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ON SEQUENTIAL OPTIMALITY THEOREMS FOR CONVEX OPTIMIZATION PROBLEMS

JAE HYOUNG LEE AND GUE MYUNG LEE*

Abstract. In this paper, we give two kinds of sequential optimality theorems for a convex optimization problem, which are expressed in terms of sequences of $\epsilon-$subgradients and subgradients of involved functions. The involved functions of the problem are proper, lower semicontinuous and convex. We give sufficient conditions for the closedness of characterization cones for the problem. Moreover, we characterize the solution set for the problem. Several numerical examples are presented to illustrate results.

1. Introduction

Consider the following convex programming problem

$$\text{(CP)} \quad \min \ f(x) \quad \text{s.t.} \ g_i(x) \leqq 0, \ i = 1, \dots, m,$$

where $\overline{\mathbb{R}} = [-\infty, +\infty]$ and f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \dots, m$, are proper lower semicontinuous convex functions.

Recently, new sequential Lagrange multiplier conditions characterizing optimality without any constraint qualification for convex programs were presented in terms of the subgradients and the $\epsilon-$subgradients ([8, 9, 12]). It was also shown how the sequential conditions are related to the standard Lagrange multiplier condition ([9, 12]).

The characterization of the solution set of all optimal solutions of optimization problems is very important for understanding the behavior of solution methods for optimization programming problems that have multiple solutions ([3, 4, 10, 11, 13, 15]). Recently, various characterizations of the solution set of the convex optimization problem have been developed ([3, 7, 13]).

In this paper, we give two kinds of sequential optimality theorems for a convex optimization problem, which are expressed in terms of sequences of $\epsilon-$subgradients and subgradients of involved functions. The involved functions of the problem are proper, lower semi-continuous and convex functions. We give sufficient conditions

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*Corresponding Author.

for the closedness of characterization cones related to constraint qualifications for the problem. Moreover, we give a characterization of the solution set for the problem.

The paper is organized as follows: in Section 2, some basic definitions and preliminary results are given. In Section 3 and 4, we establish two kinds of sequential optimality theorems for a convex optimization problem. In Section 5, we give sufficient conditions for the closedness of a characteristic cone. In Section 6, we give a characterization of the solution sets for the convex optimization problem.

2. Preliminaries

Let us first recall some notations and preliminary results which will be used throughout this thesis.

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geqq 0\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set A in $\mathbb{R}^n$ is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$.

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by dom$f$, that is, $\text{dom} f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$. The epigraph of f, epi$f$, is defined as $\text{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqq r\}$, and f is said to be convex if epi$f$ is convex. The function f is said to be concave whenever $-f$ is convex.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

More generally, for any $\epsilon \geqq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \epsilon,\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We say that f is a lower semicontinuous function if $\liminf_{y \to x} f(y) \geqq f(x)$ for all $x \in \mathbb{R}^n$.

As usual, for any proper convex function g on $\mathbb{R}^n$, its conjugate function $g^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) \mid x \in \mathbb{R}^n\}$ for any $x^* \in \mathbb{R}^n$.

For a given set $A \subset \mathbb{R}^n$, we denote the closure, the convex hull, and the conical hull generated by A, by cl$A$, co$A$, and cone$A$, respectively. The indicator function δ_A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

The following proposition, which describes the relationship between the epigraph of a conjugate function and the ϵ-subdifferential and plays a key role in deriving the main results, was recently given in [5].

Proposition 2.1 ([5])**.** *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and if $a \in \mathrm{dom} f$, then*

$$\mathrm{epi} f^* = \bigcup_{\epsilon \geqq 0} \{(v, \langle v, a\rangle + \epsilon - f(a)) \mid v \in \partial_\epsilon f(a)\}.$$

Proposition 2.2 ([6])**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. If $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$, then*

$$\mathrm{epi}(f+g)^* = \mathrm{cl}(\mathrm{epi} f^* + \mathrm{epi} g^*).$$

Moreover, if one of the functions f and g is continuous, then

$$\mathrm{epi}(f+g)^* = \mathrm{epi} f^* + \mathrm{epi} g^*.$$

We recall a version of the Brondsted-Rockafellar theorem which was established in [14].

Proposition 2.3 ([1,14, Brondsted-Rockafellar Theorem])**.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then for any real number $\epsilon > 0$ and any $x^* \in \partial_\epsilon f(\bar{x})$ there exist $x_\epsilon \in \mathbb{R}^n$, $x^*_\epsilon \in \partial f(x_\epsilon)$ such that*

$$\|x_\epsilon - \bar{x}\| \leq \sqrt{\epsilon}, \quad \|x^*_\epsilon - x^*\| \leq \sqrt{\epsilon} \quad \text{and} \quad |f(x_\epsilon) - x^*_\epsilon(x_\epsilon - \bar{x}) - f(\bar{x})| \leq 2\epsilon.$$

3. Sequential optimality theorems I

Now we give sequential optimality theorems for (CP), which are expressed sequences of epsilon subgradients of involved functions. The involved functions of the problem are proper, lower semi-continuous and convex functions.

Theorem 3.1. *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \ldots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP)*;*

(ii) *there exist $\delta_k \geqq 0$, $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $\xi_k \in \partial_{\delta_k} f(\bar{x})$, $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that*

$$\lim_{k\to\infty} (\xi_k + \zeta_k) = 0, \ \lim_{k\to\infty} (\delta_k + \gamma_k) = 0$$
$$\text{and} \ \lim_{k\to\infty} (\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) = 0.$$

Proof. Assume that $\bar{x}$ is an optimal solution of (CP). It means that $f(x) \geqq f(\bar{x})$ for any $x \in A$, that is,

$$f(x) + \delta_A(x) \geqq f(\bar{x}) + \delta_A(\bar{x}), \ \forall x \in \mathbb{R}^n.$$

Since $\langle 0, x\rangle - (f(x) + \delta_A(x)) \leqq \langle 0, x\rangle - (f(\bar{x}) + \delta_A(\bar{x}))$, for any $x \in \mathbb{R}^n$,

$$(f + \delta_A)^*(0) = -(f(\bar{x}) + \delta_A(\bar{x})) = -f(\bar{x}).$$

It means that

$$(0, -f(\bar{x})) \in \mathrm{epi}(f + \delta_A)^*.$$

From Proposition 2.2, equivalently,

$$(0, -f(\bar{x})) \in \text{cl}(\text{epi}f^* + \text{epi}\delta_A^*). \tag{3.1}$$

Since $\text{epi}\delta_A^* = \text{cl}\bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*$, from (3.1),

$$(0, -f(\bar{x})) \in \text{cl}(\text{epi}f^* + \text{cl}\bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*).$$

Since $\text{cl}(\text{epi}f^* + \text{cl}\bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*) = \text{cl}(\text{epi}f^* + \bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*)$, we have,

$$(0, -f(\bar{x})) \in \text{cl}(\text{epi}f^* + \bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*).$$

By Proposition 2.1, we see that

$$\begin{aligned} (0, -f(\bar{x})) \in\ & \text{cl}(\bigcup_{\epsilon \geqq 0}\{(\xi, \langle \xi, \bar{x}\rangle + \epsilon - f(\bar{x})) \mid \xi \in \partial_\epsilon f(\bar{x})\} \\ & + \bigcup_{\epsilon \geqq 0}\{(\zeta, \langle \zeta, \bar{x}\rangle + \epsilon - \sum_{i=1}^m \lambda_i g_i(\bar{x})) \mid \zeta \in \partial_\epsilon(\sum_{i=1}^m \lambda_i g_i)(\bar{x})\}). \end{aligned} \tag{3.2}$$

From (3.2), we see that there exist $\delta_k \geqq 0$, $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $\xi_k \in \partial_{\delta_k} f(\bar{x})$, $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that $(\xi_k, \langle \xi_k, \bar{x}\rangle + \delta_k - f(\bar{x})) + (\zeta_k, \langle \zeta_k, \bar{x}\rangle + \gamma_k - \sum_{i=1}^m \lambda_i^k g_i(\bar{x})) \to (0, -f(\bar{x}))$ as $k \to \infty$. Since $\lambda_i^k g_i(\bar{x}) \leqq 0$, $i = 1, \ldots, m$, there exist $\delta_k \geqq 0$, $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $\xi_k \in \partial_{\delta_k} f(\bar{x})$, $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that

$$\lim_{k\to\infty}(\xi_k + \zeta_k) = 0, \ \lim_{k\to\infty}(\delta_k + \gamma_k) = 0$$
$$\text{and} \quad \lim_{k\to\infty}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) = 0.$$

□

Now we give an example illustrating Theorem 3.1:

Example 3.2. Let $f(x) = \begin{cases} x\log x, & x > 0, \\ 0, & x = 0, \\ +\infty, & x < 0 \end{cases}$ and $g(x) = \max\{0, x\}$. Then $f^*(y) = e^{y-1}$ for all $y \in \mathbb{R}$ and $g^*(y) = \begin{cases} 0, & 0 \leqq y \leqq 1, \\ +\infty, & \text{otherwise.} \end{cases}$ So, $\text{epi}f^* = \{(y, \alpha) \in \mathbb{R} \times \mathbb{R} : e^{y-1} \leqq \alpha\}$ and $\bigcup_{\lambda \geqq 0} \text{epi}(\lambda g)^* = \mathbb{R}_+^2$. Hence, we see that $\text{epi}f^*$ and $\bigcup_{\lambda \geqq 0} \text{epi}(\lambda g)^*$ are closed. However, $\text{epi}f^* + \bigcup_{\lambda \geqq 0} \text{epi}(\lambda g)^* = \mathbb{R} \times (0, +\infty)$ is not closed.

Now we consider the following convex optimization problem $(\text{CP})_1$

$$\begin{aligned} (\text{CP})_1 \quad \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \leqq 0. \end{aligned}$$

Then the feasible set of $(\mathrm{CP})_1$ is $(-\infty, 0]$. So, we can easily see that $\bar{x} = 0$ is an optimal solution of $(\mathrm{CP})_1$. For each $k \in \mathbb{N}$, if we take $\delta_k = e^{-k} \geqq 0$, $\gamma_k = \frac{1}{k} \geqq 0$ and $\lambda^k = k - 1 \geqq 0$, then we can easily calculate that

$$\begin{aligned} \partial_{\delta_k} f(\bar{x}) &= (-\infty, \log \delta_k + 1] = (-\infty, -k+1] \text{ and} \\ \partial_{\gamma_k}(\lambda^k g)(\bar{x}) &= [0, \lambda^k] = [0, k-1]. \end{aligned}$$

Let $\xi_k = -k + 1 \in \partial_{\delta_k} f(\bar{x})$ and $\zeta_k = k - 1 \in \partial_{\gamma_k}(\lambda^k g)(\bar{x})$. Then we can easily see that

$$\lim_{k\to\infty}(\xi_k + \zeta_k) = 0, \ \lim_{k\to\infty}(\delta_k + \gamma_k) = 0 \quad \text{and} \quad \lim_{k\to\infty}(\lambda^k g)(\bar{x}) = 0.$$

Thus, Theorem 3.1 holds.

Theorem 3.3. *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0, \ i = 1, \ldots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$. Assume that* $\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ *is closed. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP)*;*
(ii) *there exist $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $\xi \in \partial f(\bar{x})$, $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that*

$$\xi + \lim_{k\to\infty} \zeta_k = 0, \ \lim_{k\to\infty} \gamma_k = 0 \text{ and } \lim_{k\to\infty}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) = 0.$$

Proof. Assume that $\bar{x}$ is an optimal solution of (CP). It means that $f(x) \geqq f(\bar{x})$ for any $x \in A$, that is,

$$f(x) + \delta_A(x) \geqq f(\bar{x}) + \delta_A(\bar{x}), \ \forall x \in \mathbb{R}^n.$$

Since $\langle 0, x\rangle - (f(x) + \delta_A(x)) \leqq \langle 0, x\rangle - (f(\bar{x}) + \delta_A(\bar{x}))$, for any $x \in \mathbb{R}^n$,

$$(f + \delta_A)^*(0) = -(f(\bar{x}) + \delta_A(\bar{x})) = -f(\bar{x}).$$

It means that

$$(0, -f(\bar{x})) \in \mathrm{epi}(f + \delta_A)^*.$$

From Proposition 2.2, equivalently,

$$(0, -f(\bar{x})) \in \mathrm{cl}(\mathrm{epi} f^* + \mathrm{epi}\delta_A^*). \tag{3.3}$$

Since $\mathrm{epi}\delta_A^* = \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$, from (3.3),

$$(0, -f(\bar{x})) \in \mathrm{cl}(\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*).$$

Since $\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ is closed,

$$(0, -f(\bar{x})) \in \mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*.$$

By Proposition 2.1, we see that

$$(3.4)(0,-f(\bar{x})) \in \bigcup_{\epsilon\geqq 0}\{(\xi,\langle\xi,\bar{x}\rangle+\epsilon-f(\bar{x})) \mid \xi\in\partial_\epsilon f(\bar{x})\} + \mathrm{cl}(\bigcup_{\epsilon\geqq 0}\{(\zeta,\langle\zeta,\bar{x}\rangle+\epsilon-\sum_{i=1}^m \lambda_i g_i(\bar{x})) \mid \zeta\in\partial_\epsilon(\sum_{i=1}^m \lambda_i g_i)(\bar{x})\}).$$

From (3.4), we see that there exist $\delta \geqq 0$, $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i=1,\dots,m$, $\xi \in \partial_\delta f(\bar{x})$, $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that $(\xi,\langle\xi,\bar{x}\rangle+\delta-f(\bar{x}))+(\zeta_k,\langle\zeta_k,\bar{x}\rangle+\gamma_k-\sum_{i=1}^m \lambda_i^k g_i(\bar{x}))\to(0,-f(\bar{x}))$ as $k\to\infty$. Since $\lambda_i^k g_i(\bar{x}) \leqq 0$, $i=1,\dots,m$, equivalently, there exist $\delta \geqq 0$, $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i=1,\dots,m$, $\xi\in\partial_\delta f(\bar{x})$, $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that

$$\xi + \lim_{k\to\infty}\zeta_k = 0 \text{ and } \lim_{k\to\infty}(\delta+\gamma_k-(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})) = 0.$$

Since $\delta \geqq 0$, $\gamma_k \geqq 0$ and $-(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) \geqq 0$, we have $\delta = 0$ and $\lim_{k\to\infty}(\gamma_k - (\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})) = 0$. Thus, we obtain the desired result. □

Now we give an example illustrating Theorem 3.3:

Example 3.4. Let $f(x) = \begin{cases} \sqrt{x^2+1}, & x \geqq 0, \\ +\infty, & x<0 \end{cases}$ and $g(x)=\frac{1}{2}x^2$. Then $f^*(y) = \begin{cases} -1, & y<0, \\ -\sqrt{1-y^2}, & 0\leqq y\leqq 1, \\ +\infty, & y>1 \end{cases}$ and $g^*(y)=\frac{1}{2}y^2$ for all $y\in\mathbb{R}$. So, epi$f^* = \{(y,\alpha)\in\mathbb{R}\times\mathbb{R} : -\sqrt{1-y^2}\leqq\alpha,\ 0\leqq y\leqq 1\}\cup(-\infty,0)\times[-1,\infty)$ and $\bigcup_{\lambda\geqq 0}\mathrm{epi}(\lambda g)^* = \mathbb{R}\times(0,\infty)\cup\{0\}\times\mathbb{R}_+$. Hence epi$f^* + \bigcup_{\lambda\geqq 0}\mathrm{epi}(\lambda g)^* = (-\infty,0]\times[-1,\infty)\cup(0,\infty)\times(-1,\infty)$ is not closed. But, we can easily see that epi$f^* + \mathrm{cl}\bigcup_{\lambda\geqq 0}\mathrm{epi}(\lambda g)^* = \mathbb{R}\times[-1,\infty)$ is closed.

Now we consider the following convex optimization problem $(\mathrm{CP})_2$

$$(\mathrm{CP})_2 \quad \min \quad f(x) \qquad \text{s.t.} \quad g(x)\leqq 0.$$

Then the feasible set of $(\mathrm{CP})_2$ is $\{0\}$. So, we can easily see that $\bar{x}=0$ is an optimal solution of $(\mathrm{CP})_2$. For each $k\in\mathbb{N}$, if we take $\gamma_k = \frac{1}{2k^2} \geqq 0$ and $\lambda^k = k-1 \geqq 0$, then we can easily calculate that

$$\partial f(\bar{x}) = (-\infty,0] \text{ and}$$
$$\partial_{\gamma_k}(\lambda^k g)(\bar{x}) = [-\sqrt{2\gamma_k\lambda^k},\sqrt{2\gamma_k\lambda^k}] = [-\sqrt{\tfrac{k-1}{k^2}},\sqrt{\tfrac{k-1}{k^2}}]$$

Let $\xi = 0\in\partial f(\bar{x})$ and $\zeta_k = -\sqrt{\frac{k-1}{k^2}} \in \partial_{\gamma_k}(\lambda^k g)(\bar{x})$. Then we can easily see that

$$\xi+\lim_{k\to\infty}\zeta_k = 0,\ \lim_{k\to\infty}\gamma_k = 0 \text{ and } \lim_{k\to\infty}(\lambda^k g)(\bar{x}) = 0.$$

Thus, Theorem 3.3 holds.

Theorem 3.5. *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \dots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \dots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Let $A \cap \mathrm{dom} f \neq \emptyset$. Assume that $\mathrm{epi} f^* + \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ is closed. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP);

(ii) *there exist $\bar{\lambda}_i \geqq 0$, $i = 1, \dots, m$, such that*

$$0 \in \partial f(\bar{x}) + \partial(\sum_{i=1}^m \bar{\lambda}_i g_i)(\bar{x}) \quad \text{and} \quad \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) = 0.$$

Proof. Assume that $\bar{x}$ is an optimal solution of (CP). It means that $f(x) \geqq f(\bar{x})$ for any $x \in A$, that is,

$$f(x) + \delta_A(x) \geqq f(\bar{x}) + \delta_A(\bar{x}),\ \forall x \in \mathbb{R}^n.$$

Since $\langle 0, x \rangle - (f(x) + \delta_A(x)) \leqq \langle 0, x \rangle - (f(\bar{x}) + \delta_A(\bar{x}))$, for any $x \in \mathbb{R}^n$,

$$(f + \delta_A)^*(0) = -(f(\bar{x}) + \delta_A(\bar{x})) = -f(\bar{x}).$$

It means that

$$(0, -f(\bar{x})) \in \mathrm{epi}(f + \delta_A)^*.$$

From Proposition 2.2, equivalently,

$$(0, -f(\bar{x})) \in \mathrm{cl}(\mathrm{epi} f^* + \mathrm{epi} \delta_A^*). \tag{3.5}$$

Since $\mathrm{epi}\delta_A^* = \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$, from (3.5),

$$(0, -f(\bar{x})) \in \mathrm{cl}(\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*). \tag{3.6}$$

Since $\mathrm{cl}(\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*) = \mathrm{cl}(\mathrm{epi} f^* + \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*)$ and $\mathrm{epi} f^* + \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ is closed, (3.6) is equivalent to the condition that

$$(0, -f(\bar{x})) \in \mathrm{epi} f^* + \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*.$$

By Proposition 2.1, we see that

$$\begin{aligned}(0, -f(\bar{x})) \in & \bigcup_{\delta \geqq 0} \{(\xi, \langle \xi, \bar{x} \rangle + \delta - f(\bar{x})) \mid \xi \in \partial_\delta f(\bar{x})\} \\ & + \bigcup_{\gamma \geqq 0} \{(\zeta, \langle \zeta, \bar{x} \rangle + \gamma - \sum_{i=1}^m \lambda_i g_i(\bar{x})) \mid \zeta \in \partial_\gamma \sum_{i=1}^m \lambda_i g_i(\bar{x})\}.\end{aligned} \tag{3.7}$$

From (3.7), we see that there exist $\bar{\delta} \geqq 0$, $\bar{\gamma} \geqq 0$, $\bar{\lambda}_i \geqq 0$, $i = 1, \dots, m$, $\bar{\xi} \in \partial_{\bar{\delta}} f(\bar{x})$, $\bar{\zeta} \in \partial_{\bar{\gamma}}(\sum_{i=1}^m \lambda_i g_i)(\bar{x})$ such that $(\bar{\xi}, \langle \bar{\xi}, \bar{x} \rangle + \bar{\delta} - f(\bar{x})) + (\bar{\zeta}, \langle \bar{\zeta}, \bar{x} \rangle + \bar{\gamma} - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x})) = (0, -f(\bar{x}))$ i.e., $\bar{\xi} + \bar{\zeta} = 0$ and $\bar{\delta} + \bar{\gamma} - (\sum_{i=1}^m \bar{\lambda}_i g_i)(\bar{x}) = 0$. Since $\bar{\lambda}_i g_i(\bar{x}) \leqq 0$, $i = 1, \dots, m$, equivalently, there exist $\bar{\lambda}_i \geqq 0$, $i = 1, \dots, m$, such that

$$0 \in \partial f(\bar{x}) + \partial(\sum_{i=1}^m \bar{\lambda}_i g_i)(\bar{x}) \quad \text{and} \quad \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) = 0.$$

□

Now we give an example illustrating Theorem 3.5:

Example 3.6. Let $f(x) = \begin{cases} x\log x, & x > 0, \\ 0, & x = 0, \\ +\infty, & x < 0 \end{cases}$ and $g(x) = \frac{1}{2}x^2 - 1$. Then $f^*(y) = e^{y-1}$ for all $y \in \mathbb{R}$ and $g^*(y) = \frac{1}{2}y^2 + 1$ for all $y \in \mathbb{R}$. So, $\text{epi}f^* = \{(y,\alpha) \in \mathbb{R} \times \mathbb{R} : e^{y-1} \leqq \alpha\}$ and $\bigcup_{\lambda \geqq 0} \text{epi}(\lambda g)^* = \{(y,\alpha) \in \mathbb{R} \times \mathbb{R} : |y| \leqq \alpha\}$. Hence $\text{epi}f^* + \bigcup_{\lambda \geqq 0} \text{epi}(\lambda g)^* = \{(y,\alpha) \in \mathbb{R} \times \mathbb{R} : e^{y-1} \leqq \alpha,\ y \leqq 1\} \cup \{(y,\alpha) \in \mathbb{R} \times \mathbb{R} : |y| \leqq \alpha,\ y > 1\}$ is closed.

Now we consider the following convex optimization problem $(\text{CP})_3$

$$(\text{CP})_3 \quad \min \quad f(x)$$
$$\text{s.t.} \quad g(x) \leqq 0.$$

Then the feasible set of $(\text{CP})_3$ is $[-\sqrt{2}, \sqrt{2}]$. So, we can easily see that $\bar{x} = e^{-1}$ is an optimal solution of $(\text{CP})_3$. Moreover, we can easily see that $\partial f(\bar{x}) = \{0\}$ and $\partial(\bar{\lambda} g)(\bar{x}) = \{\bar{\lambda} e^{-1}\}$. If we take $\bar{\lambda} = 0$, then we have

$$0 \in \partial f(\bar{x}) + \partial(\bar{\lambda} g)(\bar{x}) \ \text{ and } \ \bar{\lambda} g(\bar{x}) = 0.$$

Thus, Theorem 3.5 holds.

Remark 3.7. Theorem 3.5 can be regarded as one which is sharper than Theorem 4.2 in [2] in the case that the involved geometric set is empty.

Now we give an example illustrating that the Slater condition may not imply the closedness of the set $\text{epi}f^* + \bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*$.

Example 3.8. Let $f(x) = \begin{cases} x\log x, & x > 0, \\ 0, & x = 0, \\ +\infty, & x < 0 \end{cases}$ and $g(x) = x(x+1)$. Clearly, Slater condition holds. $f^*(y) = e^{y-1}$ for all $y \in \mathbb{R}$ and $g^*(y) = \frac{(y-1)^2}{4}$ for all $y \in \mathbb{R}$. So, $\text{epi}f^* = \{(y,\alpha) \in \mathbb{R} \times \mathbb{R} : e^{y-1} \leqq \alpha\}$ and $\bigcup_{\lambda \geqq 0} \text{epi}(\lambda g)^* = \{(y,\alpha) \in \mathbb{R} \times \mathbb{R} : y \leqq \alpha,\ y < 0\} \cup \mathbb{R}^2_+$. Hence $\text{epi}f^* + \bigcup_{\lambda \geqq 0} \text{epi}(\lambda g)^* = \mathbb{R} \times (0, +\infty)$ is not closed.

4. Sequential optimality theorems II

By using Proposition 2.3(a version of Brondsted-Rockafellar Theorem) and Theorem 3.1, we can obtain the following sequential optimality theorem for (CP) which involve only the subgradients at nearby points to a minimizer of (CP). So the sequential optimality condition in Theorem 3.1 is different from the one in the following theorem. However, we omit the proof of the following theorem:

Theorem 4.1. *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \ldots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \text{dom}f \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP)*;*

(ii) *there exist $x_k \in \mathbb{R}^n$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $\bar{\xi}_k \in \partial f(x_k)$, $\bar{\zeta}_k \in \partial(\sum_{i=1}^m \lambda_i^k g_i)(x_k)$ such that*

$$\lim_{k\to\infty} x_k = \bar{x},\ \lim_{k\to\infty} (\bar{\xi}_k + \bar{\zeta}_k) = 0,$$

$$\textit{and} \quad \lim_{k\to\infty}\left[f(x_k) + (\sum_{i=1}^{m}\lambda_i^k g_i)(x_k) - f(\bar{x})\right] = 0.$$

Now we give an example illustrating Theorem 4.1:

Example 4.2. Consider the following convex optimization problem $(\mathrm{CP})_1$

$$(\mathrm{CP})_1 \quad \min \quad f(x) \qquad \text{s.t.} \quad g(x) \leqq 0,$$

where $f(x) = \begin{cases} x\log x, & x > 0, \\ 0, & x = 0, \\ +\infty, & x < 0 \end{cases}$ and $g(x) = \max\{0, x\}$. Then the feasible set of $(\mathrm{CP})_1$ is $(-\infty, 0]$. So, we can easily see that $\bar{x} = 0$ is an optimal solution of $(\mathrm{CP})_1$. For each $k \in \mathbb{N}$, if we take $x_k = e^{-k}$ and $\lambda^k = k - 1 \geqq 0$, then we can easily calculate that

$$\begin{aligned} \partial f(x_k) &= (-\infty, \log x_k + 1] = (-\infty, -k+1] \ \text{ and} \\ \partial(\lambda^k g)(x_k) &= \{k-1\}. \end{aligned}$$

Let $\xi_k = -k + 1 \in \partial f(x_k)$ and $\zeta_k = k - 1 \in \partial(\lambda^k g)(x_k)$. Then we can easily see that

$$\begin{aligned} &\lim_{k\to\infty} x_k = \lim_{k\to\infty} e^{-k} = 0 = \bar{x}, \ \lim_{k\to\infty}(\xi_k + \zeta_k) = 0 \\ \text{and} \quad &\lim_{k\to\infty}\left[f(x_k) + (\lambda^k g)(x_k) - f(\bar{x})\right] = \lim_{k\to\infty}(-e^{-k}) = 0. \end{aligned}$$

Thus, Theorem 4.1 holds.

By using Proposition 2.3 and Theorem 3.3, we can obtain the following sequential optimality theorem for (CP). The sequential optimality condition in Theorem 3.3 is different from the one in the following theorem. We omit the proof of the following theorem:

Theorem 4.3. *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0, \ i = 1, \ldots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$ and*

$$\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{n}\lambda_j g_i)^*$$

is closed. Then the following statements are equivalent:

(i) *$\bar{x}$ is an optimal solution of (CP);*

(ii) *there exist $x_k \in \mathbb{R}^n$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $\bar{\xi} \in \partial f(\bar{x})$, $\bar{\zeta}_k \in \partial(\sum_{i=1}^m \lambda_i^k g_i)(x_k)$ such that*

$$\lim_{k\to\infty} x_k = \bar{x}, \ \bar{\xi} + \lim_{k\to\infty}\bar{\zeta}_k = 0 \ \textit{and} \ \lim_{k\to\infty}(\sum_{i=1}^{m}\lambda_i^k g_i)(x_k) = 0.$$

Now we give an example illustrating Theorem 4.3:

Example 4.4. Consider the following convex optimization problem $(\mathrm{CP})_2$

$$(\mathrm{CP})_2 \quad \min \quad f(x) \qquad \text{s.t.} \quad g(x) \leqq 0,$$

where $f(x) = \begin{cases} \sqrt{x^2+1}, & x \geqq 0, \\ +\infty, & x < 0 \end{cases}$ and $g(x) = \frac{1}{2}x^2$. Then the feasible set of $(\mathrm{CP})_2$ is $\{0\}$. So, we can easily see that $\bar{x} = 0$ is an optimal solution of $(\mathrm{CP})_2$. For each $k \in \mathbb{N}$, if we take $x_k = \frac{1}{k^2}$ and $\lambda^k = k - 1 \geqq 0$, then we can easily calculate that

$$\begin{aligned} \partial f(\bar{x}) &= (-\infty, 0] \text{ and} \\ \partial(\lambda^k g)(x_k) &= \{\lambda^k x_k\} = \left\{ \frac{k-1}{k^2} \right\}. \end{aligned}$$

Let $\xi = 0 \in \partial f(x_k)$ and $\zeta_k = \frac{k-1}{k^2} \in \partial(\lambda^k g)(x_k)$. Then we can easily see that

$$\lim_{k\to\infty} x_k = 0 = \bar{x}, \ \xi + \lim_{k\to\infty} \zeta_k = 0$$
$$\text{and} \quad \lim_{k\to\infty} (\lambda^k g)(x_k) = 0.$$

Thus, Theorem 4.3 holds.

5. Closedness of characterization cones

The set $\bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ is called the characterization cone of (CP). Closedness of the set is important in Theorem 3.5 since the set is related to the constraint qualification for (CP) ([9]). Now we give sufficient conditions for the set to be closed.

Proposition 5.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ be convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0, \ i = 1, \ldots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $h : \mathbb{R}^n \to \mathbb{R}$ is a positive homogeneous convex function such that $g^* \geqq h$ and $0 \notin \partial h(0)$. Then*

$$\Lambda := \bigcup_{\lambda \geqq 0} \mathrm{epi}(\lambda g)^* = \bigcup_{\lambda > 0} \mathrm{epi}(\lambda g)^* \cup \{0\} \times \mathbb{R}_+$$

is closed.

Proof. Since $g^* \geqq h$ and h is a positive homogeneous convex function,

$$\mathrm{epi} g^* \subset \mathrm{epi} h = \partial h(0) \times \mathbb{R}_+ \tag{5.1}$$

Let $\{(v_k, \alpha_k)\}$ be a sequence in the set Λ such that $(v_k, \alpha_k) \to (v^*, \alpha^*)$ as $k \to \infty$. If $(v_k, \alpha_k) \in \{0\} \times \mathbb{R}_+$ infinitely, $(v^*, \alpha^*) \in \{0\} \times \mathbb{R}_+$. Now we assume that $\{(v_k, \alpha_k)\} \subset \bigcup_{\lambda > 0} \mathrm{epi}(\lambda g)^*$ and $v^* \neq 0$. Then there exist $\lambda_k >$ and $(w_k, \beta_k) \in \mathrm{epi} g^*$ such that $(v_k, \alpha_k) = \lambda_k(w_k, \beta_k)$. Since $w_k \in \partial h(0)$ (from (5.1)), $\partial h(0)$ is compact and $0 \notin \partial h(0)$, we may assume that $w_k \to w^* \neq 0$ as $k \to \infty$. If $\lambda_k \to +\infty$, then $\lambda_k w_k$ can not converges to v^*. So, we may assume that $\lambda_k \to \lambda^* (\neq 0)$ as $k \to \infty$. Since $\lim_{k\to\infty} \frac{\lambda_k \beta_k - \alpha^*}{\lambda_k} = 0$, $\lim_{k\to\infty} \beta_k = \frac{\alpha^*}{\lambda^*}$. Since $(w_k, \beta_k) \in \mathrm{epi} g^*$ and g^* is lower semicontinuous,

$$g^*(w^*) \leqq \liminf_{k\to\infty} g^*(w_k) \leqq \liminf_{k\to\infty} \beta_k = \frac{\alpha^*}{\lambda^*},$$

and so $(w^*, \frac{\alpha^*}{\lambda^*}) \in \mathrm{epi}g^*$, that is, $(\lambda^* w^*, \alpha^*) \in \lambda^* \mathrm{epi}g^*$. Hence $(v^*, \alpha^*) \in \lambda^* \mathrm{epi}g^*$. Thus Λ is closed. □

Proposition 5.2. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive homogeneous convex function which is separable, that is , $g(x) = \sum_{i=1}^m g_i(x_i)$, where $g_i : \mathbb{R} \to \mathbb{R}$ is a function, $i = 1, 2, \dots, m$. Assume that $g_i(0) = 0$, $i = 1, 2, \dots, m$. Then $\bigcup_{\lambda \geqq 0} (\lambda g)^*$ is closed.*

Proof. Since g is positive homogeneous convex and separable, we see that

$$\begin{aligned}
&\bigcup_{\lambda \geqq 0} \mathrm{epi}(\lambda g)^* \\
&= \bigcup_{\lambda > 0} \lambda \mathrm{epi}g^* \cup \{0\} \times \mathbb{R}_+ \\
&= \bigcup_{\lambda > 0} \lambda(\partial g(0) \times \mathbb{R}_+) \cup \{0\} \times \mathbb{R}_+ \\
&= \bigcup_{\lambda > 0} \lambda(\partial g_1(0) \times \cdots \times \partial g_m(0) \times \mathbb{R}_+) \cup \{0\} \times \mathbb{R}_+ \\
&= \bigcup_{\lambda > 0} \lambda([g'_{1-}(0), g'_{1+}(0)] \times \cdots \times [g'_{m-}(0), g'_{m+}(0)] \times \mathbb{R}_+) \cup \{0\} \times \mathbb{R}_+ \\
&= \bigcup_{\lambda \geqq 0} \lambda([g'_{1-}(0), g'_{1+}(0)] \times \cdots \times [g'_{m-}(0), g'_{m+}(0)] \times \mathbb{R}_+) \\
&= \{\lambda_1 g'_{1-}(0) + \tilde{\lambda}_1 g'_{1+}(0) + \cdots + \lambda_m g'_{m-}(0) + \tilde{\lambda}_m g'_{m+}(0) \mid \lambda_i \geqq 0,\ \tilde{\lambda}_i \geqq 0,\ i = 1, \dots, m\}.
\end{aligned}$$

Thus $\bigcup_{\lambda \geqq 0} (\lambda g)^*$ is closed. □

Proposition 5.3. *Let $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, be a function such that $g_i = \max\{a_i^j x_i + b_i^j : j = 1, 2\}$, $i = 1, 2$. Let $g = g_1 + g_2$. Then $\bigcup_{\lambda \geqq 0} \mathrm{epi}(\lambda g)^*$ is closed*

Proof. Notice that $\bigcup_{\lambda \geqq 0} \mathrm{epi}(\lambda g)^* = \bigcup_{\lambda \geqq 0} \lambda \mathrm{epi}g^* \cup \{0\} \times \mathbb{R}_+$. Since $g = g_1 + g_2$ and g_i, $i = 1, 2$, is a sublinear, we can easily see that

$$\mathrm{epi}g^* = \mathrm{epi}g_1^* + \mathrm{epi}g_2^* \tag{5.2}$$

Now, we will prove that

$$\begin{aligned}
\mathrm{epi}g_1^* &= \mathrm{co}\{(a_1^1, 0, -b_1^1), (a_1^2, 0, -b_1^2)\} + \{0\} \times \mathbb{R}_+ \\
\mathrm{epi}g_2^* &= \mathrm{co}\{(0, a_2^1, -b_2^1), (0, a_2^2, -b_2^2)\} + \{0\} \times \mathbb{R}_+.
\end{aligned}$$

Since $g_i = \max\{a_i^j x_i + b_i^j : j = 1, 2\}$, $i = 1, 2$,

$$g_i^* = \left(\max_{j \in \{1,2\}} (a_i^j x_i + b_i^j) \right)^* = \mathrm{cl} \left(\mathrm{co} \left(\inf_{j \in \{1,2\}} (a_i^j x_i + b_i^j)^* \right) \right).$$

Then we can easily see that

$$\inf_{j \in \{1,2\}} (a_i^j x_i + b_i^j)^*(\xi_1, \xi_2) = \begin{cases} -b_i^1, & \text{if } \xi_i = a_i^1 \\ -b_i^2, & \text{if } \xi_i = a_i^2 \\ +\infty, & \text{otherwise.} \end{cases}.$$

So, we have

$$\text{co}\left(\inf_{j\in\{1,2\}}(a_i^j x_i + b_i^j)^*(\xi_1,\xi_2)\right) = \begin{cases} -\alpha_i^1 b_i^1 - \alpha_i^2 b_i^2, & \text{if } \xi_i = \alpha_i^1 a_i^1 + \alpha_i^2 a_i^2, \\ & \alpha_i^1, \alpha_i^2 \geqq 0,\ \alpha_i^1 + \alpha_i^2 = 1 \\ +\infty, & \text{otherwise.} \end{cases}.$$

Since $\text{co}\left(\inf_{j\in\{1,2\}}(a_i^j x_i + b_i^j)^*\right)$ is lower semi-continuous,

$$g_i^* = \text{cl}\left(\text{co}\left(\inf_{j\in\{1,2\}}(a_i^j x_i + b_i^j)^*\right)\right) = \text{co}\left(\inf_{j\in\{1,2\}}(a_i^j x_i + b_i^j)^*\right).$$

So, we have

$$\begin{aligned} \text{epi} g_1^* &= \text{co}\{(a_1^1, 0, -b_1^1), (a_1^2, 0, -b_1^2)\} + \{0\} \times \mathbb{R}_+ \\ \text{epi} g_2^* &= \text{co}\{(0, a_2^1, -b_2^1), (0, a_2^2, -b_2^2)\} + \{0\} \times \mathbb{R}_+. \end{aligned}$$

Hence, from (5.2), we can easily see that

$$\begin{aligned} &\bigcup_{\lambda\geqq 0} \text{epi}(\lambda g)^* \\ =& \bigcup_{\lambda>0} \lambda\left[\text{co}\{(a_1^1, 0, -b_1^1), (a_1^2, 0, -b_1^2)\} + \text{co}\{(0, a_2^1, -b_2^1), (0, a_2^2, -b_2^2)\} + \{0\} \times \mathbb{R}_+\right] \\ & \cup \{0\} \times \mathbb{R}_+ \\ =& \bigcup_{\lambda>0} \lambda\left[\text{co}\{(a_1^1, a_2^1, -b_1^1 - b_2^1), (a_1^1, a_2^2, -b_1^1 - b_2^2), (a_1^2, a_2^1, -b_1^2 - b_2^1), (a_1^2, a_2^2, -b_1^2 - b_2^2)\}\right. \\ & \left. + \{0\} \times \mathbb{R}_+\right] \cup \{0\} \times \mathbb{R}_+ \\ =& \lambda\left\{\left(\sum_{j=1}^2 \alpha_1^j a_1^j, \sum_{j=1}^2 \alpha_2^j a_2^j, -\sum_{j=1}^2 \alpha_1^j b_1^j - \sum_{j=1}^2 \alpha_2^j b_2^j\right) \mid \alpha_i^j \geqq 0,\ \sum_{j=1}^m \alpha_i^j = 1,\right. \\ & \left. i, j = 1, 2, \right\} \cup \{0\} \times \mathbb{R}_+ \end{aligned}$$

□

6. Solution Sets

Consider the following convex optimization problem of which objective function is finite-valued convex functions:

$$\begin{aligned} \text{(CP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & x \in A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \ldots, m\} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, 2, \ldots, m$, are proper, lower semicontinuous and convex functions.

Let S be the set of all optimal solutions for (CP). We assume that $S \neq \emptyset$. Let $\bar{x} \in S$. Since the function f is continuous and $\text{epi}\delta_A^* = \text{cl}\bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*$, it follows from Proposition 2.3 that $\text{epi}(f + \delta)^*$ is closed and $\text{epi}(f + \delta_A)^* = \text{epi} f^* + \text{cl}\bigcup_{\lambda_i \geqq 0} \text{epi}(\sum_{i=1}^m \lambda_i g_i)^*$. So it follows from Theorem 3.3 that there exist $\bar{u} \in \partial f(\bar{x})$, $\{\bar{\lambda}_i^k\} \subset \mathbb{R}_+$, $\epsilon_k \geqq 0$, $\bar{v}_k \in \partial_{\epsilon_k}\left(\sum_{i=1}^m \bar{\lambda}_i^k g_i\right)(\bar{x})$ such that

$$\bar{u} + \bar{v}_k \to 0 \text{ as } k \to \infty,$$

$$\epsilon_k \to 0 \text{ as } k \to \infty,$$
$$\text{and} \quad \lim_{k\to\infty} \left(\sum_{i=1}^{m} \bar{\lambda}_i^k g_i \right)(\bar{x}) = 0.$$

Using the above optimality conditions for $\bar{x} \in S$, we can characterize the solution set S as follows;

Theorem 6.1.

$$\begin{aligned} S &= \{x \in A \mid \liminf_{k\to\infty} \left(\sum_{i=1}^{n} \bar{\lambda}_i^k g_i \right)(x) = 0,\ \bar{u} \in \partial f(x),\ \bar{u}^T(x-\bar{x}) = 0\} \\ &= \{x \in A \mid \bar{u} \in \partial f(x),\ \bar{u}^T(x-\bar{x}) = 0\}. \end{aligned}$$

Proof. Let $S_1 = \{x \in A \mid \liminf_{k\to\infty} \left(\sum_{i=1}^{n} \bar{\lambda}_i^k g_i\right)(x) = 0,\ \bar{u} \in \partial f(x),\ \bar{u}^T(x - \bar{x}) = 0\}$. Let $x \in S$. Then $\sum_{i=1}^{n} \bar{\lambda}_i^k g_i(x) \leqq 0$. Since $\bar{u} \in \partial f(\bar{x})$ and $v_k \in \partial_{\epsilon_k}\left(\sum_{i=1}^{m} \lambda_i^k g_i\right)(\bar{x})$, we have

$$\begin{aligned} & f(x) - f(\bar{x}) \geqq \bar{u}^T(x-\bar{x}) \\ \text{and} \quad & \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x) - \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(\bar{x}) \geqq \bar{v}_k^T(x-\bar{x}) - \epsilon_k, \end{aligned}$$

and hence,

$$\begin{aligned} & \liminf_{k\to\infty} \left\{ f(x) + \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x) - (f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(\bar{x})) \right\} \\ \geqq\ & \liminf_{k\to\infty} \left\{ (\bar{u} + \bar{v}_k)^T (x-\bar{x}) - \epsilon_k \right\} \\ =\ & 0. \end{aligned}$$

So, we have

$$\begin{aligned} 0 &\leqq \liminf_{k\to\infty} \left\{ f(x) + \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x) \right\} + \limsup_{k\to\infty} \left\{ -f(\bar{x}) - \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(\bar{x}) \right\} \\ &= f(x) - f(\bar{x}) + \liminf_{k\to\infty} \left\{ \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x) \right\} \\ &= \liminf_{k\to\infty} \left\{ \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x) \right\}. \end{aligned}$$

Since $\sum_{i=1}^{n} \bar{\lambda}_i^k g_i(x) \leqq 0$, $\liminf_{k\to\infty} \left\{\sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x)\right\} = 0$. So, we have

$$\begin{aligned} 0 &= \liminf_{k\to\infty} \left\{ \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x) \right\} - \lim_{k\to\infty} \sum_{i=1}^{n} \bar{\lambda}_i^k g_i(\bar{x}) \\ &= \liminf_{k\to\infty} \left\{ \sum_{i=1}^{m} \bar{\lambda}_i^k g_i(x) \right\} + \limsup_{k\to\infty} \left(-\sum_{i=1}^{n} \bar{\lambda}_i^k g_i(\bar{x}) \right) \end{aligned}$$

$$\begin{aligned}
&\geqq \liminf_{k\to\infty}\left\{\sum_{i=1}^{m}\bar{\lambda}_i^k(g_i(x)-g_i(\bar{x}))\right\}\\
&\geqq \liminf_{k\to\infty}\left\{\bar{v}_k^T(x-\bar{x})-\epsilon_k\right\}\\
&= -\bar{u}^T(x-\bar{x}).
\end{aligned}$$

Hence $0 \leqq \bar{u}^T(x-\bar{x})$. Since f is convex, $\bar{u}\in\partial f(\bar{x})$and $f(x)=f(\bar{x})$. So,

$$0 = f(x) - f(\bar{x}) \geqq \bar{u}^T(x-\bar{x}).$$

Thus, we have $\bar{u}^T(x-\bar{x})=0$. Moreover, for any $y\in\mathbb{R}^n$,

$$\begin{aligned}
f(y)-f(x) &= f(y)-f(\bar{x})\\
&\geqq \bar{u}^T(y-\bar{x})\\
&= \bar{u}^T(y-x)+\bar{u}^T(x-\bar{x})\\
&= \bar{u}^T(y-x).
\end{aligned}$$

Hence $\bar{u}\in\partial f(x)$. So, $S\subset S_1$.

Conversely, if $x\in S_1$, then $x\in A$, $\bar{u}\in\partial f(\bar{x})$ and $\bar{u}^T(x-\bar{x})=0$, and hence $f(\bar{x})-f(x)\geqq\bar{u}^T(\bar{x}-x)=0$. So, $x\in S$. Hence $S_1\subset S$. □

Example 6.2. Let $f(x,y)=x$ and $g(x,y)=\sqrt{x^2+y^2}-y$. Consider the following convex optimization problem:

$$\begin{aligned}
\text{(CP)}\quad &\min \quad f(x,y)\\
&\text{s.t.} \quad (x,y)\in A:=\{(x,y)\in\mathbb{R}^2 \mid g(x,y)\leqq 0\}.
\end{aligned}$$

Then $A=\{(x,y)\in\mathbb{R}^2 \mid y\geqq 0,\ x=0\}$ and $S=\{(x,y)\in\mathbb{R}^2\ |y\geqq 0,\ x=0\}$, where S is the set of all optimal solutions of (CP). Let $(\bar{x},\bar{y})=(0,1)$. For $\lambda>0$. Then $\partial f(\bar{x},\bar{y})+\lambda\partial g(\bar{x},\bar{y})=(1,0)^T+\lambda\{(\xi_1,\xi_2)^T-(0,1)^T \mid \xi_1^2+\xi_2^2\leqq 1\}$, and so, $(0,0)^T\notin\partial f(\bar{x},\bar{y})+\lambda\partial g(\bar{x},\bar{y})$. However, for any $\lambda>0$ and $\epsilon>0$,

$$\partial_\epsilon(\lambda g)(\bar{x},\bar{y})=\{(v_1,v_2)\in\mathbb{R}^2 \mid v_1^2+(v_2+\lambda)^2\leqq\lambda^2,\ v_2\geqq-\epsilon\}.$$

Let $u=(1,0)$. For each $k\in\mathbb{N}$, we let $\epsilon_k=\frac{1}{k}$, $\lambda_k=\frac{1}{2}(k+\frac{2}{k})+1$ and $v_k=(-1-\frac{1}{k},-\frac{1}{k})$. Then $v_k\in\partial_{\epsilon_k}(\lambda_k g)(\bar{x},\bar{y})$, $u+v_k\to 0$ as $k\to\infty$, $\epsilon_k\to 0$ as $k\to\infty$ and $\lambda_k g(\bar{x},\bar{y})=0$. Moreover, we see that

$$\begin{aligned}
&\{(x,y)\in A \mid \liminf_{k\to\infty}(\lambda_k g)(x,y)=0,\ u\in\partial f(x,y),\ u^T((x,y)-(\bar{x},\bar{y}))=0\}\\
&= \{(x,y)\in A \mid \liminf_{k\to\infty}(\frac{1}{2}(k+\frac{2}{k})+1)(\sqrt{x^2+y^2}-y)=0,\ x-\bar{x}=0, y\in\mathbb{R}\}\\
&= \{(x,y)\in A \mid \sqrt{x^2+y^2}-y=0,\ x=0,\ y\in\mathbb{R}\}\\
&= \{(x,y)\in A \mid x=0,\ y\geqq 0\}\\
&= A\\
&= S.
\end{aligned}$$

Hence Theorem 6.1 holds.

Example 6.3. Let $f(x,y) = \begin{cases} y, & x \leqq 1, \\ x-1+y, & x>1 \end{cases}$ and $g(x,y) = \sqrt{x^2+y^2} - x$. Consider the following convex optimization problem:

$$\begin{array}{lll} \text{(CP)} & \min & f(x,y) \\ & \text{s.t.} & (x,y) \in A := \{(x,y) \in \mathbb{R}^2 \mid g(x,y) \leqq 0\}. \end{array}$$

Then $A = \{(x,y) \in \mathbb{R}^2 \mid x \geqq 0,\ y = 0\}$ and $S = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqq x \leqq 1,\ y = 0\}$, where S is the set of all optimal solutions of (CP). Let $(\bar{x},\bar{y}) = (1,0)$. Then $(\bar{x},\bar{y}) \in S$. For each $k \in \mathbb{N}$, we let $\epsilon_k = \frac{1}{k}$, $\lambda_k = \frac{1}{2}(k+\frac{2}{k})+1$ and $v_k = (-\frac{1}{k}, -1-\frac{1}{k})$. Then $v_k \in \partial_{\epsilon_k}(\lambda_k g)(\bar{x},\bar{y})$ $k \in \mathbb{N}$. Let $u = (0,1)$. Then $u \in \partial f(0,1)$. Moreover, $u + v_k \to 0$ as $k \to \infty$, $\epsilon_k \to 0$ as $k \to \infty$ and $\lambda_k g(\bar{x},\bar{y}) = 0$, $k \in \mathbb{N}$. We have

$$\begin{aligned} & \{(x,y) \in A \mid \liminf_{k\to\infty} (\lambda_k g)(x,y) = 0,\ u \in \partial f(x,y),\ u^T((x,y)-(\bar{x},\bar{y})) = 0\} \\ = \ & \{(x,y) \in A \mid g(x,y) = 0,\ x \leqq 1,\ y - \bar{y} = 0\} \\ = \ & \{(x,y) \in A \mid 0 \leqq x \leqq 1,\ y = 0\} \\ = \ & S. \end{aligned}$$

Hence Theorem 6.1 holds.

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Jae Hyoung Lee
Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea
E-mail address: mc7558@naver.com

Gue Myung Lee
Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea
E-mail address: gmlee@pknu.ac.kr