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EXISTENCE AND APPROXIMATION OF FIXED POINTS OF CHATTERJEA MAPPINGS WITH BREGMAN DISTANCES

FUMIAKI KOHSAKA AND TOMONARI SUZUKI

Abstract. We introduce the concept of Chatterjea mappings on a nonempty set and then obtain fixed point theorems and convergence theorems for Chatterjea mappings with respect to a Bregman distance and its symmetrization associated with the power of a norm in a Banach space. The class of Chatterjea mappings with respect to a Bregman distance includes that of nonspreading mappings in Hilbert spaces.

1. Introduction

Many nonlinear problems such as convex minimization problems, variational inequality problems, saddle point problems, equilibrium problems, and so on can be formulated as the problem of solving $0 \in Au$ for a maximal monotone operator A of a Banach space X into 2^{X^*}; see, for instance, [1, 14–16, 20].

The concept of nonspreading mappings proposed in [10] is closely related to the problem of finding zero points of maximal monotone operators in Banach spaces. In fact, in a smooth, strictly convex, and reflexive Banach space X, the set of all points $u \in X$ such that $0 \in Au$ is identical with the fixed point set of the nonspreading mapping T of X into itself defined by $Tx = (J_2 + A)^{-1} J_2 x$ for all $x \in X$; see [9, 10] for more details.

Let C be a nonempty subset of a smooth Banach space X and T a nonspreading mapping [10] of C into itself, that is,

$$\phi_2(Tx, Ty) + \phi_2(Ty, Tx) \leq \phi_2(Tx, y) + \phi_2(Ty, x) \tag{1.1}$$

for all $x, y \in C$, where ϕ_2 is the function defined as in (2.8). Kohsaka and Takahashi [10, Theorem 4.1] showed that if C is a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space, then the fixed point set $\mathcal{F}(T)$ of T is nonempty if and only if $\{T^n x\}$ is bounded for some $x \in C$. Further, Kurokawa and Takahashi [11, Theorem 3.1] showed that if C is a nonempty closed convex subset of a Hilbert space and $\mathcal{F}(T)$ is nonempty, then the sequence

$$\left\{ \frac{1}{n} \left(x + Tx + \cdots + T^{n-1} x \right) \right\} \tag{1.2}$$

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converges weakly to an element of $\mathcal{F}(T)$ for each $x \in C$. In the case where X is a Hilbert space, T is nonspreading if and only if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|x - Ty\|^2 \tag{1.3}$$

for all $x, y \in C$.

Recently, motivated by [5, 10], Suzuki [17] proposed the concept of Chatterjea mappings in Banach spaces and obtained existence and convergence theorems for such a mapping. Since the class of nonspreading mappings in Hilbert spaces is nothing but that of Chatterjea mappings with $t \mapsto t^2$, the results due to Suzuki [17] implies that the sequence $\{T^n x\}$ in the above result by Kurokawa and Takahashi [11] is actually weakly convergent to an element of $\mathcal{F}(T)$ without the convexity of C.

Let (C, d) be a metric space, η a continuous and strictly increasing function of $[0, \infty)$ into itself with $\eta(0) = 0$, and T a mapping of C into itself which is Chatterjea with η [17], that is,

$$2\eta\big(d(Tx, Ty)\big) \le \eta\big(d(Tx, y)\big) + \eta\big(d(x, Ty)\big) \tag{1.4}$$

for all $x, y \in C$. Suzuki [17, Theorems 15 and 18] showed that if X is a Banach space, C is a nonempty boundedly weakly compact subset of X with the Opial property, and T is a Chatterjea mapping of C into itself with η, then T has a fixed point if and only if $\{T^n x\}$ is bounded for some $x \in C$. In this case, the sequence $\{T^n x\}$ converges weakly to an element of $\mathcal{F}(T)$ for each $x \in C$.

In this paper, we propose the concept of ρ-Chatterjea mappings on a nonempty set in Definition 2.1 and obtain fixed point theorems and convergence theorems for mappings which are ϕ_p-Chatterjea or Φ_p-Chatterjea, where p is a real number such that $p > 1$ and the functions ϕ_p and Φ_p are defined by (2.8) and (2.9), respectively. Let C be a nonempty subset of a smooth Banach space X. Following Definition 2.1, a mapping T of C into itself is said to be

- ϕ_p-Chatterjea if
$$2\phi_p(Tx, Ty) \le \phi_p(Tx, y) + \phi_p(x, Ty) \tag{1.5}$$
for all $x, y \in C$;
- Φ_p-Chatterjea if
$$2\Phi_p(Tx, Ty) \le \Phi_p(Tx, y) + \Phi_p(x, Ty) \tag{1.6}$$
for all $x, y \in C$.

If X is a Hilbert space and $p = 2$, then the conditions (1.1), (1.5), and (1.6) are equivalent to (1.3) for each $x, y \in C$. Thus the results obtained in Sections 5 and 6 generalize the following result in Hilbert spaces to Banach spaces:

Theorem 1.1 (See [17, Theorem 15 and Corollary 20]). *Let C be a nonempty weakly closed subset of a Hilbert space X and T a nonspreading mapping of C into itself. Then T has a fixed point if and only if $\{T^n x\}$ is bounded for some $x \in C$. In this case, the sequence $\{T^n x\}$ converges weakly to an element of $\mathcal{F}(T)$ for all $x \in C$.*

2. Preliminaries

Throughout this paper, every Banach space is real. The set of all positive integers is denoted by $\mathbb{N}$. Let X be a Banach space with its conjugate space X^*. The value of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^* \rangle$. For a sequence $\{x_n\}$ in X and a point x in X, the strong convergence and the weak convergence of $\{x_n\}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote by $x_n^* \overset{*}{\rightharpoonup} x^*$ the weak* convergence of a sequence $\{x_n^*\}$ in X^* to $x^* \in X^*$. Let C be a nonempty subset of X and S a mapping of C into itself. We denote by $\mathcal{F}(S)$ the set of all fixed points of S. The mapping S is said to be demiclosed at 0 if $Sz = 0$ whenever $\{x_n\}$ is a sequence in C, z is an element of C, $x_n \rightharpoonup z$, and $Sx_n \to 0$. It is also said to be asymptotically regular at $x \in C$ if $S^n x - S^{n+1} x \to 0$. We denote by I the identity mapping on C.

We give the definition of a ρ-Chatterjea mapping of a nonempty set into itself.

Definition 2.1. Let C be a nonempty set and ρ a function of $C \times C$ into $[0, \infty)$ such that $\rho(x, x) = 0$ for all $x \in C$. A mapping T of C into itself is said to be ρ-Chatterjea if

$$2\rho(Tx, Ty) \leq \rho(Tx, y) + \rho(x, Ty) \tag{2.1}$$

for all $x, y \in C$.

Remark 2.2. If (C, d) is a metric space, η is a continuous and strictly increasing function of $[0, \infty)$ into itself with $\eta(0) = 0$, and T is a Chatterjea mapping of C into itself with η (see (1.4)), then T is $\eta \circ d$-Chatterjea.

Lemma 2.3. *Let α, β and γ be nonnegative real numbers satisfying $2\alpha \leq \beta + \gamma$. Then $2\alpha^r \leq \beta^r + \gamma^r$ for any real number $r > 1$.*

Proof. Since the function $t \mapsto t^r$ is nondecreasing and convex on $[0, \infty)$ and $\alpha \leq (\beta + \gamma)/2$, we obtain the desired result. □

Lemma 2.3 implies the following:

Lemma 2.4. *Let r be a real number such that $r > 1$, C a nonempty set, ρ a function of $C \times C$ into $[0, \infty)$ such that $\rho(x, x) = 0$ for all $x \in C$. If T is a ρ-Chatterjea mapping of C into itself, then it is $\rho(\cdot, \cdot)^r$-Chatterjea.*

The following lemma is of fundamental importance; see also [7, 18]:

Lemma 2.5 ([17, Lemma 11]). *Put*

$$\begin{aligned} I_0 &= \{(m, n) : m, n \in \mathbb{N} \cup \{0\},\ m \leq n\}; \\ I &= \{(m, n) : m, n \in \mathbb{N},\ m < n\}. \end{aligned} \tag{2.2}$$

If A is a function of I_0 into $[0, \infty)$ such that

- $A(0, n) \leq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$;
- $A(n, n) = 0$ *for all* $n \in \mathbb{N}$;
- $2A(m, n) \leq A(m, n - 1) + A(m - 1, n)$ *for all* $(m, n) \in I$,

then $\lim_n A(n, n + 1) = 0$.

Using Lemma 2.5, we can prove the following:

Lemma 2.6. *Let C be a nonempty set, ρ a function of $C \times C$ into $[0, \infty)$ such that $\rho(x, x) = 0$ for all $x \in C$, and T a ρ-Chatterjea mapping of C into itself. If x is an element of C such that $\sup_{m,n} \rho(T^m x, T^n x) < \infty$, then $\rho(T^n x, T^{n+1} x) \to 0$.*

Proof. By assumption, there exists a positive real number M such that

$$\rho(T^m x, T^n x) \leq M \tag{2.3}$$

for all $m, n \in \mathbb{N} \cup \{0\}$. Let I_0 be the same as in Lemma 2.5 and A a function of I_0 into $[0, \infty)$ defined by

$$A(m, n) = \frac{1}{M} \rho(T^m x, T^n x) \tag{2.4}$$

for all $(m, n) \in I_0$. Then all the assumptions in Lemma 2.5 hold. Thus we obtain $\lim_n A(n, n+1) = 0$, which implies the conclusion. □

Let p be a real number such that $p > 1$, X a Banach space, and S_X the unit sphere of X. The duality mapping J_p of X into X^* with weight $t \mapsto t^{p-1}$ is defined by

$$J_p x = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\| \, \|x^*\| \, , \; \|x^*\| = \|x\|^{p-1} \right\} \tag{2.5}$$

for all $x \in X$. The space X is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.6}$$

exists for all $x, y \in S_X$. The norm on X is said to be uniformly Gâteaux differentiable if (2.6) converges uniformly in $x \in S_X$ for each $y \in S_X$. The space X is said to be strictly convex if $\|x + y\| < 2$ for all distinct $x, y \in S_X$. It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\| \leq 2(1 - \delta)$ whenever $x, y \in S_X$ and $\|x - y\| \geq \varepsilon$; see [6,12,19,21] on geometry of Banach spaces. It is known that the following hold; see, for instance, [6, 21]:

- $J_p x = \|x\|^{p-1} J_2 (x / \|x\|)$ if $x \neq 0$ and $J_p x = \{0\}$ if $x = 0$;
- if X is smooth, then J_p is single valued;
- if X is strictly convex, then J_p is one-to-one;
- if X is strictly convex, then J_p is strictly monotone, that is,

$$\langle x - y, x^* - y^* \rangle > 0 \tag{2.7}$$

  whenever $x, y \in X$, $x \neq y$, $x^* \in J_p x$, and $y^* \in J_p y$.

The mapping J_p in a smooth Banach space X is said to be weakly sequentially continuous if $J_p x_n \overset{*}{\rightharpoonup} J_p x$ whenever $\{x_n\}$ is a sequence in X and x is an element of X such that $x_n \rightharpoonup x$. It is known that if $1 < p < \infty$ and $X = l^p$, then J_p is weakly sequentially continuous; see, for instance, [6, Proposition 4.14 in Chapter II].

Let p be a real number such that $p > 1$, X a smooth Banach space, and J_p the duality mapping of X into X^* with weight $t \mapsto t^{p-1}$. We denote by ϕ_p the Bregman distance associated with the convex function $\| \cdot \|^p$ defined by

$$\phi_p(x, y) = \|x\|^p - p \langle x - y, J_p y \rangle - \|y\|^p \tag{2.8}$$

for all $x, y \in X$. This concept was originally proposed by Bregman [2]; see also [3,4] for more details on Bregman distances. It is clear that the following hold:

- $\phi_p(x,y) = \|x\|^p - p\langle x, J_p y\rangle + (p-1)\|y\|^p$ for all $x, y \in X$;
- $\phi_p(x,y) \geq 0$ for all $x, y \in X$;
- $\phi_p(x,x) = 0$ for all $x \in X$.

If X is also strictly convex, then $\|\cdot\|^p$ is strictly convex; see [21, Theorem 3.7.2]. Thus, in this case, $\phi_p(x,y) > 0$ for all distinct $x, y \in X$; see [3, Proposition 1.1.4]. We denote by Φ_p the symmetrization of ϕ_p defined by

$$\Phi_p(x,y) = \phi_p(x,y) + \phi_p(y,x) \tag{2.9}$$

for all $x, y \in X$. It is clear that

$$\Phi_p(x,y) = p\langle x - y, J_p x - J_p y\rangle \tag{2.10}$$

for all $x, y \in X$.

Choosing the power of the norm on a smooth and uniformly convex Banach space as a convex function considered in [8, Lemma 3.1], we obtain the following:

Lemma 2.7 ([8, Lemma 3.1])**.** *Let p be a real number such that $p > 1$ and X a smooth and uniformly convex Banach space. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in X such that $\phi_p(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

We also know the following:

Lemma 2.8 ([13, Lemma 2.2])**.** *If the norm on a Banach space X is uniformly Gâteaux differentiable, then J_2 is uniformly norm-to-weak* continuous on each bounded subset of X, that is, $\lim_n \langle z, J_2 x_n - J_2 y_n\rangle = 0$ for all $z \in X$ whenever $\{x_n\}$ and $\{y_n\}$ are bounded sequences in X such that $x_n - y_n \to 0$.*

Using Lemma 2.8, we can prove the following:

Corollary 2.9. *If X is the same as in Lemma 2.8 and $1 < p < \infty$, then J_p is uniformly norm-to-weak* continuous on each bounded subset of X.*

Proof. Since J_p is identical with J_2 on S_X, Lemma 2.8 ensures that J_p is uniformly norm-to-weak* continuous on S_X. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in X such that $x_n - y_n \to 0$ and let $z \in X$ be given. Note that the sequence $\{\gamma_n\}$ defined by $\gamma_n = \langle z, J_p x_n - J_p y_n\rangle$ is bounded. Let γ be any cluster point of $\{\gamma_n\}$. Then there exists a subsequence $\{\gamma_{n_i}\}$ of $\{\gamma_n\}$ tending to γ.

If $x_{n_i} \to 0$ or $y_{n_i} \to 0$, then we can see that $J_p x_{n_i} \to 0$ and $J_p y_{n_i} \to 0$ and hence $\gamma = \lim_i \gamma_{n_i} = 0$. Thus we consider the case where neither $\{x_{n_i}\}$ nor $\{y_{n_i}\}$ converges strongly to 0. Then there exist a positive real number δ and a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$\left\|x_{n_{i_j}}\right\| \geq \delta \quad \text{and} \quad \left\|y_{n_{i_j}}\right\| \geq \delta \tag{2.11}$$

for all $j \in \mathbb{N}$. Set $u_j = x_{n_{i_j}}$ and $v_j = y_{n_{i_j}}$ and let M be a positive real number such that $\|x_n\| \le M$ and $\|y_n\| \le M$ for all $n \in \mathbb{N}$. Then we have

$$
\begin{aligned}
|\langle z, J_p u_j - J_p v_j\rangle| &= \left|\left\langle z, \|u_j\|^{p-1} J_2\left(\frac{u_j}{\|u_j\|}\right) - \|v_j\|^{p-1} J_2\left(\frac{v_j}{\|v_j\|}\right)\right\rangle\right| \\
&\le \left|\left\langle z, \|u_j\|^{p-1}\left(J_2\left(\frac{u_j}{\|u_j\|}\right) - J_2\left(\frac{v_j}{\|v_j\|}\right)\right)\right\rangle\right| \\
&\quad + \left|\left\langle z, \left(\|u_j\|^{p-1} - \|v_j\|^{p-1}\right) J_2\left(\frac{v_j}{\|v_j\|}\right)\right\rangle\right| \\
&\le M^{p-1}\left|\left\langle z, J_p\left(\frac{u_j}{\|u_j\|}\right) - J_p\left(\frac{v_j}{\|v_j\|}\right)\right\rangle\right| \\
&\quad + \left|\|u_j\|^{p-1} - \|v_j\|^{p-1}\right| \|z\|
\end{aligned} \tag{2.12}
$$

for all $j \in \mathbb{N}$. Since

$$
\begin{aligned}
\left\|\frac{u_j}{\|u_j\|} - \frac{v_j}{\|v_j\|}\right\| &\le \frac{1}{\|u_j\|\,\|v_j\|}\Big(\|v_j\|\,\|u_j - v_j\| + \big|\|v_j\| - \|u_j\|\big|\,\|v_j\|\Big) \\
&\le \frac{1}{\delta^2}\Big(\|v_j\|\,\|u_j - v_j\| + \big|\|v_j\| - \|u_j\|\big|\,\|v_j\|\Big) \to 0
\end{aligned} \tag{2.13}
$$

and J_p is uniformly norm-to-weak* continuous on S_X, we also know that

$$
\left\langle z, J_p\left(\frac{u_j}{\|u_j\|}\right) - J_p\left(\frac{v_j}{\|v_j\|}\right)\right\rangle \to 0. \tag{2.14}
$$

On the other hand, since $t \mapsto t^{p-1}$ is uniformly continuous on $[0, M]$ and $\|u_j\| - \|v_j\| \to 0$, we have $\|u_j\|^{p-1} - \|v_j\|^{p-1} \to 0$. Thus, by (2.12), we have

$$
\gamma = \lim_{j\to\infty} \gamma_{n_{i_j}} = \lim_{j\to\infty} \langle z, J_p u_j - J_p v_j\rangle = 0. \tag{2.15}
$$

Therefore, we conclude that $\gamma_n \to 0$. □

3. Lemmas

In this section, we obtain some fundamental lemmas on ϕ_p and Φ_p.

Lemma 3.1. *Let p be a real number such that $p > 1$, X a smooth Banach space, U a subset of X, and z an element of X. Then the following are equivalent:*

- (i) *U is bounded;*
- (ii) *$\{\phi_p(x, y) : x, y \in U\}$ is bounded;*
- (iii) *$\{\phi_p(x, z) : x \in U\}$ is bounded;*
- (iv) *$\{\phi_p(z, x) : x \in U\}$ is bounded.*

Proof. Since

$$
\phi_p(x, y) \le \|x\|^p + p\,\|x\|\,\|y\|^{p-1} + (p-1)\,\|y\|^p, \tag{3.1}
$$

$$
\|x\|\left(\|x\|^{p-1} - p\,\|y\|^{p-1}\right) + (p-1)\,\|y\|^p \le \phi_p(x, y), \tag{3.2}
$$

and

$$
\|x\|^p + \|y\|^{p-1}\left(-p\,\|x\| + (p-1)\,\|y\|\right) \le \phi_p(x, y) \tag{3.3}
$$

for all $x, y \in X$, the conclusion clearly holds. □

Lemma 3.2. *Let p be a real number such that $p > 1$, X a smooth Banach space, both $\{x_n\}$ and $\{y_n\}$ sequences in X, and both z and w elements of X. Then the following hold:*

(i) *If $x_n \rightharpoonup z$, then*

$$\phi_p(x_n, z) - \phi_p(x_n, w) \to -\phi_p(z, w);$$

(ii) *if $x_n - y_n \to 0$ and $\{x_n\}$ is bounded, then $\phi_p(x_n, z) - \phi_p(y_n, z) \to 0$;*

(iii) *if $x_n \rightharpoonup z$ and J_p is weakly sequentially continuous, then*

$$\phi_p(z, x_n) - \phi_p(w, x_n) \to -\phi_p(w, z);$$

(iv) *if $x_n - y_n \to 0$, $\{x_n\}$ is bounded, and the norm on X is uniformly Gâteaux differentiable, then $\phi_p(z, x_n) - \phi_p(z, y_n) \to 0$.*

Proof. We first prove (i). Suppose that $x_n \rightharpoonup z$. Then we have

$$\begin{aligned} &\phi_p(x_n, z) - \phi_p(x_n, w) \\ &\quad = p\,\langle x_n, J_p w - J_p z\rangle + (p-1)\big(\|z\|^p - \|w\|^p\big) \\ &\quad \to p\,\langle z, J_p w - J_p z\rangle + (p-1)\big(\|z\|^p - \|w\|^p\big) = -\phi_p(z, w). \end{aligned} \tag{3.4}$$

We next prove (ii). Suppose that $x_n - y_n \to 0$ and $\{x_n\}$ is bounded. Then there exists a positive real number M such that $\|x_n\| \le M$ and $\|y_n\| \le M$ for all $n \in \mathbb{N}$. The uniform continuity of $t \mapsto t^p$ on $[0, M]$ implies that $\|x_n\|^p - \|y_n\|^p \to 0$ and hence

$$\phi_p(x_n, z) - \phi_p(y_n, z) = \|x_n\|^p - \|y_n\|^p - p\,\langle x_n - y_n, J_p z\rangle \to 0. \tag{3.5}$$

We next prove (iii). Suppose that $x_n \rightharpoonup z$ and J_p is weakly sequentially continuous. Then it follows from the weak* convergence of $\{J_p x_n\}$ to $J_p z$ that

$$\begin{aligned} \phi_p(z, x_n) - \phi_p(w, x_n) &= \|z\|^p - \|w\|^p + p\,\langle w - z, J_p x_n\rangle \\ &\to \|z\|^p - \|w\|^p + p\,\langle w - z, J_p z\rangle = -\phi_p(w, z). \end{aligned} \tag{3.6}$$

We finally prove (iv). Suppose that $x_n - y_n \to 0$, $\{x_n\}$ is bounded, and the norm on X is uniformly Gâteaux differentiable. Then we have $\|x_n\|^p - \|y_n\|^p \to 0$. Corollary 2.9 also implies that $\langle z, J_p x_n - J_p y_n\rangle \to 0$. Thus we have

$$\phi_p(z, x_n) - \phi_p(z, y_n) = -p\,\langle z, J_p x_n - J_p y_n\rangle + (p-1)\big(\|x_n\|^p - \|y_n\|^p\big) \to 0. \tag{3.7}$$

This completes the proof. □

Lemma 3.3. *Let p be a real number such that $p > 1$, X a smooth and strictly convex Banach space, both $\{x_n\}$ and $\{y_n\}$ sequences in X, and both $\{x_{n_i}\}$ and $\{x_{m_j}\}$ subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ and $x_{m_j} \rightharpoonup w$. Then the following hold:*

(i) *If both $\{\phi_p(x_n, z)\}$ and $\{\phi_p(x_n, w)\}$ are convergent, then $z = w$;*

(ii) *if both $\{\phi_p(z, x_n)\}$ and $\{\phi_p(w, x_n)\}$ are convergent and J_p is weakly sequentially continuous, then $z = w$;*

(iii) *if both $\{\Phi_p(x_n, z)\}$ and $\{\Phi_p(x_n, w)\}$ are convergent and J_p is weakly sequentially continuous, then $z = w$.*

Proof. We first prove (i). Suppose that both $\{\phi_p(x_n, z)\}$ and $\{\phi_p(x_n, w)\}$ are convergent and X is strictly convex. Since

$$\phi_p(x_n, w) - \phi_p(x_n, z) = p\,\langle x_n, J_p z - J_p w\rangle + (p-1)\big(\|w\|^p - \|z\|^p\big) \tag{3.8}$$

for all $n \in \mathbb{N}$, we know that $\big\{\langle x_n, J_p z - J_p w\rangle\big\}$ is also convergent. Hence we have

$$\begin{aligned}\langle z, J_p z - J_p w\rangle &= \lim_{i\to\infty} \langle x_{n_i}, J_p z - J_p w\rangle \\ &= \lim_{n\to\infty} \langle x_n, J_p z - J_p w\rangle \\ &= \lim_{j\to\infty} \big\langle x_{m_j}, J_p z - J_p w\big\rangle = \langle w, J_p z - J_p w\rangle\end{aligned} \tag{3.9}$$

and hence

$$\langle z - w, J_p z - J_p w\rangle = 0. \tag{3.10}$$

The strict convexity of X implies that $z = w$.

We next prove (ii). Suppose that both $\{\phi_p(z, x_n)\}$ and $\{\phi_p(w, x_n)\}$ are convergent, J_p is weakly sequentially continuous, and X is strictly convex. Since

$$\phi_p(w, x_n) - \phi_p(z, x_n) = \|w\|^p - \|z\|^p + p\,\langle z - w, J_p x_n\rangle \tag{3.11}$$

for all $n \in \mathbb{N}$, we know that $\big\{\langle z - w, J_p x_n\rangle\big\}$ is also convergent. Since J_p is weakly sequentially continuous, we have

$$\begin{aligned}\langle z - w, J_p z\rangle &= \lim_{i\to\infty} \langle z - w, J_p x_{n_i}\rangle \\ &= \lim_{n\to\infty} \langle z - w, J_p x_n\rangle \\ &= \lim_{j\to\infty} \big\langle z - w, J_p x_{m_j}\big\rangle = \langle z - w, J_p w\rangle\end{aligned} \tag{3.12}$$

and hence

$$\langle z - w, J_p z - J_p w\rangle = 0. \tag{3.13}$$

The strict convexity of X implies that $z = w$.

We finally prove (iii). Suppose that both $\{\Phi_p(x_n, z)\}$ and $\{\Phi_p(x_n, w)\}$ are convergent, J_p is weakly sequentially continuous, and X is strictly convex. Since

$$\begin{aligned}&\Phi_p(x_n, w) - \Phi_p(x_n, z) \\ &\quad = \phi_p(x_n, w) - \phi_p(x_n, z) + \phi_p(w, x_n) - \phi_p(z, x_n) \\ &\quad = p\big(\|w\|^p - \|z\|^p\big) + p\,\langle x_n, J_p z - J_p w\rangle + p\,\langle z - w, J_p x_n\rangle\end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N}$, we know that

$$\big\{\langle x_n, J_p z - J_p w\rangle + \langle z - w, J_p x_n\rangle\big\} \tag{3.15}$$

is convergent. Since J_p is weakly sequentially continuous, we have

$$\begin{aligned}\langle z, J_p z - J_p w\rangle + \langle z - w, J_p z\rangle &= \lim_{i\to\infty} \big\{\langle x_{n_i}, J_p z - J_p w\rangle + \langle z - w, J_p x_{n_i}\rangle\big\} \\ &= \lim_{n\to\infty} \big\{\langle x_n, J_p z - J_p w\rangle + \langle z - w, J_p x_n\rangle\big\} \\ &= \lim_{j\to\infty} \big\{\big\langle x_{m_j}, J_p z - J_p w\big\rangle + \big\langle z - w, J_p x_{m_j}\big\rangle\big\} \\ &= \langle w, J_p z - J_p w\rangle + \langle z - w, J_p w\rangle\end{aligned} \tag{3.16}$$

and hence

$$2\langle z - w, J_p z - J_p w\rangle = 0. \tag{3.17}$$

The strict convexity of X implies that $z = w$. □

4. ϕ_p-Chatterjea mappings and Φ_p-Chatterjea mappings

In this section, we obtain some preliminary results for ϕ_p-Chatterjea mappings and Φ_p-Chatterjea mappings in Banach spaces.

Definition 2.1 readily implies the following:

Lemma 4.1. *Let C be a nonempty set, ρ a function of $C \times C$ into $[0, \infty)$ such that $\rho(x, x) = 0$ for all $x \in C$, and T a ρ-Chatterjea mapping of C into itself such that $\mathcal{F}(T)$ is nonempty. Then $\rho(Tx, y) \le \rho(x, y)$ and $\rho(y, Tx) \le \rho(y, x)$ for all $x \in C$ and $y \in \mathcal{F}(T)$.*

Every ϕ_p-Chatterjea mapping is also Φ_p-Chatterjea.

Lemma 4.2. *Let p be a real number such that $p > 1$, C a nonempty subset of a smooth Banach space X, and T a mapping of C into itself. If T is ϕ_p-Chatterjea, then T is Φ_p-Chatterjea.*

Proof. Suppose that T is ϕ_p-Chatterjea. If $x, y \in C$, then we have

$$2\phi_p(Tx, Ty) \le \phi_p(Tx, y) + \phi_p(x, Ty) \tag{4.1}$$

and

$$2\phi_p(Ty, Tx) \le \phi_p(Ty, x) + \phi_p(y, Tx). \tag{4.2}$$

Adding these inequalities, we have

$$2\Phi_p(Tx, Ty) \le \Phi_p(Tx, y) + \Phi_p(x, Ty). \tag{4.3}$$

Thus T is Φ_p-Chatterjea. □

Using a mapping T found by Suzuki [17, Example 7], we can prove the following:

Example 4.3. *Let p and q be real numbers such that $q > p > 1$, w a nonzero element of a smooth Banach space X, and T a mapping of X into itself defined by*

$$Tx = \begin{cases} 0 & (x \ne w); \\ 2^{-1/q} w & (x = w). \end{cases} \tag{4.4}$$

Then T is ϕ_q-Chatterjea and T is not Φ_p-Chatterjea.

Proof. Let $x, y \in X$ be given. If either $x \ne w$ and $y \ne w$, or $x = y = w$ hold, then we have

$$2\phi_q(Tx, Ty) = 0 \le \phi_q(Tx, y) + \phi_q(x, Ty). \tag{4.5}$$

If $x = w$ and $y \ne w$, then $Tx = 2^{-1/q} w$ and $Ty = 0$ and hence

$$2\phi_q(Tx, Ty) = \|w\|^q = \phi_q(x, Ty) \le \phi_q(Tx, y) + \phi_q(x, Ty). \tag{4.6}$$

If $x \ne w$ and $y = w$, then $Tx = 0$ and $Ty = 2^{-1/q} w$ and hence

$$2\phi_q(Tx, Ty) = (q - 1)\|w\|^q = \phi_q(Tx, y) \le \phi_q(Tx, y) + \phi_q(x, Ty). \tag{4.7}$$

Thus T is ϕ_q-Chatterjea.

Since $T^2 w = 0$, $1 - p/q > 0$, and $w \neq 0$, we also know that

$$\begin{aligned} 2\Phi_p(Tw, T^2w) &= 2\big(\phi_p(Tw, T^2w) + \phi_p(T^2w, Tw)\big) \\ &= 2\Big(\left\|2^{-1/q}w\right\|^p + (p-1)\left\|2^{-1/q}w\right\|^p\Big) \\ &= 2^{1-p/q} p \|w\|^p \\ &> p\|w\|^p = \Phi_p(Tw, Tw) + \Phi_p(w, T^2w) \end{aligned} \tag{4.8}$$

and hence T is not Φ_p-Chatterjea. □

By using Lemmas 2.6 and 2.7, we can prove the following:

Lemma 4.4. *Let p be a real number such that $p > 1$, C a nonempty subset of a smooth and uniformly convex Banach space X, T a Φ_p-Chatterjea mapping of C into itself, and x an element of C such that $\{T^n x\}$ is bounded. Then T is asymptotically regular at x.*

Proof. Since $\{T^n x\}$ is bounded, it follows from Lemma 3.1 that

$$\sup_{m,n} \Phi_p(T^m x, T^n x) \leq 2 \sup_{m,n} \phi_p(T^m x, T^n x) < \infty. \tag{4.9}$$

Thus Lemma 2.6 ensures that

$$0 \leq \phi_p(T^n x, T^{n+1}x) \leq \Phi_p(T^n x, T^{n+1}x) \to 0 \tag{4.10}$$

and hence $\phi_p(T^n x, T^{n+1}x) \to 0$. Since X is uniformly convex, Lemma 2.7 implies that $\left\|T^n x - T^{n+1}x\right\| \to 0$. Thus T is asymptotically regular at x. □

We next obtain the following lemma:

Lemma 4.5. *Let p be a real number such that $p > 1$, C a nonempty weakly closed subset of a smooth and uniformly convex Banach space X, and T a Φ_p-Chatterjea mapping of C into itself such that $I - T$ is demiclosed at 0. Then $\mathcal{F}(T)$ is nonempty if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

Proof. Since the only if part is obvious, it is sufficient to prove the if part. Suppose that $\{T^n x\}$ is bounded for some $x \in C$ and set $x_n = T^n x$ for all $n \in \mathbb{N}$. Then, by Lemma 4.4, we know that

$$\|x_n - Tx_n\| = \left\|T^n x - T^{n+1}x\right\| \to 0. \tag{4.11}$$

Since $\{x_n\}$ is bounded and X is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is weakly convergent to some $z \in X$. The weak closedness of C implies that $z \in C$. Since $I - T$ is demiclosed at 0 by assumption, we know that $(I - T)z = 0$ and hence $z \in \mathcal{F}(T)$. □

5. Existence of fixed points

In this section, we give fixed point theorems for ϕ_p-Chatterjea mappings and Φ_p-Chatterjea mappings in Banach spaces.

We first obtain the following two demiclosedness principles:

Lemma 5.1. *Let p be a real number such that $p > 1$, C a nonempty subset of a smooth and strictly convex Banach space X, and T a ϕ_p-Chatterjea mapping of C into itself. Then $I - T$ is demiclosed at 0.*

Proof. Let $\{x_n\}$ be a sequence in C and z an element of C such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Since T is ϕ_p-Chatterjea, we have

$$0 \leq \phi_p(Tx_n, z) - \phi_p(Tx_n, Tz) + \phi_p(x_n, Tz) - \phi_p(Tx_n, Tz) \tag{5.1}$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, we have $Tx_n \rightharpoonup z$. Then it follows from (i) and (ii) of Lemma 3.2 that

$$\phi_p(Tx_n, z) - \phi_p(Tx_n, Tz) \to -\phi_p(z, Tz) \tag{5.2}$$

and

$$\phi_p(x_n, Tz) - \phi_p(Tx_n, Tz) \to 0, \tag{5.3}$$

respectively. Thus, letting $n \to \infty$ in (5.1), we obtain $0 \leq -\phi_p(z, Tz)$ and hence we have $\phi_p(z, Tz) = 0$. By the strict convexity of X, we obtain $Tz = z$. □

Lemma 5.2. *Let p be a real number such that $p > 1$, C a nonempty subset of a strictly convex Banach space X such that the norm on X is uniformly Gâteaux differentiable and J_p is weakly sequentially continuous, and T a Φ_p-Chatterjea mapping of C into itself. Then $I - T$ is demiclosed at 0.*

Proof. Let $\{x_n\}$ be a sequence in C and z an element of C such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Since T is Φ_p-Chatterjea, we have

$$0 \leq \Phi_p(Tx_n, z) - \Phi_p(Tx_n, Tz) + \Phi_p(x_n, Tz) - \Phi_p(Tx_n, Tz) \tag{5.4}$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, we have $Tx_n \rightharpoonup z$. Since J_p is weakly sequentially continuous, it follows from (i) and (iii) of Lemma 3.2 that

$$\begin{aligned} &\Phi_p(Tx_n, z) - \Phi_p(Tx_n, Tz) \\ &= \phi_p(Tx_n, z) - \phi_p(Tx_n, Tz) + \phi_p(z, Tx_n) - \phi_p(Tz, Tx_n) \\ &\to -\phi_p(z, Tz) - \phi_p(Tz, z) = -\Phi_p(Tz, z). \end{aligned} \tag{5.5}$$

Since the norm on X is uniformly Gâteaux differentiable, $\{x_n\}$ is bounded, and $x_n - Tx_n \to 0$, it follows from (ii) and (iv) of Lemma 3.2 that

$$\begin{aligned} &\Phi_p(x_n, Tz) - \Phi_p(Tx_n, Tz) \\ &= \phi_p(x_n, Tz) - \phi_p(Tx_n, Tz) + \phi_p(Tz, x_n) - \phi_p(Tz, Tx_n) \to 0. \end{aligned} \tag{5.6}$$

Thus, letting $n \to \infty$ in (5.4), we obtain $0 \leq -\Phi_p(Tz, z)$ and hence we have $\Phi_p(Tz, z) = 0$. It follows from (2.10) that

$$p \langle z - Tz, J_p z - J_p Tz \rangle = 0. \tag{5.7}$$

By the strict convexity of X, we obtain $Tz = z$. □

As a consequence of Lemmas 4.2, 4.5, and 5.1, we obtain the following fixed point theorem for ϕ_p-Chatterjea mappings:

Theorem 5.3. *Let p be a real number such that $p > 1$, C a nonempty weakly closed subset of a smooth and uniformly convex Banach space X, and T a ϕ_p-Chatterjea mapping of C into itself. Then $\mathcal{F}(T)$ is nonempty if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

As a consequence of Lemmas 4.5 and 5.2, we obtain the following fixed point theorem for Φ_p-Chatterjea mappings:

Theorem 5.4. *Let p be a real number such that $p > 1$, C a nonempty weakly closed subset of a uniformly convex Banach space X such that the norm on X is uniformly Gâteaux differentiable and J_p is weakly sequentially continuous, and T a Φ_p-Chatterjea mapping of C into itself. Then $\mathcal{F}(T)$ is nonempty if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

6. Convergence to fixed points

In this section, we give convergence theorems for ϕ_p-Chatterjea mappings and Φ_p-Chatterjea mappings in Banach spaces.

We first obtain the following convergence theorem for ϕ_p-Chatterjea mappings:

Theorem 6.1. *Let p be a real number such that $p > 1$, C a nonempty weakly closed subset of a smooth and uniformly convex Banach space X, and T a ϕ_p-Chatterjea mapping of C into itself such that $\mathcal{F}(T)$ is nonempty. Then $\{T^n x\}$ converges weakly to an element of $\mathcal{F}(T)$ for all $x \in C$.*

Proof. Let $x \in C$ be given and set $x_n = T^n x$ for all $n \in \mathbb{N}$. Let $z \in \mathcal{F}(T)$ be given. Since T is ϕ_p-Chatterjea, it follows from Lemma 4.1 that

$$\phi_p(x_{n+1}, z) \leq \phi_p(x_n, z) \tag{6.1}$$

for all $n \in \mathbb{N}$. Thus $\{\phi_p(x_n, z)\}$ is convergent. Since $\{\phi_p(x_n, z)\}$ is bounded, it follows from Lemma 3.1 that $\{x_n\}$ is bounded. Then the reflexivity of X implies the existence of a weakly convergent subsequence of $\{x_n\}$.

Lemma 4.2 implies that T is also Φ_p-Chatterjea. Since X is uniformly convex, Lemma 4.4 implies that $x_n - Tx_n \to 0$. Let w be any weak subsequential limit of $\{x_n\}$. Since C is weakly closed, we have $w \in C$. By Lemma 5.1, the mapping $I - T$ is demiclosed at 0 and hence $w \in \mathcal{F}(T)$. Thus each weak subsequential limit of $\{x_n\}$ belongs to $\mathcal{F}(T)$.

Let u, u' be weak subsequential limits of $\{x_n\}$. Then we know that $u, u' \in \mathcal{F}(T)$ and hence $\{\phi_p(x_n, u)\}$ and $\{\phi_p(x_n, u')\}$ are convergent. Since X is strictly convex, it follows from (i) of Lemma 3.3 that $u = u'$. Thus, the sequence $\{x_n\}$ is weakly convergent to an element of $\mathcal{F}(T)$. □

We finally obtain the following convergence theorem for Φ_p-Chatterjea mappings:

Theorem 6.2. *Let p be a real number such that $p > 1$, C a nonempty weakly closed subset of a uniformly convex Banach space X such that the norm on X is uniformly Gâteaux differentiable and J_p is weakly sequentially continuous, and T a Φ_p-Chatterjea mapping of C into itself such that $\mathcal{F}(T)$ is nonempty. Then $\{T^n x\}$ converges weakly to an element of $\mathcal{F}(T)$ for all $x \in C$.*

Proof. Let $x \in C$ be given and set $x_n = T^n x$ for all $n \in \mathbb{N}$. Let $z \in \mathcal{F}(T)$ be given. Since T is Φ_p-Chatterjea, it follows from Lemma 4.1 that

$$\Phi_p(x_{n+1}, z) \leq \Phi_p(x_n, z) \tag{6.2}$$

for all $n \in \mathbb{N}$. Thus $\{\Phi_p(x_n, z)\}$ is convergent. Since $\{\Phi_p(x_n, z)\}$ is bounded and

$$\phi_p(x_n, z) \leq \Phi_p(x_n, z) \tag{6.3}$$

for all $n \in \mathbb{N}$, it follows from Lemma 3.1 that $\{x_n\}$ is bounded. Hence there exists a weakly convergent subsequence of $\{x_n\}$.

Since X is uniformly convex, it follows from Lemma 4.4 that $x_n - Tx_n \to 0$. Let w be any weak subsequential limit of $\{x_n\}$. Then the weak closedness of C implies that $w \in C$. By Lemma 5.2, the mapping $I - T$ is demiclosed at 0 and hence $w \in \mathcal{F}(T)$. Thus each weak subsequential limit of $\{x_n\}$ belongs to $\mathcal{F}(T)$.

Let u, u' be weak subsequential limits of $\{x_n\}$. Then we know that $u, u' \in \mathcal{F}(T)$ and hence $\{\Phi_p(x_n, u)\}$ and $\{\Phi_p(x_n, u')\}$ are convergent. Since J_p is weakly sequentially continuous and X is strictly convex, it follows from (iii) of Lemma 3.3 that $u = v$. Thus, the sequence $\{x_n\}$ is weakly convergent to an element of $\mathcal{F}(T)$. □

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F. Kohsaka
Department of Mathematical Sciences, Tokai University, Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan
E-mail address: `f-kohsaka@tsc.u-tokai.ac.jp`

T. Suzuki
Department of Basic Sciences, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan
E-mail address: `suzuki-t@mns.kyutech.ac.jp`