L^0(\Omega)$, $i = 1, 2, 3$, be quasi-Banach ideal spaces. Assume that E_3 has the Fatou property* (3.5)*. If generalized Hölder's inequality*

$$(3.6) \qquad \|fg\|_{E_2} \le C\|f\|_{E_1}\|g\|_{E_3}, \quad f \in E_1,\ g \in E_3,$$

holds, and if, for any finitely simple function g contained in E_3 with $g \neq 0$, there exists $f \in E_1$ with $f \neq 0$ such that

$$(3.7) \qquad \|fg\|_{E_2} \ge C'\|f\|_{E_1}\|g\|_{E_3},$$

then

$$\mathrm{PWM}(E_1, E_2) = E_3 \quad \textit{and} \quad C'\|g\|_{E_3} \le \|g\|_{\mathrm{Op}} \le C\|g\|_{E_3}.$$

We give two kinds of proofs. The first proof uses Theorem 3.2 (the uniform boundedness theorem) and the second uses Corollary 2.8 (the closed graph theorem).

The first proof of Theorem 3.4. Generalized Hölder's inequality (3.6) shows that

$$\mathrm{PWM}(E_1, E_2) \supset E_3, \quad \|g\|_{\mathrm{Op}} \le C\|g\|_{E_3}.$$

For $g \in E_3$, take a sequence of finitely simple functions $g_j \in E_3$, $j = 1, 2, \dots$, such that $g_j \nearrow |g|$. Then by (3.7), the lattice property of E_2 and the Fatou property of E_3 we have

$$C'\|g_j\|_{E_3} \le \|g_j\|_{\mathrm{Op}} \le \|g\|_{\mathrm{Op}} \quad \text{and} \quad C'\|g\|_{E_3} \le \|g\|_{\mathrm{Op}} \quad \text{for all } g \in E_3.$$

Therefore, we have the conclusion by Theorem 3.2. □

The second proof of Theorem 3.4. Generalized Hölder's inequality (3.6) shows that

$$\mathrm{PWM}(E_1,E_2)\supset E_3,\quad \|g\|_{\mathrm{Op}}\le C\|g\|_{E_3}.$$

Conversely, let $g\in\mathrm{PWM}(E_1,E_2)$. Then g is a bounded operator by Corollary 2.8. By Proposition 2.1 we can take a sequence of finitely simple functions $g_j\in E_3$, $j=1,2,\dots$, such that $g_j\nearrow|g|$ a.e. Then by (3.7) and the lattice property of E_2 we have

$$C'\|g_j\|_{E_3}\le\|g_j\|_{\mathrm{Op}}\le\|g\|_{\mathrm{Op}}.$$

Then by the Fatou property of E_3 we have $g\in E_3$ and

$$C'\|g\|_{E_3}\le\|g\|_{\mathrm{Op}}.$$

Therefore we have the conclusion. □

Example 3.1. Let $p_i\in(0,\infty]$ $(i=1,2,3)$ and $1/p_1+1/p_3=1/p_2$. Then

$$\mathrm{PWM}(L^{p_1}(\Omega),L^{p_2}(\Omega))=L^{p_3}(\Omega)\quad\text{and}\quad\|g\|_{\mathrm{Op}}=\|g\|_{L^{p_3}(\Omega)}. \tag{3.8}$$

Actually, by Hölder's inequality, we have

$$\|fg\|_{L^{p_2}(\Omega)}\le\|f\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_3}(\Omega)},\quad f\in L^{p_1}(\Omega),\ g\in L^{p_3}(\Omega).$$

If $p_3\ne\infty$, then for any $g\in L^{p_3}(\Omega)$ with $g\ne0$, take $f=|g|^{p_3/p_1}$ ($f=1$ if $p_1=\infty$). Then $f\in L^{p_1}(\Omega)$ and

$$\|fg\|_{L^{p_2}(\Omega)}=\|f\|_{L^{p_1}(\Omega)}\|g\|_{L^{p_3}(\Omega)}.$$

If $p_3=\infty$ and $p_1=p_2$, then taking h_η as in the proof of Corollary 3.3, we have

$$\|h_\eta g\|_{L^{p_2}(\Omega)}\ge\eta\|h_\eta\|_{L^{p_1}(\Omega)}=\frac{\eta}{\|g\|_{L^\infty(\Omega)}}\|h_\eta\|_{L^{p_1}(\Omega)}\|g\|_{L^\infty(\Omega)},$$

for any η with $0<\eta<\|g\|_{L^\infty(\Omega)}$. Then we have (3.8) by Theorem 3.4. (In case of $p_3=\infty$ and $p_1=p_2$ the conclusion (3.8) also follows from Corollary 3.3).

4. Lorentz, Orlicz and Morrey spaces

In this section we state results on Lorentz, Orlicz, Musielak-Orlicz and Morrey spaces without proofs. These function spaces have the lattice property.

4.1. **Lorentz spaces.** Let $\Omega=(\Omega,\mu)$ be a complete σ-finite measure space. For $f\in L^0(\Omega)$ and $s,t\in[0,\infty)$, let

$$\mu(f,s)=\mu(\{x\in\Omega:|f(x)|>s\}),$$
$$f^*(t)=\inf\{s>0:\mu(f,s)\le t\}.$$

Definition 4.1 (Lorentz space)**.** For $p,q\in(0,\infty]$, let $L^{p,q}(\Omega)$ be the set of all $f\in L^0(\Omega)$ such that $\|f\|_{L^{p,q}(\Omega)}<\infty$, where

$$\|f\|_{L^{p,q}(\Omega)}=\begin{cases}\left(\displaystyle\int_0^\infty t^{(q/p)-1}(f^*(t))^q\,dt\right)^{1/q}, & 0<p\le\infty,\ 0<q<\infty,\\ \sup\limits_{t>0}t^{1/p}f^*(t), & 0<p\le\infty,\ q=\infty.\end{cases}$$

Then $\|\cdot\|_{L^{p,q}(\Omega)}$ is a quasi-norm and thereby $L^{p,q}(\Omega)$ is a complete quasi-normed linear space with the lattice property and the Fatou property. If $p=\infty$ and $0<q<\infty$, then $L^{p,q}(\Omega)=\{0\}$. Note that

$$L^{p,p}(\Omega)=L^p(\Omega) \quad \text{and} \quad \|f\|_{L^{p,p}(\Omega)}=\|f\|_{L^p(\Omega)}, \qquad 0<p\le\infty.$$

By the inequality $(fg)^*(t)\le f^*(t/2)g^*(t/2)$ and Hölder's inequality we have the following proposition:

Proposition 4.1. *Let $p_i\in(0,\infty)$, $q_i\in(0,\infty]$ $(i=1,2,3)$. If $1/p_1+1/p_3=1/p_2$ and $1/q_1+1/q_3=1/q_2$, then*

$$\|fg\|_{L^{p_2,q_2}(\Omega)}\le 2^{1/p_2}\|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_3,q_3}(\Omega)}.$$

If $1/p_1+1/p_3=1/p_2$ and $p_1/q_1=p_2/q_2=p_3/q_3$, then, for $g\in L^{p_3,q_3}(\Omega)$, setting $f=|g|^{p_3/p_1}$, we have

$$\|fg\|_{L^{p_2,q_2}(\Omega)}=\|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_3,q_3}(\Omega)}.$$

By Theorem 3.4 we have the following theorem:

Theorem 4.2 ([40])**.** *Let $p_i\in(0,\infty)$ and $q_i\in(0,\infty]$. If $1/p_1+1/p_3=1/p_2$ and $p_1/q_1=p_2/q_2=p_3/q_3$, then*

$$\mathrm{PWM}(L^{p_1,q_1}(\Omega),L^{p_2,q_2}(\Omega))=L^{p_3,q_3}(\Omega).$$

and

$$\|g\|_{L^{p_3,q_3}(\Omega)}\le\|g\|_{\mathrm{Op}}\le 2^{1/p_2}\|g\|_{L^{p_3,q_3}(\Omega)}.$$

Remark 4.1. In the above, if $p_2\ge q_2$, then we have $\|g\|_{L^{p_3,q_3}(\Omega)}=\|g\|_{\mathrm{Op}}$, see [39].

4.2. **Orlicz and Musielak-Orlicz spaces.** Let $\bar{\varPhi}$ be the set of all functions $\Phi:[0,\infty]\to[0,\infty]$ such that

$$\lim_{t\to0+0}\Phi(t)=\Phi(0)=0 \quad \text{and} \quad \lim_{t\to\infty}\Phi(t)=\Phi(\infty)=\infty. \tag{4.1}$$

Let

$$a(\Phi)=\sup\{r\ge0:\Phi(r)=0\}, \quad b(\Phi)=\inf\{r\ge0:\Phi(r)=\infty\}.$$

Definition 4.2. A function $\Phi\in\bar{\varPhi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is nondecreasing on $[0,\infty)$ and convex on $[0,b(\Phi))$, and

$$\lim_{r\to b(\Phi)-0}\Phi(r)=\Phi(b(\Phi))\ (\le\infty).$$

We denote by $\varPhi_Y$ the set of all Young functions. Any Young function is neither identically zero nor identically infinity on $(0,\infty)$.

Let $\Omega=(\Omega,\mu)$ be a complete σ-finite measure space.

Definition 4.3 (Orlicz space)**.** For a Young function Φ, let

$$L^\Phi(\Omega)=\left\{f\in L^0(\Omega):\int_\Omega\Phi(k|f(x)|)\,d\mu(x)<\infty \text{ for some } k>0\right\},$$

$$\|f\|_{L^\Phi}=\inf\left\{\lambda>0:\int_\Omega\Phi\left(\frac{|f(x)|}{\lambda}\right)d\mu(x)\le1\right\}.$$

Then $\|f\|_{L^\Phi}$ is a norm and thereby $L^\Phi(\Omega)$ is a Banach space.

For example, if $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^\Phi(\Omega) = L^p(\Omega)$. If $\Phi(r) = 0$ $(0 \le r \le 1)$ and $\Phi(r) = \infty$ $(r > 1)$, then $L^\Phi(\Omega) = L^\infty(\Omega)$.

Next we recall the generalized inverse of Young function Φ in the sense of O'Neil [58, Definition 1.2]. For a Young function Φ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \inf\{t \ge 0 : \Phi(t) > u\},$$

where $\inf \emptyset = \infty$. Then $\Phi^{-1}(u)$ is finite for all $u \in [0,\infty)$. If Φ is bijective from $[0,\infty)$ to itself, then Φ^{-1} is the usual inverse function of Φ.

Theorem 4.3 ([58])**.** *Let Φ_i be Young functions, $i = 1,2,3$. If*

$$\Phi_1^{-1}(r)\Phi_3^{-1}(r) \le C\,\Phi_2^{-1}(r) \quad \textit{for all } r > 0,$$

then

$$\|fg\|_{L^{\Phi_2}(\Omega)} \le 2C\,\|f\|_{L^{\Phi_1}(\Omega)}\|g\|_{L^{\Phi_3}(\Omega)}.$$

Description of the space of pointwise multipliers $\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega))$ for Orlicz spaces was given by Maligranda and Persson [35, pp. 332–334] (see also [31, Chapter 10, pp. 69–79]) under some conditions either on measure or on Young functions Φ_1, Φ_2. In general, result was proved by Maligranda and Nakai [34].

Theorem 4.4 ([34])**.** *Let Φ_i be Young functions, $i = 1,2,3$. If*

$$C_1\,\Phi_2^{-1}(r) \le \Phi_1^{-1}(r)\Phi_3^{-1}(r) \le C_2\,\Phi_2^{-1}(r) \quad \textit{for all } r > 0,$$

then

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_3}(\Omega)$$

and

$$C_1\|g\|_{L^{\Phi_3}(\Omega)} \le \|g\|_{\mathrm{Op}} \le 2C_2\|g\|_{L^{\Phi_3}(\Omega)}.$$

Result in Theorem 4.4 is not showing how for given Young functions Φ_1, Φ_2 we can find another Young function Φ_3 with the above equivalence. We consider the conjugate (complementary) function to Φ_1 with respect to Φ_2 by the formula

$$\Phi_2 \ominus \Phi_1(u) = \sup\{\Phi_2(tu) - \Phi_1(t) : t > 0\}, \quad u > 0.$$

In particular, if $\Phi_2(u) = u$, then $\Phi_2 \ominus \Phi_1 = \widetilde{\Phi}_1$ is the usual conjugate (complementary) function (in sense of Young) to Φ_1. This operation on the class of N-functions was defined by Ando [2, p. 180] and on the class of Young functions by O'Neil [58, p. 325] and he referred to Ando [2]. Kolwicz, Leśnik and Maligranda [19, Theorem 8] proved that under some additional assumptions on Young functions we have identification $\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_3}(\Omega)$, where $\Phi_3 = \Phi_2 \ominus \Phi_1$.

On the other hand, if we restrict supremum in the above operation to $(0,1]$, that is,

$$(\Phi_2 \ominus \Phi_1)_0(u) = \sup\{\Phi_2(tu) - \Phi_1(t) : 0 < t \le 1\}, \quad u > 0.$$

then Djakov and Ramanujan [10], in the case of Orlicz sequence spaces, showed the following identification $\mathrm{PWM}(\ell^{\Phi_1}, \ell^{\Phi_2}) = \ell^{\Phi_3}$, where $\Phi_3 = (\Phi_2 \ominus \Phi_1)_0$ without any restrictions on Young functions. Of course, the function $(\Phi_2 \ominus \Phi_1)_0$ is smaller than $\Phi_2 \ominus \Phi_1$ and it can be different than $\Phi_2 \ominus \Phi_1$.

Leśnik and Tomaszewski [25] proved recently how we should understand supremum in the definition of operation $\Phi_2 \ominus \Phi_1$ that in a non-atomic measure case we have identification $\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_2 \ominus \Phi_1}(\Omega)$.

Next we generalize Young functions to the following:

Definition 4.4. Let $\mathit{\Phi}_Y^v$ be the set of all $\Phi : \Omega \times [0,\infty] \to [0,\infty]$ such that $\Phi(x,\cdot)$ is a Young function for every $x \in \Omega$, and that $\Phi(\cdot,t)$ is measurable on Ω for every $t \in [0,\infty]$. Assume also that, for any subset $A \subset \Omega$ with finite measure, there exists $t \in (0,\infty)$ such that $\Phi(\cdot,t)\chi_A$ is integrable.

Definition 4.5. (i) Let $\mathit{\Phi}_{GY}$ be the set of all $\Phi \in \bar{\mathit{\Phi}}$ such that $\Phi((\cdot)^{1/\ell})$ is in $\mathit{\Phi}_Y$ for some $\ell \in (0,1]$.

(ii) Let $\mathit{\Phi}_{GY}^v$ be the set of all $\Phi : \Omega \times [0,\infty] \to [0,\infty]$ such that $\Phi(\cdot,(\cdot)^{1/\ell})$ is in $\mathit{\Phi}_Y^v$ for some $\ell \in (0,1]$.

For $\Phi, \Psi \in \bar{\mathit{\Phi}}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \le \Psi(t) \le \Phi(Ct) \quad \text{for all } t \in (0,\infty).$$

For $\Phi, \Psi : \Omega \times [0,\infty] \to [0,\infty]$, we also write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(x,C^{-1}t) \le \Psi(x,t) \le \Phi(x,Ct) \quad \text{for all } (x,t) \in \Omega \times (0,\infty).$$

Definition 4.6. Let $\bar{\mathit{\Phi}}_Y$, $\bar{\mathit{\Phi}}_Y^v$, $\bar{\mathit{\Phi}}_{GY}$ and $\bar{\mathit{\Phi}}_{GY}^v$ be the sets of all Φ such that $\Phi \approx \Psi$ for some Ψ in $\mathit{\Phi}_Y$, $\mathit{\Phi}_Y^v$, $\mathit{\Phi}_{GY}$ and $\mathit{\Phi}_{GY}^v$, respectively.

Lemma 4.5. *Let $\Phi \in \mathit{\Phi}_{GY}^v$. For a subset $A \subset \Omega$ with $0 < \mu(A) < \infty$, let $\Phi^A(t) = \int_A \Phi(x,t)\,d\mu(x)$. Then $\Phi^A \in \mathit{\Phi}_{GY}$.*

Definition 4.7 (Musielak-Orlicz space)**.** For a function $\Phi \in \bar{\mathit{\Phi}}_{GY}^v$, let

$$L^\Phi(\Omega) = \left\{ f \in L^0(\Omega) : \int_\Omega \Phi(x,k|f(x)|)\,d\mu(x) < \infty \text{ for some } k > 0 \right\},$$
$$\|f\|_{L^\Phi} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\}.$$

Then each function $f \in L^\Phi(\Omega)$ satisfies $|f(x)| < \infty$ a.e. $x \in \Omega$. By the assumption in Definition 4.4 all simple functions are in $L^\Phi(\Omega)$. Moreover, $\|\cdot\|_{L^\Phi(\Omega)}$ is a quasi-norm and thereby $L^\Phi(\Omega)$ is a complete quasi-normed space with the lattice property and the Fatou property. If $\Phi \in \mathit{\Phi}_Y^v$, then $\|\cdot\|_{L^\Phi}$ is a norm.

Example 4.1. Let $p = p(\cdot)$ be a variable exponent, that is, it is a measurable function defined on Ω valued in $(0,\infty]$, and let $\Phi(x,t) = t^{p(x)}$. Let $p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x)$. If $p_- > 0$, then $\Phi \in \mathit{\Phi}_{GY}^v$ and $\Phi(x,(\cdot)^{\max(1,1/p_-)}) \in \mathit{\Phi}_Y^v$. In this case we denote $L^\Phi(\Omega)$ by $L^{p(\cdot)}(\Omega)$ which is a generalized Lebesgue space with variable exponent $p(\cdot)$.

For the function spaces with variable exponent, see for example [13, 57].

Example 4.2. Let w be a weight function, that is, a measurable function defined on Ω valued in $(0,\infty)$ a.e., and $\int_A w(x)\,d\mu(x) < \infty$ for any $A \subset \Omega$ with finite measure. Let p be a variable exponent, and let

$$\Phi(x,t) = t^{p(x)} w(x).$$

If $\operatorname*{ess\,inf}_{x\in\Omega} p(x) > 0$, then $\Phi \in \mathit{\Phi}_{GY}^{v}$. In this case we denote $L^{\Phi}(\Omega)$ by $L^{p(\cdot)}_{w}(\Omega)$.

Example 4.3. Let p be a variable exponent, and let

$$\Phi(x,t) = \begin{cases} 1/\exp(1/t^{p(x)}), & t \in [0,1], \\ \exp(t^{p(x)}), & t \in (1,\infty]. \end{cases}$$

If $\operatorname*{ess\,inf}_{x\in\Omega} p(x) > 0$, then $\Phi \in \bar{\mathit{\Phi}}_{Y}^{v}$. In this case we denote $L^{\Phi}(\Omega)$ by $\exp(L^{p(\cdot)})(\Omega)$.

Next proposition is a generalized Hölder's inequality for Musielak-Orlicz spaces, which can be proven in the same way as in O'Neil's paper [58].

Proposition 4.6. *Let $\Phi_i \in \bar{\mathit{\Phi}}_{GY}^{v}$, $i = 1,2,3$. Assume that there exists a constant $C > 0$ such that*

$$\Phi_1^{-1}(x,t)\Phi_3^{-1}(x,t) \le C\,\Phi_2^{-1}(x,t) \quad \textit{for} \quad (x,t) \in \Omega \times (0,\infty). \tag{4.2}$$

If $f \in L^{\Phi_1}(\Omega)$ and $g \in L^{\Phi_3}(\Omega)$, then $fg \in L^{\Phi_2}(\Omega)$ and

$$\|fg\|_{L^{\Phi_2}} \le C' \|f\|_{L^{\Phi_1}} \|g\|_{L^{\Phi_3}},$$

where C' is a positive constant dependent only on Φ_i, $i = 1,2,3$, and C.

We define three subsets of Young functions $\mathcal{Y}^{(i)}$ $(i = 1,2,3)$ as

$$\begin{aligned} \mathcal{Y}^{(1)} &= \{\Phi \in \mathit{\Phi}_Y : b(\Phi) = \infty\}, \\ \mathcal{Y}^{(2)} &= \{\Phi \in \mathit{\Phi}_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) = \infty\}, \\ \mathcal{Y}^{(3)} &= \{\Phi \in \mathit{\Phi}_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) < \infty\}. \end{aligned}$$

Then we have the following theorem:

Theorem 4.7 ([48])**.** *Let $\Phi_i \in \bar{\mathit{\Phi}}_{GY}^{v}$, $i = 1,2,3$. Assume that*

$$\frac{1}{C}\Phi_2^{-1}(x,t) \le \Phi_1^{-1}(x,t)\Phi_3^{-1}(x,t) \le C\Phi_2^{-1}(x,t) \quad \textit{for} \quad (x,t) \in \Omega \times (0,\infty). \tag{4.3}$$

Assume also that there exists $\Psi_3 \in \mathit{\Phi}_{GY}^{v}$ such that $\Phi_3 \approx \Psi_3$ and $\Psi_3^{A}((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for some $\ell \in (0,1]$ and for any $A \subset \Omega$ with $0 < \mu(A) < \infty$, where $\Psi_3^{A}(t) = \int_A \Psi_3(x,t)\,d\mu(x)$. Then

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_3}(\Omega).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega))$ is comparable to $\|g\|_{L^{\Phi_3}}$.

Example 4.4. Let p_i be variable exponents, $i = 1,2,3$, and

$$\Omega_\infty = \{x \in \Omega : p_3(x) = \infty\}.$$

Assume that $\operatorname*{ess\,inf}_{x\in\Omega} p_i(x) > 0$, $i = 1, 2, 3$, $\operatorname*{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)}. \tag{4.4}$$

Then

$$\mathrm{PWM}(L^{p_1(\cdot)}(\Omega), L^{p_2(\cdot)}(\Omega)) = L^{p_3(\cdot)}(\Omega) \quad \text{and} \quad \|g\|_{\mathrm{Op}} \sim \|g\|_{L^{p_3(\cdot)}(\Omega)}.$$

Example 4.5. Let p_i be variable exponents, w_i be weight functions, $i = 1, 2, 3$, and

$$\Omega_\infty = \{x \in \Omega : p_3(x) = \infty\}.$$

Assume that $\operatorname*{ess\,inf}_{x\in\Omega} p_i(x) > 0$, $i = 1, 2, 3$, $\operatorname*{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)}, \quad w_1(x)^{1/p_1(x)} w_3(x)^{1/p_3(x)} = w_2(x)^{1/p_2(x)}. \tag{4.5}$$

Then

$$\mathrm{PWM}(L^{p_1(\cdot)}_{w_1}(\Omega), L^{p_2(\cdot)}_{w_2}(\Omega)) = L^{p_3(\cdot)}_{w_3}(\Omega) \quad \text{and} \quad \|g\|_{\mathrm{Op}} \sim \|g\|_{L^{p_3(\cdot)}_{w_3}(\Omega)}.$$

Example 4.6. Let p_i be variable exponents, $i = 1, 2, 3$, and

$$\Omega_\infty = \{x \in \Omega : p_3(x) = \infty\}.$$

Assume that $\operatorname*{ess\,inf}_{x\in\Omega} p_i(x) > 0$, $i = 1, 2, 3$, $\operatorname*{ess\,sup}_{x\in\Omega\setminus\Omega_\infty} p_3(x) < \infty$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)}. \tag{4.6}$$

Then

$$\mathrm{PWM}(\exp(L^{p_1(\cdot)})(\Omega), \exp(L^{p_2(\cdot)})(\Omega)) = \exp(L^{p_3(\cdot)})(\Omega)$$

and

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\exp(L^{p_3(\cdot)})(\Omega)}.$$

4.3. **Morrey spaces.** Let $B(x, r) \subset \mathbb{R}^n$ be the ball with center $x \in \mathbb{R}^n$ and radius $r > 0$. That is, $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. Let $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. For a measurable set $A \subset \mathbb{R}^n$, we denote its Lebesgue measure and characteristic function by $|A|$ and χ_A, respectively.

Definition 4.8 (generalized Morrey space)**.** For $p \in (0, \infty)$ and $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, let $L_{p,\phi}(\mathbb{R}^n)$ be the set of all $f \in L^0(\mathbb{R}^n)$ such that $\|f\|_{L_{p,\phi}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{L_{p,\phi}(\mathbb{R}^n)} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(x)|^p \, dx \right)^{1/p}.$$

The supremum above is taken over all balls B in $\mathbb{R}^n$.

Then $\|\cdot\|_{L_{p,\phi}(\mathbb{R}^n)}$ is a quasi-norm and thereby $L_{p,\phi}(\mathbb{R}^n)$ is a complete quasi-normed spaces with the lattice property and the Fatou property. If $1 \le p < \infty$, then $L_{p,\phi}(\mathbb{R}^n)$ is a Banach space. If $\phi(B) = |B|^{-1/p}$, then $L_{p,\phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\phi(B) \equiv 1$, then $L_{p,\phi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $\phi(x, r) = r^{(\lambda-n)/p}$, then $L_{p,\phi}(\mathbb{R}^n)$ coincides with the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey [36]. See also [1]. That is,

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{B(x,r)} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p},$$

where the supremum is taken over all balls $B(x,r)$ in $\mathbb{R}^n$.

By Hölder's inequality we have the following proposition:

Proposition 4.8. *Let $p_i \in (0,\infty)$ $(i = 1,2,3)$. If $1/p_1 + 1/p_3 = 1/p_2$ and $\phi_1\phi_3 \le C\phi_2$, then*

$$\|fg\|_{L_{p_2,\phi_2}(\mathbb{R}^n)} \le C\|f\|_{L_{p_1,\phi_1}(\mathbb{R}^n)}\|g\|_{L_{p_3,\phi_3}(\mathbb{R}^n)}.$$

Corollary 4.9. *If $1/p_1 + 1/p_3 = 1/p_2$ and $\lambda_1/p_1 + \lambda_3/p_3 = \lambda_2/p_2$, then*

$$\|fg\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)} \le \|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_3,\lambda_3}(\mathbb{R}^n)}.$$

For functions θ_1, $\theta_2 : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we write $\theta_1 \sim \theta_2$ if there exists a positive constant C such that

$$C^{-1}\theta_1(x,r) \le \theta_2(x,r) \le C\theta_1(x,r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{4.7}$$

A function $\theta : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is almost increasing (almost decreasing) if there exists a positive constant C such that

$$\theta(x,r) \le C\theta(x,s) \quad (\theta(x,r) \ge C\theta(x,s)) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r < s. \tag{4.8}$$

If ϕ is almost decreasing, $\phi(x,r)|B(x,r)|^{1/p}$ is almost increasing and $\inf_{x\in\mathbb{R}^n} \phi(x,1) > 0$, then $\chi_B \in L_{p,\phi}(\mathbb{R}^n)$ for all balls B. Moreover, we see that $\phi(x,r) \sim \phi(x,2r)$, that is, ϕ satisfies the doubling condition. We also consider the following condition; there exists a positive constant C such that

$$C^{-1}\phi(x,r) \le \phi(y,r) \le C\phi(x,r) \quad \text{for all } x,y \in \mathbb{R}^n \text{ with } |x-y| < r. \tag{4.9}$$

Theorem 4.10 ([42])**.** *Let $p_i \in (0,\infty)$ and $\phi_i : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ $(i = 1,2,3)$. Suppose that ϕ_i is almost decreasing, $\phi_i(x,r)|B(x,r)|^{1/p_i}$ is almost increasing, $\inf_{x\in\mathbb{R}^n} \phi_i(x,1) > 0$ and ϕ_i satisfies* (4.9) *$(i = 1,2,3)$. Suppose also that $1/p_1 + 1/p_3 = 1/p_2$ and $\phi_1\phi_3 \sim \phi_2$. If $\phi_2^{p_2/p_1}/\phi_1$ is almost increasing, then*

$$\mathrm{PWM}(L_{p_1.\phi_1}(\mathbb{R}^n), L_{p_2.\phi_2}(\mathbb{R}^n)) = L_{p_3.\phi_3}(\mathbb{R}^n).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L_{p_1.\phi_1}(\mathbb{R}^n), L_{p_2.\phi_2}(\mathbb{R}^n))$ is comparable to $\|g\|_{L_{p_3.\phi_3}(\mathbb{R}^n)}$.

The above result is valid for spaces of homogeneous type (X,d,μ), see [42]. We cannot remove the almost increasingness of $\phi_2^{p_2/p_1}/\phi_1$, see [43].

For the classical Morrey spaces, we have the following theorem:

Theorem 4.11 ([43])**.** *Let $p_i \in (0,\infty)$ and $\lambda_i \in (0,n)$. Suppose that $1/p_1 + 1/p_3 = 1/p_2$ and $\lambda_1/p_1 + \lambda_3/p_3 = \lambda_2/p_2$. Then*

$\mathrm{PWM}(L^{p_1,\lambda_1}(\mathbb{R}^n), L^{p_2,\lambda_2}(\mathbb{R}^n))$

$$\begin{cases} = \{0\}, & p_1 < p_2, \\ = \{0\}, & p_1 = p_2 \ \textit{and} \ \lambda_1 < \lambda_2, \\ = L^\infty(\mathbb{R}^n), & p_1 = p_2 \ \textit{and} \ \lambda_1 = \lambda_2, \\ \supsetneq \{0\}, & p_1 = p_2 \ \textit{and} \ \lambda_1 > \lambda_2, \\ = \{0\}, & p_1 > p_2 \ \textit{and} \ n + (\lambda_1 - n)p_2/p_1 < \lambda_2, \\ = L^\infty(\mathbb{R}^n), & p_1 > p_2 \ \textit{and} \ \lambda_2 = n + (\lambda_1 - n)p_2/p_1, \\ = L^{p_3,\lambda_3}(\mathbb{R}^n), & p_1 > p_2 \ \textit{and} \ \lambda_1 \le \lambda_2 < n + (\lambda_1 - n)p_2/p_1, \\ \supsetneq L^{p_3,\lambda_3}(\mathbb{R}^n), & p_1 > p_2 \ \textit{and} \ \lambda_1 p_2/p_1 < \lambda_2 < \lambda_1, \\ \supsetneq L^{p_3}(\mathbb{R}^n), & p_1 > p_2 \ \textit{and} \ \lambda_2 = \lambda_1 p_2/p_1, \\ \supsetneq \{0\}, & p_1 > p_2 \ \textit{and} \ \lambda_2 < \lambda_1 p_2/p_1. \end{cases}$$

For Musielak-Orlicz-Morrey spaces, see [49].

5. Campanato spaces with variable growth condition

In this section we concentrate on generalized Campanato spaces on $\mathbb{R}^n$. The results are also valid for spaces of homogeneous type (X, d, μ).

5.1. Definition and connections with Morrey and Hölder spaces.

For $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r)$ be a ball centered at x and radius r, or, a cube centered at x and sidelength r. For $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ and $B = B(x, r)$, we write $\phi(B)$ in place of $\phi(x, r)$. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and for a ball (cube) B, let

$$f_B = \frac{1}{|B|} \int_B f(x)\, dx,$$

where $|B|$ is the Lebesgue measure of B.

Definition 5.1 ([44])**.** For $p \in [1, \infty)$ and $\phi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$, Campanato spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, Morrey spaces $L_{p,\phi}(\mathbb{R}^n)$ and Hölder spaces $\Lambda_\phi(\mathbb{R}^n)$ are defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$, $\|f\|_{L_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p\, dx \right)^{1/p},$$

$$\|f\|_{L_{p,\phi}(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(x)|^p\, dx \right)^{1/p},$$

$$\|f\|_{\Lambda_\phi(\mathbb{R}^n)} = \sup_{x,y \in \mathbb{R}^n,\, x \ne y} \frac{2|f(x) - f(y)|}{\phi(x, |x-y|) + \phi(y, |x-y|)}.$$

We regard $L_{p,\phi}(\mathbb{R}^n)$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ as spaces of functions modulo null-functions (that is, two functions are considered equal if they are equal almost everywhere, as usual) and $\Lambda_\phi(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$. Moreover, regarding $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$ as spaces modulo constant functions, $\|\cdot\|_{L_{p,\phi}(\mathbb{R}^n)}$, $\|\cdot\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)}$ and $\|\cdot\|_{\Lambda_\phi(\mathbb{R}^n)}$ are norms, and thereby $L_{p,\phi}(\mathbb{R}^n)$, $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$ are Banach spaces, respectively.

If $\phi(x,r)=r^\alpha$, then

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{B=B(z,r)\subset\mathbb{R}^n} \frac{1}{r^\alpha}\left(\frac{1}{|B|}\int_B |f(x)-f_B|^p\,dx\right)^{1/p}, \quad -n/p\le\alpha\le 1,$$

$$\|f\|_{L_{p,\phi}} = \sup_{B=B(z,r)\subset\mathbb{R}^n} \frac{1}{r^\alpha}\left(\frac{1}{|B|}\int_B |f(x)|^p\,dx\right)^{1/p}, \quad -n/p\le\alpha\le 0,$$

$$\|f\|_{\Lambda_\phi} = \|f\|_{\mathrm{Lip}_\alpha} = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \quad 0<\alpha\le 1,$$

which are the usual Campanato, Morrey and Hölder (Lipschitz) spaces. If $p=1$ and $\phi\equiv 1$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n)=\mathrm{BMO}(\mathbb{R}^n)$. For $\phi(x,r)=r^\alpha$, the following properties are known.

$$\begin{aligned}
\mathcal{L}_{p,\phi} &= \mathcal{L}_{1,\phi} = \mathrm{BMO}, \quad \alpha=0,\ 1\le p<\infty,\\
\mathcal{L}_{p,\phi} &= \Lambda_\phi, \quad 0<\alpha\le 1,\ 1\le p<\infty,\\
\mathcal{L}_{p,\phi}/\mathcal{C} &= L_{p,\phi}, \quad -n/p\le\alpha<0,\ 1\le p<\infty,\\
\mathcal{L}_{p,\phi}/\mathcal{C} &= L_{p,\phi} = L^p, \quad \alpha=-n/p,\ 1\le p<\infty,
\end{aligned}$$

where $\mathcal{L}_{p,\phi}/\mathcal{C}=L_{p,\phi}$ means that the Campanato space $\mathcal{L}_{p,\phi}$ modulo constant functions can be identified with the Morrey space $L_{p,\phi}$.

For $p(\cdot)\in L^0(\mathbb{R}^n;\mathbb{R})$, that is, a measurable function from $\mathbb{R}^n$ to $\mathbb{R}$, let

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

Definition 5.2 ([46])**.** Let α_* be a constant in $[0,\infty)$ and $\alpha(\cdot)\in L^0(\mathbb{R}^n;\mathbb{R})$ satisfy $0\le\alpha_-\le\alpha_+<\infty$.

(i) For $\phi(x,r)=r^{\alpha(x)}$, denote $\Lambda_\phi(\mathbb{R}^n)$ by $\mathrm{Lip}_{\alpha(\cdot)}(\mathbb{R}^n)$. In this case,

$$\|f\|_{\mathrm{Lip}_{\alpha(\cdot)}(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{2|f(x)-f(y)|}{|x-y|^{\alpha(x)}+|y-x|^{\alpha(y)}}.$$

(ii) For

$$\phi(x,r) := \begin{cases} r^{\alpha(x)}, & 0<r<1/2,\\ r^{\alpha_*}, & 1/2\le r<\infty, \end{cases} \tag{5.1}$$

denote $\Lambda_\phi(\mathbb{R}^n)$ by $\mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n)$. In this case,

$$\|f\|_{\mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n)} = \max\left\{\sup_{0<|x-y|<1/2} \frac{2|f(x)-f(y)|}{|x-y|^{\alpha(x)}+|x-y|^{\alpha(y)}},\ \sup_{|x-y|\ge 1/2} \frac{|f(x)-f(y)|}{|x-y|^{\alpha_*}}\right\}.$$

Theorem 5.1 ([44])**.** *Let $1\le p<\infty$ and $\phi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$. Assume that there exists a positive constant C such that, for all $x\in\mathbb{R}^n$ and $r\in(0,\infty)$,*

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le C\phi(x,r).$$

Then

$$\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = L_{p,\phi}(\mathbb{R}^n).$$

Moreover, if also $\phi(B) = |B|^{-1/p}$*, then*

$$\mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = L_{p,\phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

Theorem 5.2 ([45])**.** *Let* $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$*. Assume that there exist positive constants* A_i*,* $i = 1, 2, 3$*, such that, for all* $x, y \in \mathbb{R}^n$ *and* $r, s \in (0,\infty)$*,*

$$\frac{1}{A_1} \le \frac{\phi(x,s)}{\phi(x,r)} \le A_1, \quad \text{if} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{5.2}$$

$$\frac{1}{A_2} \le \frac{\phi(x,r)}{\phi(y,r)} \le A_2, \quad \text{if} \quad |x-y| \le r, \tag{5.3}$$

$$\phi(x,r) \le A_3 \phi(x,s), \qquad \text{if} \quad r < s. \tag{5.4}$$

Then

$$\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n), \quad p \in [1,\infty).$$

Theorem 5.3 ([44])**.** *Let* $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$*. Assume that* ϕ *satisfies* (5.2)–(5.4) *and that there exists a positive constant* C *such that, for all* $x \in \mathbb{R}^n$ *and* $r \in (0,\infty)$*,*

$$\int_0^r \frac{\phi(x,t)}{t}\, dt \le C\phi(x,r). \tag{5.5}$$

Then

$$\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n), \quad p \in [1,\infty).$$

We consider the local log-Hölder continuity condition;

$$|p(x) - p(y)| \le \frac{c_*}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \le \frac{1}{2},\ x, y \in \mathbb{R}^n, \tag{5.6}$$

and a log-Hölder type decay condition at infinity;

$$|p(x) - p_\infty| \le \frac{c^*}{\log(e + |x|)} \quad \text{for} \quad x \in \mathbb{R}^n, \tag{5.7}$$

where c_*, c^* and p_∞ are positive constants independent of x and y. Let

$$LH_0 = \{p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R}) : p(\cdot) \text{ satisfies (5.6)}\},$$
$$LH_\infty = \{p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R}) : p(\cdot) \text{ satisfies (5.7)}\},$$
$$LH = LH_0 \cap LH_\infty.$$

Example 5.1. For a constant $\alpha_* \in [0,\infty)$ and for $\alpha(\cdot) \in LH_0$ satisfying $0 \le \alpha_- \le \alpha_+ < \infty$, let

$$\phi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r < 1/2, \\ r^{\alpha_*}, & 1/2 \le r < \infty. \end{cases}$$

Then

$$\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n), \quad p \in [1,\infty).$$

Moreover, if $\alpha_- > 0$, then

$$\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n), \quad p \in [1,\infty).$$

Example 5.2. For $p(\cdot) \in LH$ and $(n-1)/n \le p_- \le p_+ \le 1$. Let

$$\phi(x,r) = \begin{cases} r^{\alpha(x)}, & 0 < r < 1/2, \\ r^{\alpha_*}, & 1/2 \le r < \infty, \end{cases} \quad \text{where} \quad \begin{cases} \alpha(x) = n(1/p(x) - 1), \\ \alpha_* = n(1/p_\infty - 1). \end{cases}$$

Then

$$(H^{p(\cdot)}(\mathbb{R}^n))^* = \mathcal{L}_{1,\phi}(\mathbb{R}^n).$$

Moreover, if $p_+ < 1$, then

$$(H^{p(\cdot)}(\mathbb{R}^n))^* = \mathrm{Lip}_{\alpha(\cdot)}^{\alpha_*}(\mathbb{R}^n).$$

In the above, we can take $0 < p_- \le p_+ \le 1$ instead of $(n-1)/n \le p_- \le p_+ \le 1$, by using higher dimensional Campanato spaces. For Hardy spaces with variable exponent and these facts, see Nakai and Sawano [53] (2012).

Theorem 5.4 ([46]). *Let $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ and $p(\cdot)/(p(\cdot)-1) \in LH_0$. Define*

$$\phi(x,r) := \begin{cases} r^{-n/p(x)}, & 0 < r < 1/2, \\ r^{-n/p_+}, & 1/2 \le r < \infty. \end{cases}$$

Then $L^{p(\cdot)}(\mathbb{R}^n) \subset L_{1,\phi}(\mathbb{R}^n) \subset \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{L}_{1,\phi}} + |f_{B(0,1)}| \le C_1 \|f\|_{L_{1,\phi}} \le C_2 \|f\|_{L^{p(\cdot)}}.$$

5.2. **Pointwise multipliers on Campanato spaces.** If we regard $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$ as spaces modulo constant functions, then $\|\cdot\|_{L_{p,\phi}(\mathbb{R}^n)}$ and $\|\cdot\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)}$ are norms and thereby $L_{p,\phi}(\mathbb{R}^n)$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ are Banach spaces, respectively. However, pointwise multipliers are not well defined on a space modulo constant functions, since, for $g \in L^0(\Omega)$ and for the constant function 1, pointwise multiplication $1g$ is not a constant function in general.

To consider pointwise multipliers on Campanato and Hölder spaces, we regard these spaces as spaces of functions. Moreover, we define norms

$$\|f\|_{\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)} + |f_{B(0,1)}| \quad \text{and} \quad \|f\|_{\Lambda^\natural_\phi(\mathbb{R}^n)} = \|f\|_{\Lambda_\phi(\mathbb{R}^n)} + |f(0)|,$$

and we denote $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\Lambda_\phi(\mathbb{R}^n)$ equipped with these norms by $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $\Lambda^\natural_\phi(\mathbb{R}^n)$, respectively.

Note that $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ fails the lattice property. But we have the following proposition. Therefore, all pointwise multipliers on Campanato spaces are bounded operator, see Theorem 2.5.

Proposition 5.5. *For any $p \in [1,\infty)$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, the following property holds:*

$$\lim_{j\to\infty} \|f_j\|_{\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)} = 0 \implies f_j \to 0 \text{ in measure on each } B(0,2^m).$$

Proof. For any m, we have

$$|f_{B(0,2^m)} - f_{B(0,1)}| = \left| \frac{1}{|B(0,1)|} \int_{B(0,1)} f(x)\,dx - f_{B(0,2^m)} \right|$$

$$\begin{aligned}
&\le \frac{1}{|B(0,1)|}\int_{B(0,1)} \left|f(x)-f_{B(0,2^m)}\right|\,dx \\
&\le \frac{|B(0,2^m)|}{|B(0,1)|}\frac{1}{|B(0,2^m)|}\int_{B(0,2^m)} \left|f(x)-f_{B(0,2^m)}\right|\,dx \\
&\le 2^{mn}\left(\frac{1}{|B(0,2^m)|}\int_{B(0,2^m)} \left|f(x)-f_{B(0,2^m)}\right|^p\,dx\right)^{1/p} \\
&\le 2^{mn}\phi(0,2^m)\|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)}.
\end{aligned}$$

Then

$$|f_{B(0,2^m)}| \le (1+2^{mn}\phi(0,2^m))\,\|f\|_{\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)}.$$

Next, using Chebyshev's inequality and the inequality

$$|f|_B \le \phi(B)\|f\|_{\mathcal{L}_{p,\phi}(\mathbb{R}^n)} + |f_B|,$$

we have

$$\epsilon\,|\{x\in B(0,2^m): |f(x)|\ge \epsilon\}| \le |B(0,2^m)||f|_{B(0,2^m)} \le C_{\phi,m}\|f\|_{\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)},$$

where the constant $C_{\phi,m}$ is dependent only on ϕ, m and n. Therefore, we have the conclusion. □

Definition 5.3. (i) For $\phi(x,r)=r^\alpha$ ($\alpha>0$), let

$$\mathrm{Lip}_\alpha(\mathbb{R}^n)=\Lambda_\phi(\mathbb{R}^n),\quad \mathrm{Lip}^{\natural}_\alpha(\mathbb{R}^n)=\Lambda^{\natural}_\phi(\mathbb{R}^n).$$

(ii) For $p=1$, let

$$\mathrm{BMO}_\phi(\mathbb{R}^n)=\mathcal{L}_{1,\phi}(\mathbb{R}^n),\quad \mathrm{BMO}^{\natural}_\phi(\mathbb{R}^n)=\mathcal{L}^{\natural}_{1,\phi}(\mathbb{R}^n).$$

(iii) For $\phi\equiv 1$, let

$$\mathrm{BMO}(\mathbb{R}^n)=\mathrm{BMO}_\phi(\mathbb{R}^n),\quad \mathrm{BMO}^{\natural}(\mathbb{R}^n)=\mathrm{BMO}^{\natural}_\phi(\mathbb{R}^n).$$

BMO is the space of functions with bounded mean oscillation introduced by John and Nirenberg [16] in 1961. It is known that $\log|x-a|$ is in $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{BMO}^{\natural}(\mathbb{R}^n)$ for all $a\in\mathbb{R}^n$.

Theorem 5.6 ([55])**.** *Let*

$$\phi(x,r)=\frac{1}{\log(r+1/r+|x|)},\quad x\in\mathbb{R}^n,\ r\in(0,\infty).$$

Then

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}(\mathbb{R}^n))=\mathrm{BMO}_\phi(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$$

and

$$\|g\|_{\mathrm{Op}}\sim\|g\|_{\mathrm{BMO}_\phi}+\|g\|_{L^\infty(\mathbb{R}^n)}.$$

For example,

$$g_1(x):=\sin\left(\chi_{B(0,1/e)}(x)\log\log(|x|^{-1})\right), \tag{5.8}$$

$$g_2(x):=\sin\left(\chi_{B(0,e)^\complement}(x)\log\log|x|\right) \tag{5.9}$$

are in $\mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$. Note that g_1 is not continuous and that $\lim_{|x|\to\infty} g_2(x)$ doesn't exist. Example (5.8) was given by Janson [15] and Stegenga [64], and example (5.9) by Nakai and Yabuta [55].

Theorem 5.7 ([37]). *Let $p \in [1,\infty)$ and there exists a positive constant A such that, for all $x, y \in \mathbb{R}^n$, $r \in (0,\infty)$, $s \in [1,\infty)$,*

(i) $A^{-1} \le \phi(x,r)/\phi(x,2r) \le A$,

(ii) $\displaystyle\int_0^r \phi(x,t)t^{n/p-1}\,dt \le A\phi(x,r)r^{n/p}$,

(iii) $|x-y| \le r \Rightarrow A^{-1} \le \phi(x,r)/\phi(y,r) \le A$,

(iv) $\phi(x,sr) \le As\phi(x,r)$.

Then

$$\mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)) = \mathcal{L}_{p,\psi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

and

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)},$$

where $\psi = \phi/(\Phi^* + \Phi^{**})$, and

$$\Phi^*(x,r) = \int_1^{\max(2,|x|,r)} \frac{\phi(0,t)}{t}\,dt,\ \ \Phi^{**}(x,r) = \int_r^{\max(2,|x|,r)} \frac{\phi(x,t)}{t}\,dt. \tag{5.10}$$

Remark 5.1 ([37]). Under four conditions in Theorem 5.7, let

$$f_a(x) = \int_{|x-a|}^1 \frac{\phi(a,t)}{t}\,dt.$$

Then f_a is in $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ for all $a \in \mathbb{R}^n$.

For ϕ_i $(i = 1, 2)$, we define Φ_i^* and Φ_i^{**} by (5.10).

Theorem 5.8 ([41]). *Suppose that ϕ_1 and ϕ_2 are almost increasing and satisfy four conditions in Theorem 5.7. Suppose also that*

$$\int_1^r \frac{\phi_2(x,t)}{\phi_1(x,t)}t^{n/p-1}\,dt \le A\frac{\phi_2(x,r)}{\phi_1(x,r)}r^{n/p}, \quad r > 1, \tag{v}$$

for some $p > 1$. Then

$$\mathrm{PWM}(\mathrm{BMO}^\natural_{\phi_1}(\mathbb{R}^n), \mathrm{BMO}^\natural_{\phi_2}(\mathbb{R}^n)) = \mathrm{BMO}_{\phi_3}(\mathbb{R}^n) \cap L_{\phi_2/\phi_1}(\mathbb{R}^n)$$

and

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathrm{BMO}_{\phi_3}(\mathbb{R}^n)} + \|g\|_{L_{\phi_2/\phi_1}(\mathbb{R}^n)},$$

where $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$.

Theorem 5.9 ([41]). *Let $1 < p_2 < p_1 < \infty$ and $p_1 + p_2 \le p_1p_2$. Suppose that ϕ_1 and ϕ_2 satisfy* (v) *for $p = p_2$ and four conditions in Theorem 5.7, and that $(\Phi_2^* + \Phi_2^{**})/\phi_2 \le C(\Phi_1^* + \Phi_1^{**})/\phi_1$. If $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$ is almost increasing, then*

$$\mathrm{PWM}(\mathcal{L}^\natural_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^\natural_{p_2,\phi_2}(\mathbb{R}^n)) = \mathrm{BMO}_{\phi_3}(\mathbb{R}^n) \cap L_{\phi_2/\phi_1}(\mathbb{R}^n)$$

and

$$\|g\|_{\mathrm{Op}} \sim \|g\|_{\mathrm{BMO}_{\phi_3}(\mathbb{R}^n)} + \|g\|_{L_{\phi_2/\phi_1}(\mathbb{R}^n)},$$

See also [30, 71] for related results.

5.3. **Examples.** In theorems in the previous subsection we can replace $\mathbb{R}^n$ with spaces of homogeneous type (X, d, μ). If $\mu(X) < \infty$, then we can omit the condition (v). For example, X is a cube $Q \subset \mathbb{R}^n$ or $\mathbb{T}^n$. All of examples in this subsection were given in [41]. Let α_0 be the constant with respect to the continuity of quasi-distance d (see page 36 in [41]). If d is a distance, then $\alpha_0 = 1$.

5.3.1. *The case $\mu(X) < \infty$.*

Example 5.3. For $0 \le \beta < \alpha < 1$,

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}_{(\log(1/r))^{-\alpha}}(X), \mathrm{BMO}^{\natural}_{(\log(1/r))^{-\beta}}(X)) = \mathrm{BMO}^{\natural}_{(\log(1/r))^{\alpha-\beta-1}}(X).$$

For $\alpha = 1/2$ and $\beta = 0$ in particular,

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}_{(\log(1/r))^{-1/2}}(X), \mathrm{BMO}^{\natural}(X)) = \mathrm{BMO}^{\natural}_{(\log(1/r))^{-1/2}}(X).$$

Example 5.4.

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}_{(\log(1/r))^{-1}}(X), \mathrm{BMO}^{\natural}(X)) = \mathrm{BMO}^{\natural}_{(\log\log(1/r))^{-1}}(X).$$

Example 5.5.

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}_{(\log\log(1/r))^{-1}}(X), \mathrm{BMO}^{\natural}(X)) = \mathrm{BMO}_{(\mathrm{li}(\log(1/r)))^{-1}}(X) \cap L_{(\log\log(1/r))}(X),$$

where $\mathrm{li}(R) = \int_e^R 1/(\log t)\,dt$.

Example 5.6.

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}(X)) = \mathrm{BMO}_{(\log(1/r))^{-1}}(X) \cap L^{\infty}(X).$$

If $X = \mathbb{T}^n$, $d(x,y) = |x-y|$ and μ is Lebesgue measure, then the example above is known (Janson [15] and Stegenga [64]).

Example 5.7. For $\alpha > 1$,

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}_{(\log(1/r))^{-\alpha}}(X), \mathrm{BMO}^{\natural}(X)) = \mathrm{BMO}^{\natural}(X).$$

Example 5.8. For $0 < \beta \le \alpha \le \alpha_0$,

$$\mathrm{PWM}(\mathrm{BMO}^{\natural}_{r^{\alpha}}(X), \mathrm{BMO}^{\natural}_{r^{\beta}}(X)) = \mathrm{BMO}^{\natural}_{r^{\beta}}(X).$$

If $X = \mathbb{T}^n$, $d(x,y) = |x-y|$ and μ is Lebesgue measure, then $\mathrm{BMO}^{\natural}_{r^{\alpha}}(\mathbb{T}^n) = \mathrm{Lip}^{\natural}_{\alpha}(\mathbb{T}^n)$. Therefore, for $0 < \beta \le \alpha \le 1$,

$$\mathrm{PWM}(\mathrm{Lip}^{\natural}_{\alpha}(\mathbb{T}^n), \mathrm{Lip}^{\natural}_{\beta}(\mathbb{T}^n)) = \mathrm{Lip}^{\natural}_{\beta}(\mathbb{T}^n).$$

Example 5.9. For $-1 < \alpha < \beta \le \alpha + 1,\ 1 < p_2 < p_1 < \infty,\ p_1 p_2 \ge p_1 + p_2$,

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,(\log(1/r))^{\alpha}}(X), \mathcal{L}^{\natural}_{p_2,(\log(1/r))^{\beta}}(X)) = \mathrm{BMO}^{\natural}_{(\log(1/r))^{\beta-\alpha-1}}(X).$$

5.3.2. *The case $\mu(X)=\infty$, fix $x_0 \in X$.*

Example 5.10.

$$\mathrm{PWM}(\mathrm{BMO}^\natural(X), \mathrm{BMO}^\natural(X)) = \mathrm{BMO}_{(\log(d(x_0,x)+r+1/r))^{-1}}(X) \cap L^\infty(X).$$

If $X=\mathbb{R}^n$, $d(x,y)=|x-y|$ and μ is Lebesgue measure, then the example above is Theorem 5.6.

Example 5.11. For $0<\beta\le\alpha\le\alpha_0$,

$$\mathrm{PWM}(\mathrm{BMO}^\natural_{r^\alpha}(X), \mathrm{BMO}^\natural_{r^\beta}(X)) = \mathrm{BMO}_{\frac{r^\beta}{(2+d(x_0,x)+r)^\alpha}}(X) \cap L_{r^{\beta-\alpha}}(X).$$

Example 5.12. For $0<\alpha\le\alpha_0,\ \beta\ge 0,\ \beta-\alpha+\delta>0$,

$$\begin{aligned}\mathrm{PWM}(&\mathrm{BMO}^\natural_{(2+d(x_0,x)+r)^\alpha}(X), \mathrm{BMO}^\natural_{(2+d(x_0,x)+r)^\beta}(X)) \\ &= \mathrm{BMO}_{\frac{(2+d(x_0,x)+r)^{\beta-\alpha}}{\log(d(x_0,x)+r+1/r)}}(X) \cap L_{(2+d(x_0,x)+r)^{\beta-\alpha}}(X).\end{aligned}$$

Example 5.13. For $1<p<\infty$,

$$\mathrm{PWM}(\mathrm{BMO}^\natural(X), \mathcal{L}^\natural_{p,\log(d(x_0,x)+r+1/r)}(X)) = \mathrm{BMO}^\natural(X).$$

Example 5.14. Let w be an $A_{p'}$-weight on $\mathbb{R}^n$. Then

$$\phi(a,r) = \left(\int_{B(a,r)} w(x)\,dx\right)^\alpha$$

satisfies four conditions in Theorem 5.7 for $-1/(pp')<\alpha\le 1/(np')$, and is almost increasing for $\alpha\ge 0$. Let

$$\phi_i(a,r) = \left(\int_{B(a,r)} w(x)\,dx\right)^{\alpha_i}, i=1,2, \qquad 0<\alpha_2\le\alpha_1.$$

Then $(\Phi_2^*+\Phi_2^{**})/\phi_2 \le C(\Phi_1^*+\Phi_1^{**})/\phi_1$.

6. Applications

In this section we state applications of pointwise multipliers on $\mathrm{BMO}(\mathbb{R}^n)$.

6.1. **Hardy-Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$.** Let M be the Hardy-Littlewood maximal operator, that is, for $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\,dy,$$

where the supremum is taken over all balls B containing x. For the definition of $L^{p(\cdot)}(\mathbb{R}^n)$, see Example 4.1. Let $\mathcal{B}_M(\mathbb{R}^n)$ be the set of all variable exponents $p(\cdot)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. That is,

$$p(\cdot)\in\mathcal{B}_M(\mathbb{R}^n) \quad\Leftrightarrow\quad M\in B(L^{p(\cdot)}(\mathbb{R}^n)).$$

Note that $p(x)=4\chi_{(-\infty,0)}(x)+2\chi_{[0,\infty)}(x)\notin\mathcal{B}_M(\mathbb{R})$ and $p(x)=3+\cos(2\pi x)\notin\mathcal{B}_M(\mathbb{R})$.

Remark 6.1 ([24]). Let $1 < p_- \le p_+ < \infty$. Then

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \quad \text{for all } f \in L^{p(\cdot)}(\mathbb{R}^n),$$

if and only if $p(\cdot)$ is a constant.

Recall that

$$LH_0 = \{p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R}) : p(\cdot) \text{ satisfies (5.6)}\},$$
$$LH_\infty = \{p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R}) : p(\cdot) \text{ satisfies (5.7)}\},$$
$$LH = LH_0 \cap LH_\infty.$$

Diening [9] and Cruz-Uribe, Fiorenza and Neugebauer [6, 7] proved the following theorem:

Theorem 6.1 ([6,7,9])**.** *If $p(\cdot) \in LH$ and $1 < p_- \le p_+ < \infty$, then $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^n)$.*

On the other hand, Lerner [23] showed that, for $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^n)$, we don't need the continuity of $p(\cdot)$ or existence of $\lim_{|x|\to\infty} p(x)$.

Theorem 6.2 ([23])**.** *Let $p(\cdot) \in L^0(\mathbb{R}^n;\mathbb{R})$. If $p(\cdot) \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$, then $\alpha + p(\cdot) \in \mathcal{B}_M(\mathbb{R}^n)$ for some nonnegative constant α.*

Let

$$g_1(x) = \sin\big(\chi_{B(0,1/e)}(x) \log\log(|x|^{-1})\big),$$
$$g_2(x) = \sin\big(\chi_{B(0,e)^\complement}(x) \log\log|x|\big).$$

Then

$$g_1 \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n)) \setminus LH_0,$$
$$g_2 \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n)) \setminus LH_\infty.$$

The following inclusion relation is a special case of [55, Proposition 5.1].

$$LH = LH_0 \cap LH_\infty \subset \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n)).$$

We can also consider pointwise multipliers on martingale Campanato spaces and their applications to the boundedness of the maximal operator, see [50–52].

6.2. **Density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$.** For the density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in Sobolev spaces, the following theorem is known.

Theorem 6.3 ([54])**.** *Let $E(\mathbb{R}^n)$ be a subspace of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume the following four conditions:*

(1) *$\chi_B \in E(\mathbb{R}^n)$ for all open balls $B \subset \mathbb{R}^n$.*
(2) *If $g \in E(\mathbb{R}^n)$, $f \in L^0(\mathbb{R}^n)$ and $|f| \le |g|$ a.e., then $f \in E(\mathbb{R}^n)$.*
(3) *If $g \in E(\mathbb{R}^n)$, $f_j \in L^0(\mathbb{R}^n)$ $(j = 1, 2, \ldots)$, $|f_j| \le |g|$ a.e., and $\lim_{j\to\infty} f_j = 0$ a.e., then $\lim_{j\to\infty} \|f_j\|_{E(\mathbb{R}^n)} = 0$.*
(4) *The Hardy-Littlewood maximal operator M is bounded on $E(\mathbb{R}^n)$.*

Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $E^m(\mathbb{R}^n)$, the Sobolev space based on $E(\mathbb{R}^n)$.

Corollary 6.4. *$C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $W^{p(\cdot),m}(\mathbb{R}^n)$ if $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^n)$, in particular, $p(\cdot) \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ and p_- is large enough.*

6.3. **Calderón-Zygmund operators.**

Definition 6.1 (standard kernel of type ω, see Yabuta [70])**.** Let ω be a nonnegative nondecreasing function on $(0,\infty)$ satisfying the Dini condition $\int_0^1 \omega(t)t^{-1}dt < \infty$. A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \in \mathbb{R}^{2n}\}$ is said to be a standard kernel of type ω if the following conditions are satisfied;

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \ne y, \tag{6.1}$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le \frac{C}{|x-y|^n}\,\omega\left(\frac{|y-z|}{|x-y|}\right) \quad \text{for} \quad 2|y-z| \le |x-y|. \tag{6.2}$$

Definition 6.2 (Calderón-Zygmund operator)**.** A linear mapping $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type ω, if T is bounded on $L^2(\mathbb{R}^n)$ and there exists a standard kernel K of type ω such that, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, \quad x \notin \operatorname{supp} f. \tag{6.3}$$

Let $\mathrm{CZO}(\omega)$ be the set of all Calderón-Zygmund operators of type ω.

It is known that (Yabuta [70])

$$\mathrm{CZO}(\omega) \subset \left(\bigcap_{1<p<\infty} B(L^p(\mathbb{R}^n))\right) \cap B(L^1(\mathbb{R}^n), L^1_{\mathrm{weak}}(\mathbb{R}^n)).$$

Let $M^\sharp$ be the sharp maximal operator, that is, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$M^\sharp f(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y) - f_B|\,dy,$$

where the supremum is taken over all balls B containing x. Alvarez and Pérez [3] proved that, if $r \in (1,\infty)$, then

$$(M^\sharp(|Tf|^{1/r})(x))^r \le C_r Mf(x) \tag{6.4}$$

for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Diening and Růžička [8] proved that, if $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^n)$ and $p_+ < \infty$, then, for $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$C^{-1}\|f\|_{L^{p(\cdot)}} \le \|M^\sharp f\|_{L^{p(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}.$$

Hence, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$\begin{aligned}\|Tf\|_{L^{p(\cdot)}} = \|\,|Tf|^{1/r}\,\|^r_{L^{rp(\cdot)}} &\le C\|M^\sharp(|Tf|^{1/r})\|^r_{L^{rp(\cdot)}} \\ &\le C\|(Mf)^{1/r}\|^r_{L^{rp(\cdot)}} = C\|Mf\|_{L^{p(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}.\end{aligned}$$

By the density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ in $L^{p(\cdot)}(\mathbb{R}^n)$, we have the following theorem.

Theorem 6.5 ([8])**.** *Let $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^n)$ and $1 < p_- \le p_+ < \infty$. Then* $\mathrm{CZO}(\omega) \subset B(L^{p(\cdot)}(\mathbb{R}^n))$.

Corollary 6.6. *If $p(\cdot) \in \mathrm{PWM}(\mathrm{BMO}^\natural(\mathbb{R}^n))$ and p_- is large enough, then* $\mathrm{CZO}(\omega) \subset B(L^{p(\cdot)}(\mathbb{R}^n))$.

6.4. **Hardy-Littlewood maximal operator on** $\mathrm{BMO}(\mathbb{R}^n)$**.** The following two theorems were proven by Bennett, DeVore and Sharpley [5] and Bennett [4], respectively.

Theorem 6.7 ([5])**.** *For* $f \in \mathrm{BMO}(\mathbb{R}^n)$*, if* Mf *is not identically infinite, then* $Mf \in \mathrm{BMO}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{\mathrm{BMO}(\mathbb{R}^n)} \le c\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Theorem 6.8 ([4])**.** *For* $f \in \mathrm{BMO}(\mathbb{R}^n)$*, if* Mf *is not identically infinite, then* $Mf \in \mathrm{BLO}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{\mathrm{BLO}(\mathbb{R}^n)} \le c\|f\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where

$$\|f\|_{\mathrm{BLO}(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|}\int_Q (f(x) - \operatorname*{ess\,inf}_Q f)\, dx.$$

In the above theorems, $\mathbb{R}^n$ can be replaced by a cube $Q_0 \subset \mathbb{R}^n$. To make sure that Theorem 6.8 is an improvement of Theorem 6.7, we must fined a function $f \in \mathrm{BMO}(\mathbb{R}^n)$ such that $f \ge 0$, $\operatorname{supp} f$ is compact and $f \notin \mathrm{BLO}(\mathbb{R}^n)$.

Theorem 6.9 ([28])**.** *For* $x \in \mathbb{R}$*, let*

$$f(x) = \begin{cases} \log(2/|x|), & \textit{if} \quad |x| \le 2, \\ 0, & \textit{if} \quad |x| > 2, \end{cases}$$

$$g(x) = \begin{cases} \sin\left(1 + \displaystyle\int_{|x|}^{1} \frac{dt}{t\log(2/t)}\right), & \textit{if} \quad |x| \le 1, \\ \sin 1, & \textit{if} \quad |x| > 1. \end{cases}$$

Then $|fg| \in \mathrm{BMO}(\mathbb{R})$*, but* $|fg| \notin \mathrm{BLO}(\mathbb{R})$*.*

Proof. From $f \in \mathrm{BMO}(\mathbb{R})$ and $g \in \mathrm{PWM}(\mathrm{BMO}(\mathbb{R}^n))$, it follows that fg and $|fg|$ are in $\mathrm{BMO}(\mathbb{R})$. But an elementary calculation shows that $|fg| \notin \mathrm{BLO}(\mathbb{R})$. □

For the martingale BMO, we also have a similar result, see [52].

7. Besov and Triebel-Lizorkin spaces

In this section we give some known results of type

$$\mathrm{PWM}(E_1, E_2) \supset E_3,$$

for Besov and Triebel-Lizorkin spaces, where "$\supset$" indicates a continuous embedding. For the properties of Besov and Triebel-Lizorkin spaces, see [27, 61, 67–69], etc.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. By $\mathcal{S}'(\mathbb{R}^n)$ we denote its topological dual, the space of tempered distributions. If $\varphi \in \mathcal{S}$ then

$$\mathcal{F}\varphi(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\xi}\varphi(\xi)\, d\xi, \quad x \in \mathbb{R}^n$$

denotes the Fourier transform $\mathcal{F}\varphi$ of φ. As usual, $\mathcal{F}^{-1}\varphi$ means the inverse Fourier transform of φ. Both, $\mathcal{F}$, $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a non-negative function with

$$\begin{cases} \psi(x) = 1 & \text{if } |x| \le 1, \\ \psi(x) = 0 & \text{if } |x| \ge 3/2, \end{cases} \qquad x \in \mathbb{R}^n. \tag{7.1}$$

For $f \in \mathcal{S}'$, let

$$f^j(x) = \mathcal{F}^{-1}[\psi(2^{-j}\xi)\mathcal{F}f(\xi)](x), \quad j = 0, 1, 2, \ldots.$$

Then f^j is an entire analytic function of exponential type. Hence, the pointwise product $f^j g^j$ makes sense for any j and any $f, g \in \mathcal{S}'$. We define

$$fg = \lim_{j\to\infty} f^j g^j \quad \text{in } \mathcal{S}',$$

whenever this limit exists. In the following theorems the limit element of $f^j g^j$ is independent of the choice of ψ.

Let ψ be the function defined in (7.1), and let

$$\begin{cases} \psi_0(x) = \psi(x), \\ \psi_1(x) = \psi(x/2) - \psi(x), \\ \psi_k(x) = \psi_1(2^{-k+1}x), \quad k = 2, 3, \ldots, \end{cases} \qquad x \in \mathbb{R}^n.$$

Definition 7.1. For $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, let

$$B^s_{pq}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{pq}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{B^s_{pq}(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{sjq} \left\| \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot) \right\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q},$$

(usual modification if $q = \infty$).

Definition 7.2. For $p \in (0, \infty)$, $q \in (0, \infty]$ and $s \in \mathbb{R}$, let

$$F^s_{pq}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{F^s_{pq}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{F^s_{pq}(\mathbb{R}^n)} = \left\| \left(\sum_{j=0}^{\infty} 2^{sjq} |\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\cdot)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

(usual modification if $q = \infty$).

The spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are independent of the choice of ψ. Recall some special cases:

- $F^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $p \in (1, \infty)$ (Lebesgue spaces),
- $F^s_{p,2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n)$, $p \in (1, \infty)$, $s \in \mathbb{N}$ (Sobolev spaces),
- $F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$, $p \in (1, \infty)$, $s \in \mathbb{R}$ (fractional Sobolev spaces),
- $F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n)$, $p \in (0, \infty)$ (inhomogeneous Hardy spaces),
- $B^s_{p,q}(\mathbb{R}^n)$, $p \in (1, \infty)$, $q \in [1, \infty]$, $s \in (0, \infty)$ (classical Besov spaces),

- $B^s_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n)$, $s \in (0,\infty)$ (Hölder-Zygmund spaces).

The following theorem is one of main results in [63].

Theorem 7.1 (Sickel and Triebel [63], 1995)**.** *Let $p_i \in (0,\infty)$, $q_i \in (0,\infty]$, $i = 1,2,3$, and $s \in \mathbb{R}$. Assume that*

$$\frac{1}{r_i} = \frac{1}{p_i} - \frac{s}{n} > 0,\ i = 1,2,3, \quad \textit{and} \quad \frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} < 1.$$

Then

$$\mathrm{PWM}(B^s_{p_1,q_1}, B^s_{p_2,q_2}) \supset B^s_{p_3,q_3} \iff 0 < q_1 \le r_1,\ 0 < q_3 \le r_3 \ \textit{and}\ \max(q_1,q_3) \le q_2 \le \infty,$$

and

$$\mathrm{PWM}(F^s_{p_1,q_1}, F^s_{p_2,q_2}) \supset F^s_{p_3,q_3} \iff \max(q_1,q_3) \le q_2 \le \infty.$$

If $\mathrm{PWM}(E) \supset E$, then E is an algebra (multiplicative algebra). Let $C^0(\mathbb{R}^n)$ be the set of all complex-valued bounded and uniformly continuous functions on $\mathbb{R}^n$ with norm $\|f\|_{L^\infty(\mathbb{R}^n)}$.

Theorem 7.2 (Triebel [66], 1978)**.** *Let $p, q \in (0,\infty]$ and $s \in \mathbb{R}$. The following are equivalent:*

(i) $\mathrm{PWM}(B^s_{p,q}(\mathbb{R}^n)) \supset B^s_{p,q}(\mathbb{R}^n)$.
(ii) $B^s_{p,q}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$.
(iii) *either* $0 < p \le \infty$, $0 < q \le \infty$, $s > n/p$
or $0 < p \le \infty$, $0 < q \le 1$, $s = n/p$.

Theorem 7.3 (Franke [11], 1986)**.** *Let $p \in (0,\infty)$, $q \in (0,\infty]$ and $s \in \mathbb{R}$. The following are equivalent:*

(i) $\mathrm{PWM}(F^s_{p,q}(\mathbb{R}^n)) \supset F^s_{p,q}(\mathbb{R}^n)$.
(ii) $F^s_{p,q}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$.
(iii) *either* $0 < p < \infty$, $0 < q \le \infty$, $s > n/p$
or $0 < p \le 1$, $0 < q \le \infty$, $s = n/p$.

Theorem 7.3 contains Strichartz's result in [65] (1967):

$$\mathrm{PWM}(H^s_p(\mathbb{R}^n)) \supset H^s_p(\mathbb{R}^n) \quad \text{for } s > n/p,\ 1 < p < \infty.$$

For other results, see [12, 14, 62, 63, 67] and their references.

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References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), 765–778.

[2] T. Ando, *On products of Orlicz spaces*, Math. Ann. **140** (1960), 174–186.

[3] J. Alvarez and C. Perez, *Estimates with A_∞ weights for various singular integral operators*, Boll. Un. Mat. Ital. A (7) **8** (1994), 123–133.

[4] C. Bennett, *Another characterization of BLO*, Proc. Amer. Math. Soc. **85** (1982), 552–556.

[5] C. Bennett, R. A. DeVore and R. Sharpley, *Weak-L^∞ and* BMO, Ann. of Math. (2) **113** (1981), 601–611.

[6] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238.

[7] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, Corrections to "The maximal function on variable L^p spaces", Ann. Acad. Sci. Fenn. Math. **29** (2004), 247–249.

[8] L. Diening and M. Růžička, *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. **563** (2003), 197–220.

[9] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), 245–253.

[10] P. B. Djakov and M. S. Ramanujan, *Multipliers between Orlicz sequence spaces*, Turk. J. Math. **24** (2000), 313–319.

[11] J. Franke, *On the spaces F^s_{pq} of Triebel-Lizorkin type: pointwise multipliers and spaces on domains*, Math. Nachr. **125** (1986), 29–68.

[12] H. F. Gonçalves and H. Kempka, *Non-smooth atomic decomposition of 2-microlocal spaces and application to pointwise multipliers*, J. Math. Anal. Appl. **434** (2016), 1875–1890.

[13] M. Izuki, E. Nakai and Y. Sawano, *Function spaces with variable exponents – an introduction –*, Sci. Math. Jpn. **77** (2014), 187–315.

[14] M. Izuki and T. Noi, *Generalized Besov Morrey spaces and generalized Triebel-Lizorkin Morrey spaces on domains*, preprint.

[15] S. Janson, *On functions with conditions on the mean oscillation*, Ark. Math. **14** (1976), 189–196.

[16] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[17] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-Space Sampler*, London Mathematical Society Lecture Note Series, 89, Cambridge University Press, Cambridge, 1984.

[18] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Translated from the Russian by Howard L. Silcock, Second edition, Pergamon Press, Oxford-Elmsford, N.Y., 1982.

[19] P. Kolwicz, K. Leśnik and L. Maligranda, *Pointwise multipliers of Calderón-Lozanovskiĭ spaces*, Math. Nachr. **286** (2013), 876–907.

[20] P. Kolwicz, K. Leśnik and L. Maligranda, *Pointwise products of some Banach function spaces and factorization*, J. Funct. Anal. **266** (2014), 616–659.

[21] H. Komatsu, *Fourier Analysis (Japanese)*, Second edition, Iwanami Lectures on Fundamental Mathematics, 13, Analysis (I), vi, Iwanami Shoten, Tokyo, 1983. (First edition was published in 1978 in Japanese)

[22] S. Kuroda, *Functional analysis (Japanese)*, Kyoritsu Shuppan Co., Ltd., Tokyo, 1980.

[23] A. K. Lerner, *Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces*, Math. Z. **251** (2005), 509–521.

[24] A. K. Lerner, *On modular inequalities in variable L^p spaces*, Arch. Math. (Basel) **85** (2005), 538–543.

[25] K. Leśnik and J. Tomaszewski, *Pointwise multipliers of Orlicz function spaces and factorization*, preprint, https://arxiv.org/abs/1605.08581

[26] W. Li, E. Nakai and Do. Yang, *Pointwise multipliers on BMO spaces with non-doubling measures*, preprint.

[27] Y. Liang, Da. Yang, W. Yuan, Y. Sawano and T. Ullrich, *A new framework for generalized Besov-type and Triebel-Lizorkin-type spaces*, Dissertationes Math. (Rozprawy Mat.) **489** (2013), 114 pp.

[28] H. Lin, E. Nakai and Da. Yang, *Boundedness of Lusin-area and functions on localized BMO spaces over doubling metric measure spaces*, Bull. Sci. Math. **135** (2011), 59–88.
[29] L. Liu and Da. Yang, *Pointwise multipliers for Campanato spaces on Gauss measure spaces*, Nagoya Math. J. **214** (2014), 169–193.
[30] H. Lin and Da. Yang, *Pointwise multipliers for localized Morrey-Campanato spaces on RD-spaces*, Acta Math. Sci. Ser. B Engl. Ed. **34** (2014), 1677–1694.
[31] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Math. Vol. 5, Univ. of Campinas, Campinas SP, Brazil, 1989.
[32] L. Maligranda, *Type, cotype and convexity properties of quasi-Banach spaces*, in: Banach and Function Spaces, Yokohama Publ., Yokohama, 2004, pp. 83–120.
[33] L. Maligranda, *Tosio Aoki (1910–1989)*, in: Banach and Function Spaces II, Yokohama Publ., Yokohama, 2008, pp. 1–23.
[34] L. Maligranda and E. Nakai, *Pointwise multipliers of Orlicz spaces*, Arch. Math. (Basel) **95** (2010), 251–256.
[35] L. Maligranda and L. E. Persson, *Generalized duality of some Banach function spaces*, Indag. Math. **51** (1989), 323–338.
[36] C. B. Morrey, Jr, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
[37] E. Nakai, *Pointwise multipliers for functions of weighted bounded mean oscillation*, Studia Math. **105** (1993), 105–119.
[38] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.
[39] E. Nakai, *Pointwise multipliers*, Memoirs of The Akashi College of Technology **37** (1995), 85–94.
[40] E. Nakai, *Pointwise multipliers on the Lorentz spaces*, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci. **45** (1996), 1–7.
[41] E. Nakai, *Pointwise multipliers on weighted BMO spaces*, Studia Math. **125** (1997), 35–56.
[42] E. Nakai, *Pointwise multipliers on the Morrey spaces*, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci. **46** (1997), 1–11.
[43] E. Nakai, *A characterization of pointwise multipliers on the Morrey spaces*, Sci. Math. **3** (2000), 445–454.
[44] E. Nakai, *The Campanato, Morrey and Hölder spaces on spaces of homogeneous type*, Studia Math. **176** (2006), 1–19.
[45] E. Nakai, *A generalization of Hardy spaces H^p by using atoms*, Acta Math. Sin. (Engl. Ser.) **24** (2008), 1243–1268.
[46] E. Nakai, *Singular and fractional integral operators on Campanato spaces with variable growth conditions*, Rev. Mat. Complut. **23** (2010), 355–381.
[47] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr. **287** (2014), 339–351.
[48] E. Nakai, *Pointwise multipliers on Musielak-Orlicz spaces*, Nihonkai Math. J. **27** (2016), 135–146.
[49] E. Nakai, *Pointwise multipliers on Musielak-Orlicz-Morrey spaces*, in: Function Spaces and Inequalities, Springer, to appear.
[50] E. Nakai and G. Sadasue, *Maximal function on generalized martingale Lebesgue spaces with variable exponent*, Statist. Probab. Lett. **83** (2013), 2168–2171.
[51] E. Nakai and G. Sadasue, *Pointwise multipliers on martingale Campanato spaces*, Studia Math. **220** (2014), 87–100.
[52] E. Nakai and G. Sadasue, *Some new properties concerning BLO martingales*, Tohoku Math. J. to appear.
[53] E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), 3665–3748.
[54] E. Nakai, N. Tomita and K. Yabuta, *Density of the set of all infinitely differentiable functions with compact support in weighted Sobolev spaces*, Sci. Math. Jpn. **60** (2004), 121–127.

[55] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of bounded mean oscillation*, J. Math. Soc. Japan **37** (1985), 207–218.
[56] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type*, Math. Japon. **46** (1997), 15–28.
[57] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co., Ltd., Tokyo, 1950.
[58] R. O'Neil, *Fractional integration in Orlicz spaces. I.*, Trans. Amer. Math. Soc. **115** (1965), 300–328.
[59] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam, New York, 1980.
[60] S Rolewicz, *Metric Linear Spaces*, Second edition, Mathematics and its Applications (East European Series), 20, D. Reidel Publishing Co., Dordrecht; PWN–Polish Scientific Publishers, Warsaw, 1985.
[61] Y. Sawano, *Theory of Besov spaces*, Springer, to appear. (Japanese edition was published in 2011 by Nippon Hyoron sha Co., Ltd., Tokyo.)
[62] B. Scharf, *Atomic representations in function spaces and applications to pointwise multipliers and diffeomorphisms, a new approach*, Math. Nachr. **286** (2013), 283–305.
[63] W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type*, Z. Anal. Anwendungen **14** (1995), 105–140.
[64] D. A. Stegenga, *Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation*, Amer. J. Math. **98** (1976), 573–589.
[65] R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967), 1031–1060.
[66] H. Triebel, *Spaces of Besov-Hardy-Sobolev Type*, Teubner-Texte Math. 15, Teubner, Leipzig, 1978.
[67] H. Triebel, *Theory of Function Spaces*, Birkhäuser Verlag, Basel, 1983.
[68] H. Triebel, *Theory of Function Spaces II*, Birkhäuser Verlag, Basel, 1992.
[69] W. Yuan, W. Sickel and D. Yang, *Morrey and Campanato meet Besov, Lizorkin and Triebel*, Lecture Notes in Mathematics, 2005, Springer-Verlag, Berlin, 2010.
[70] K. Yabuta, *Generalizations of Calderón-Zygmund operators*, Studia Math. **82** (1985), 17–31.
[71] K. Yabuta, *Pointwise multipliers of weighted BMO spaces*, Proc. Amer. Math. Soc. **117** (1993), 737–744.
[72] K. Yosida, *Fanctional Analysis*, Sixth edition, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1980.

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