Linear and Nonlinear Analysis
Volume 3, Number 1, 2017, 61–72

Yokohama Publishers
ISSN 2188-8167

FIXED POINT AND WEAK CONVERGENCE THEOREMS FOR NONLINEAR HYBRID MAPPINGS IN BANACH SPACES

MAYUMI HOJO AND WATARU TAKAHASHI

Abstract. Let E be a real Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called extended generalized hybrid if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. In this paper, we first obtain some properties for extended generalized hybrid mappings in a Banach space. Then, we prove fixed point and weak convergence theorems of Mann's type for such mappings in a Banach space satisfying Opial's condition.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Then a mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [4] and Goebel and Kirk [8]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [3] and [7]. Recently, Kohsaka and Takahashi [17], and Takahashi [23] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called nonspreading [17] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.1}$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [23] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [16], Iemoto and Takahashi [11] and Takahashi and Yao

2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 47H05.

Key words and phrases. Banach space, nonexpansive mapping, nonspreading mapping, hybrid mapping, fixed point, weak convergence.

The second author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

[25]. Motivated by these mappings and results, Kocourek, Takahashi and Yao [14] introduced a broad class of nonlinear mappings in a Hilbert space. They called this the class of generalized hybrid mappings; see also Aoyama, Iemoto, Kohsaka and Takahashi [1]. Then they proved fixed point theorems and convergence theorems for generalized hybrid mappings in a Hilbert space; see also [28] and [9]. Furthermore, Hsu, Takahashi and Yao [10] extended this class in a Hilbert space to that of a Banach space and they proved fixed point theorems for such mappings in a Banach space; see also [15]. Takahashi and Yao [27] also proved weak convergence theorems of Mann's type [18] for such mappings in a Banach space satisfying Opial's conditon [19].

In this paper, we first introduce a more broad class of nonlinear mappings in a Banach space which covers generalized hybrid mappings and then deal with some properties for such mappings in a Banach space. Then, we prove fixed point and weak convergence theorems for such mappings in a Banach space satisfying Opial's condition.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [12]. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. For more details, see [21, 22]. The following result is also in [21, 22].

Theorem 2.1. *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j\rangle,$$

where $j \in Jy$.

The following result was proved by Xu [29].

Theorem 2.2 ([29]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\mu x + (1-\mu)y\|^2 \leq \mu\|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)g(\|x - y\|)$$

for all $x, y \in B_r$ and μ with $0 \leq \mu \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a Banach space. Then, E satisfies Opial's condition [19] if for any $\{x_n\}$ of E such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Let E be a Banach space and let $A \subset E \times E$. Then, A is accretive if for $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$, where J is the duality mapping of E. An accretive operator $A \subset E \times E$ is called m-accretive if $R(I + rA) = E$ for all $r > 0$, where I is the identity operator and $R(I + rA)$ is the range of $I + rA$. An accretive operator $A \subset E \times E$ is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where $\overline{D(A)}$ is the closure of the domain $D(A)$ of A. An m-accretive operator satisfies the range condition.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [21] for the proof of the existence of a Banach limit and its other elementary properties.

3. Extension of generalized hybrid mappings in Banach spaces

Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \to E$ is said to be firmly nonexpansive [5] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle,$$

for all $x, y \in C$, where $j \in J(Tx - Ty)$. It is known that the resolvent of an accretive operator satisfying the range condition in a Banach space is a firmly nonexpansive mapping of the closure of the domain into itself. In fact, let $C = \overline{D(A)}$ and $r > 0$. Define the resolvent J_r of A as follows:

$$J_r x = \{z \in D(A) : x \in z + rAz\}$$

for all $x \in \overline{D(A)}$. It is known that such $J_r x$ is a singleton; see [21]. We have that for $x_1, x_2 \in \overline{D(A)}$, $x_1 = z_1 + ry_1,\ y_1 \in Az_1$ and $x_2 = z_2 + ry_2,\ y_2 \in Az_2$. Since A is accretive, we have that $\langle y_1 - y_2, j\rangle \geq 0$, where $j \in J(z_1 - z_2)$. So, we have

$$\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \geq 0.$$

Furthermore, we have that

$$\begin{aligned}\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \geq 0 \\ \iff \langle x_1 - z_1 - (x_2 - z_2), j\rangle \geq 0 \\ \iff \langle x_1 - x_2, j\rangle \geq \|z_1 - z_2\|^2.\end{aligned}$$

From $z_1 = J_r x_1$ and $z_2 = J_r x_2$, we have that J_r is a firmly nonexpansive mapping of C into itself; see also [5], [6] and [26]. Hsu, Takahashi and Yao [10] defined a class of nonlinear mappings in a Banach space containing nonexpansive mappings, nonspreading mappings and hybrid mappings as follows: Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. They called such a mapping $(\alpha,\ \beta)$-generalized hybrid. We note that an $(\alpha,\ \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We consider an extension of generalized hybrid mappings in a Banach space: A mapping $T : C \to E$ is called extended generalized hybrid if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0 \tag{3.2}$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid. We have the following result.

Theorem 3.1. *Let E be a Banach space, let C be a nonempty subset of E and let $\lambda, \mu \in [0, 1]$ with $\lambda + \mu > 0$. Then the following hold:*

- (i) *An extended generalized hybrid mapping which has a fixed point is quasi-nonexpansive;*
- (ii) *a firmly nonexpansive mapping is $(2\mu + \lambda, -\mu, -\mu, -\lambda)$-extended generalized hybrid.*

Proof. We show (i). Since $T : C \to E$ is an extended generalized hybrid mapping, there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. Let $u \in F(T)$. Then we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + \beta\|u - Ty\|^2 + \gamma\|u - y\|^2 + \delta\|u - y\|^2 \leq 0. \tag{3.3}$$

From $\alpha+\beta+\gamma+\delta \geq 0$ and $\alpha+\beta > 0$, we have that

$$\|u-Ty\|^2 \leq \frac{-(\gamma+\delta)}{\alpha+\beta}\|u-y\|^2 \leq \|u-y\|^2$$

and hence $\|u-Ty\| \leq \|u-y\|$. This implies that T is quasi-nonexpansive.

We next show (ii). Let T be a firmly nonexpansive mapping of C into E. Then we have that for $x, y \in C$ and $j \in J(Tx-Ty)$,

$$\|Tx-Ty\|^2 \leq \langle x-y, j\rangle.$$

From Theorem 2.1 we have that

$$\begin{aligned}\|Tx-Ty\|^2 &\leq \langle x-y, j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x-Tx-(y-Ty), j\rangle \\ &\Longrightarrow 0 \leq \|x-y\|^2 - \|Tx-Ty\|^2 \\ &\Longleftrightarrow \|Tx-Ty\|^2 \leq \|x-y\|^2.\end{aligned}$$

So, for $\lambda \in [0,1]$ we have

$$\lambda\|Tx-Ty\|^2 \leq \lambda\|x-y\|^2. \tag{3.4}$$

Futhermore, we have that for $x, y \in C$ and $j \in J(Tx-Ty)$,

$$\begin{aligned}\|Tx-Ty\|^2 &\leq \langle x-y, j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x-Tx-(y-Ty), j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x-Tx, j\rangle + 2\langle Ty-y, j\rangle \\ &\Longrightarrow 0 \leq \|x-Ty\|^2 - \|Tx-Ty\|^2 + \|Tx-y\|^2 - \|Tx-Ty\|^2 \\ &\Longleftrightarrow 0 \leq \|x-Ty\|^2 + \|y-Tx\|^2 - 2\|Tx-Ty\|^2 \\ &\Longleftrightarrow 2\|Tx-Ty\|^2 \leq \|x-Ty\|^2 + \|y-Tx\|^2.\end{aligned}$$

Thus, for $\mu \in [0,1]$ we have

$$2\mu\|Tx-Ty\|^2 \leq \mu\|x-Ty\|^2 + \mu\|y-Tx\|^2. \tag{3.5}$$

Therefore, we have from (3.4) and (3.5) that

$$(2\mu+\lambda)\|Tx-Ty\|^2 \leq \mu\|x-Ty\|^2 + \mu\|y-Tx\|^2 + \lambda\|x-y\|^2$$

and hence

$$(2\mu+\lambda)\|Tx-Ty\|^2 - \mu\|x-Ty\|^2 - \mu\|y-Tx\|^2 - \lambda\|x-y\|^2 \leq 0.$$

Since $(2\mu+\lambda)-\mu-\mu-\lambda = 0$ and $(2\mu+\lambda)-\mu = \mu+\lambda > 0$, T is $(2\mu+\lambda, -\mu, -\mu, -\lambda)$-extended generalized hybrid. □

Using Takahashi and Jeong's result [24], Hsu, Takahashi and Yao [10] also proved the following lemma for nonlinear mappings in a Banach space; see also [2, 13].

Lemma 3.2 ([10])**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a mapping of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. If*

$$\mu_n\|x_n-Ty\|^2 \leq \mu_n\|x_n-y\|^2$$

for all $y \in C$, then T has a fixed point in C.

Theorem 3.3. *Let E be a uniformly convex Banach space and let C be a noempty closed convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping from C into itself. Then the following are equivalent:*

(a) $F(T) \neq \emptyset$;

(b) $\{T^n z\}$ *is bounded for some* $z \in C$.

Furthermore, a fixed point p of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ and for any $y \in C$, $\{T^n y\}$ converges strongly to p.

Proof. Suppose that T has a fixed point z. Then $\{T^n z\} = \{z\}$. Therefore $\{T^n z\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z\}$ is bounded. Since T is an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping from C into itself, we obtain that

$$\alpha \|T^{n+1}z - Ty\|^2 + \beta \|T^n z - Ty\|^2 + \gamma \|T^{n+1}z - y\|^2 + \delta \|T^n z - y\|^2 \leq 0$$

for any $n \in \mathbb{N} \cup \{0\}$ and $y \in C$. Applying a Banach limit μ to both sides of this inequality, we obtain that

$$\mu_n \big(\alpha \|T^{n+1}z - Ty\|^2 + \beta \|T^n z - Ty\|^2 + \gamma \|T^{n+1}z - y\|^2 + \delta \|T^n z - y\|^2\big) \leq 0$$

and hence

$$(\alpha + \beta)\mu_n \|T^n z - Ty\|^2 + (\gamma + \delta)\mu_n \|T^n z - y\|^2 \leq 0.$$

By $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta > 0$ we obtain that

$$\begin{aligned} \mu_n \|T^n z - Ty\|^2 &\leq \frac{-(\gamma + \delta)}{\alpha + \beta} \mu_n \|T^n z - y\|^2 \\ &\leq \mu_n \|T^n z - y\|^2 \end{aligned}$$

for all $y \in C$. By Lemma 3.2 we obtain a fixed point $p \in C$.

Suppose that $\alpha + \beta + \gamma + \delta > 0$ and p_1 and p_2 are fixed points of T. Then we have that

$$\begin{aligned} \alpha \|Tp_1 - Tp_2\|^2 &+ \beta \|p_1 - Tp_2\|^2 + \gamma \|Tp_1 - p_2\|^2 + \delta \|p_1 - p_2\|^2 \\ &= (\alpha + \beta + \gamma + \delta)\|p_1 - p_2\|^2 \leq 0 \end{aligned}$$

and hence $p_1 = p_2$. Therefore, a fixed point p of T is unique. For such a unique fixed point p of T, we have that for any $y \in C$,

$$\alpha \|Tp - Ty\|^2 + \beta \|p - Ty\|^2 + \gamma \|Tp - y\|^2 + \delta \|p - y\|^2 \leq 0$$

and hence

$$(\alpha + \beta)\|p - Ty\|^2 + (\gamma + \delta)\|p - y\|^2 \leq 0.$$

From $\alpha + \beta + \gamma + \delta > 0$ and $\alpha + \beta > 0$, we have that

$$\|p - Ty\|^2 \leq \frac{-(\gamma + \delta)}{\alpha + \beta} \|p - y\|^2.$$

If $y = p$ for all $y \in C$, $\{T^n y\}$ converges strongly to p. If $y \neq p$ for some $y \in C$, then $-(\gamma + \delta) \geq 0$ and hence

$$0 \leq \frac{-(\gamma + \delta)}{\alpha + \beta} < 1.$$

Putting $\lambda = \frac{-(\gamma+\delta)}{\alpha+\beta}$, we have that

$$\|T^n y - p\|^2 \leq \lambda^n \|y - p\|^2.$$

Thus $\{T^n y\}$ for all $y \in C$ converges strongly to p. This completes the proof. □

Using Theorem 3.3, we can prove the following fixed point theorem.

Theorem 3.4. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a generalized hybrid mapping. Then the following are equivalent:*

(a) *$F(T) \neq \emptyset$;*
(b) *$\{T^n z\}$ is bounded for some $z \in C$.*

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there exist $a, b \in \mathbb{R}$ such that

$$a\|Tx - Ty\|^2 + (1-a)\|x - Ty\|^2 \leq b\|Tx - y\|^2 + (1-b)\|x - y\|^2$$

for all $x, y \in C$. From $a + (1-a) - b - (1-b) = 0$ and $a + (1-a) = 1$, we have the desired result by using Theorem 3.3. □

Using Theorem 3.3, we can also prove the following fixed point theorems in a Banach space.

Theorem 3.5 ([21]). *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Theorem 3.6 ([2]). *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Theorem 3.7 ([2]). *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

4. Some properties of generalized hybrid mappings

Let E be a Banach space. Let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then, $p \in C$ is called an asymptotic fixed point of T [20] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $I - T$ of C into E is said to be demiclosed on C if $\hat{F}(T) = F(T)$.

Theorem 4.1. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into itself which satisfies $\beta \le 0$ and $\gamma \le 0$. Then $\hat{F}(T) = F(T)$, i.e., $I - T$ is demiclosed.*

Proof. The inclusion $F(T) \subset \hat{F}(T)$ is obvious. We show $\hat{F}(T) \subset F(T)$. Let $u \in \hat{F}(T)$ be given. Then we have a sequence $\{x_n\}$ of C such that $x_n \rightharpoonup u$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since $T : C \to C$ is an extended generalized hybrid mapping, we obtain that

$$\alpha\|Tx_n - Tu\|^2 + \beta\|x_n - Tu\|^2 + \gamma\|Tx_n - u\|^2 + \delta\|x_n - u\|^2 \le 0. \tag{4.1}$$

From $\beta \le 0$, $\gamma \le 0$ and (4.1), we have

$$\begin{aligned}\alpha\|Tx_n - Tu\|^2 + \beta(\|x_n - Tx_n\| + \|Tx_n - Tu\|)^2 + \delta\|x_n - u\|^2 \\ + \gamma(\|Tx_n - x_n\| + \|x_n - u\|)^2 \le 0.\end{aligned}$$

Then we have that

$$\begin{aligned}(\alpha + \beta)\|Tx_n - Tu\|^2 + (\gamma + \delta)\|x_n - u\|^2 + (\beta + \gamma)\|x_n - Tx_n\|^2 \\ + 2\beta\|Tx_n - Tu\|)\|x_n - Tx_n\| + 2\gamma\|x_n - u\|\|Tx_n - x_n\|.\end{aligned}$$

From $x_n \rightharpoonup u$, we obtain that $\{x_n\}$ is bounded. From $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ we also have that $\{Tx_n\}$ is bounded. Suppose $Tu \ne u$. Then we have from Opial's condition, $\alpha + \beta + \gamma + \delta \ge 0$ and $\alpha + \beta > 0$ that

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - u\|^2 &< \liminf_{n\to\infty} \|x_n - Tu\|^2 \\ &= \liminf_{n\to\infty} \|x_n - Tx_n + Tx_n - Tu\|^2 \\ &= \liminf_{n\to\infty} \|Tx_n - Tu\|^2 \\ &\le \liminf_{n\to\infty} \frac{1}{\alpha + \beta}\{-(\gamma + \delta)\|x_n - u\|^2 - (\beta + \gamma)\|x_n - Tx_n\|^2 \\ &\quad - 2\beta\|Tx_n - Tu\|\|x_n - Tx_n\| - 2\gamma\|x_n - u\|\|Tx_n - x_n\|\} \\ &\le \liminf_{n\to\infty} \|x_n - u\|^2.\end{aligned}$$

This is a contradiction. Thus we have $Tu = u$ and hence $\hat{F}(T) \subset F(T)$. □

Using Theorem 4.1, we can prove the following theorems in a Banach space.

Theorem 4.2. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. It is clear that T is a (1, 0, 0, -1)-extended generalized hybrid mapping of C into itself. Thus Theorem 4.1 implies that $I - T$ is demiclosed on C. □

Theorem 4.3. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. It is clear that T is a (2, -1, -1, 0)-extended generalized hybrid mapping of C into itself. Thus Theorem 4.1 implies that $I - T$ is demiclosed on C. □

Theorem 4.4. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. It is clear that T is a (3, -1, -1, -1)-extended generalized hybrid mapping of C into itself. Thus Theorem 4.1 implies that $I - T$ is demiclosed on C. □

We also have the following property of an extended generalized hybrid mapping.

Theorem 4.5. *Let E be a strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be an extended generalized hybrid mapping of C into itself. Then $F(T)$ is closed and convex.*

Proof. Since $T : C \to C$ is an extended generalized hybrid mapping with $F(T) \neq \emptyset$, we have from Theorem 3.1 that T is quasi-nonexpansive. From Itoh and Takahashi [12], we have that $F(T)$ is closed and convex. □

5. Weak convergence theorems

In this section, we first prove a weak convergence theorem of Mann's type [18] for extended generalized hybrid mappings in a Banach space satisfying Opial's condition.

Theorem 5.1. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$. Let $\{\gamma_n\}$ be a sequence of real numbers such that $0 < a \leq \gamma_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ of C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n)Tx_n, \quad \forall n \in \mathbb{N}.$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Proof. Since $F(T) \neq \emptyset$, we have from Theorem 3.1 that T is quasi-nonexpansive. Using this fact, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\|x_{n+1} - u\| = \|\gamma_n x_n + (1 - \gamma_n)Tx_n - u\|$$

$$
\begin{aligned}
&= \|\gamma_n(x_n - u) + (1-\gamma_n)(Tx_n - u)\| \\
&\le \gamma_n\|x_n - u\| + (1-\gamma_n)\|Tx_n - u\| \\
&\le \gamma_n\|x_n - u\| + (1-\gamma_n)\|x_n - u\| \\
&= \|x_n - u\|.
\end{aligned}
$$

Thus $\lim_{n\to\infty}\|x_n - u\|$ exists and $\{x_n\}$ is bounded. Since T is quasi-nonexpansive, $\{Tx_n\}$ is also bounded. Let

$$r = \max\{\sup_{n\in\mathbb{N}}\|x_n - u\|, \sup_{n\in\mathbb{N}}\|Tx_n - u\|\}.$$

Then, from Theorem 2.2, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and

$$\|\mu x + (1-\mu)y\|^2 \le \mu\|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)g(\|x-y\|)$$

for all $x, y \in B_r$ and μ with $0 \le \mu \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$. Then we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
\|x_{n+1} - u\|^2 &= \|\gamma_n x_n + (1-\gamma_n)Tx_n - u\|^2 \\
&= \|\gamma_n(x_n - u) + (1-\gamma_n)(Tx_n - u)\|^2 \\
&\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|Tx_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\
&\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|x_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\
&= \|x_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\
&\le \|x_n - u\|^2
\end{aligned}
$$

and hence

$$\gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Since $\lim_{n\to\infty}\|x_n - u\|^2$ exists, we have from $0 < a \le \gamma_n \le b < 1$ that

$$\lim_{n\to\infty} g(\|x_n - Tx_n\|) = 0.$$

From the properties of g, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{5.1}$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $u \in C$. Using Theorem 4.1 and (5.1), we have $Tu = u$. Let us show that the entire sequence $\{x_n\}$ converges weakly to some point of $F(T)$. To show it, let us take two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. Suppose $u \neq v$. From $u, v \in F(T)$, we know that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. Since E satisfies Opial's condition, we have that

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n - u\| &= \lim_{i\to\infty}\|x_{n_i} - u\| \\
&< \lim_{i\to\infty}\|x_{n_i} - v\| \\
&= \lim_{n\to\infty}\|x_n - v\| \\
&= \lim_{j\to\infty}\|x_{n_j} - v\|
\end{aligned}
$$

$$< \lim_{j\to\infty} \|x_{n_j} - u\| \\ = \lim_{n\to\infty} \|x_n - u\|.$$

This is a contradiction. So, we must have $u = v$. This implies that $\{x_n\}$ converges weakly to a point of $F(T)$. □

Using Theorem 5.1, we obtain the following result.

Theorem 5.2. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$ and let λ be a real number with $0 < \lambda < 1$. Define a mapping $S : C \to C$ by*

$$S = \lambda I + (1 - \lambda)T.$$

If $F(T) \neq \emptyset$, then for any $x \in C$, $S^n x$ converges weakly to an element $z \in F(T)$.

Proof. Putting $\gamma_n = \lambda$ for all $n \in \mathbb{N}$ and $S = \lambda I + (1 - \lambda)T$, we have that for any $x \in C$,

$$x_2 = Sx_1 = Sx, x_3 = S^2 x_1 = S^2 x, \ldots$$

in Theorem 5.1. So, we have from Theorem 5.1 that $S^n x$ converges weakly to an element $z \in F(T)$. This completes the proof. □

References

[1] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335–343.

[2] K. Aoyama and F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear Anal. **74** (2011), 4387–4391.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[4] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[5] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341–355.

[6] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.

[7] P.L. Combettes and A. Hirstoaga, *Equilibrium problems in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[8] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[9] M. Hojo, W. Takahashi, and J. -C. Yao, *Weak and strong mean convergence theorems for super hybrid mappings in Hilbert spaces*, Fixed Point Theory **12** (2011), 113–126.

[10] M.-H. Hsu, W. Takahashi and J.-C. Yao, *Generalized hybrid mappings in Hilbert spaces and Banach spaces*, Taiwanese J. Math. **16** (2012), 129–149.

[11] S. Iemoto and W. Takahashi, *Approximating fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082–2089.

[12] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[13] M. Kikkawa and T. Suzuki, *Fixed point theorems for new nonlinear mappings satisfying condition (CC)*, Linear Nonlinear Anal. **1** (2015), 37–52.

[14] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.
[15] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.
[16] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.
[17] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.
[18] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[19] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.
[20] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 313–318.
[21] W. Takahashi, *Nonlinear Functional Analysis,* Yokohoma Publishers, Yokohoma, 2000.
[22] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.
[23] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[24] W. Takahashi and D. H. Jeong, *Fixed point theorem for nonexpansive semigroups on Banach space*, Proc. Amer. Math. Soc. **122** (1994), 1175–1179.
[25] W. Takahashi and J. -C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.
[26] W. Takahashi and J. -C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math. **15** (2011), 787–818.
[27] W. Takahashi and J. -C. Yao, *Weak convergence theorems for generalized hybrid mappings in Banach spaces*, J. Nonlinear Anal. Optim. **2** (2011), 133–143.
[28] W. Takahashi, J.-C. Yao and P. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 547–566.
[29] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1981), 1127–1138.

Manuscript received 28 February 2016
revised 24 September 2016

Mayumi Hojo
Center for Promotion of Educational Innovation Shibaura Institute of Technology Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama337-8570, Japan
E-mail address: `mayumi-h@shibaura-it.ac.jp`

Wataru Takahashi
Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: `wataru@is.titech.ac.jp; wataru@a00.itscom.net`