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ITERATIVE APPROXIMATION WITH ERRORS OF ZERO POINTS OF MAXIMAL MONOTONE OPERATORS IN A HILBERT SPACE

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Abstract. In this paper, we study the shrinking projection method with error introduced by Kimura [8]. We obtain an iterative approximation of a zero point of a maximal monotone operator generated by the shrinking projection method with errors in a Hilbert space. Using our result, we discuss some applications.

1. Introduction

Let H be a real Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Then, the zero point problem is to find $u \in H$ such that

$$0 \in Au. \tag{1.1}$$

Such a $u \in H$ is called a zero point (or a zero) of A. The set of zero points of A is denoted by $A^{-1}0$. This problem is connected with many problems in Nonlinear Analysis and Optimization, that is, convex minimization problems, variational inequality problems, equilibrium problems and so on. A well-known method for solving (1.1) is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \dots, \tag{1.2}$$

where $\{r_n\} \subset]0, \infty[$ and $J_{r_n} = (I + r_n A)^{-1}$. This algorithm was first introduced by Martinet [12]. In 1976, Rockafellar [17] proved that if $\liminf_n r_n > 0$ and $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a solution of the zero point problem. Later, many researchers have studied this problem; see [5–7, 10, 11, 13, 18] and others.

Recently, Kimura [8] introduced the following iterative scheme for finding a fixed point of a nonexpansive mapping by the shrinking projection method with error in a Hilbert space:

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Theorem 1.1 (Kimura [8]). *Let C be a bounded closed convex subset of a Hilbert space H with $D = \operatorname{diam} C = \sup_{x,y\in C}\|x-y\| < \infty$, and let $T : C \to H$ be a nonexpansive mapping having a fixed point. Let $\{\epsilon_n\}$ be a nonnegative real sequence such that $\epsilon_0 = \limsup_n \epsilon_n < \infty$. For a given point $u \in H$, generate an iterative sequence $\{x_n\}$ as follows: $x_1 \in C$ such that $\|x_1 - u\| < \epsilon_1$, $C_1 = C$,*

$$\begin{cases} C_{n+1} = \{z \in C : \|z - Tx_n\| \le \|z - x_n\|\} \cap C_n, \\ x_{n+1} \in C_{n+1} \text{ such that } \|u - x_{n+1}\|^2 \le d(u, C_{n+1})^2 + \epsilon_{n+1}^2 \end{cases}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le 2\epsilon_0.$$

Further, if $\epsilon_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u \in F(T)$.

We remark that the original result of the theorem above deals with a family of nonexpansive mappings, and the shrinking projection method was first introduced by Takahashi, Takeuchi and Kubota [22].

In this paper, we study the shrinking projection method with error introduced by Kimura [8] (see also [9]). We obtain an iterative approximation of a zero point of a maximal monotone operator generated by the shrinking projection method with errors in a Hilbert space. Using our result, we discuss the convex minimization problem, the variational inequality problem and the equilibrium problem in a Hilbert space.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote strong convergence and weak convergence of a sequence $\{x_n\}$ to x in H by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a real Hilbert space H, we have from [21]

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Let C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. It is also known that $y = P_C x$ is equivalent to $\langle x - y, y - z\rangle \ge 0$ for all $z \in C$.

A multi-valued operator $A \subset H \times H$ with domain $D(A) = \{x \in H : Ax \ne \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \ge 0$ for any $(x_1, y_1), (x_2, y_2) \in A$. A monotone operator A is said to be maximal if $A = B$ whenever $B \subset H \times H$ is a monotone operator such that $A \subset B$. We know that a monotone operator A is maximal if and only if $R(I + rA) = H$ for all $r > 0$, where I is the identity operator on H.

Let $A \subset H \times H$ be a maximal monotone operator. It is known that $A^{-1}0$ is closed and convex, where $A^{-1}0 = \{u \in H : 0 \in Au\}$. We can define, for each $r > 0$, a single-valued mapping $(I + rA)^{-1} : H \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A for $r > 0$. It is known that

$$\frac{x - J_r x}{r} \in AJ_r x \tag{2.2}$$

for all $x \in H$ and $r > 0$ (see [15, 16, 21] for more details).

The following Lemma is easily deduced from the theorem proved by Tsukada [23] (see also [8]).

Lemma 2.1 (Tsukada [23]). *Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hilbert space H such that $C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$. Let u be a point of H. Then, if $C_0 = \cap_{n=1}^{\infty} C_n$ is nonempty, then the sequence $\{P_{C_n} u\}$ converges strongly to $P_{C_0} u$, where P_{C_i} is the metric projection of E onto C_i for each $i \in \mathbb{N} \cup \{0\}$.*

3. Main result

In this section, we obtain an iterative approximation of a zero point of a maximal monotone operator generated by the shrinking projection method with errors [8] in a Hilbert space.

Theorem 3.1. *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. Let $\{\delta_n\}$ be a bounded nonnegative real sequence such that $\delta_0 = \limsup_n \delta_n$ and let $\{r_n\}$ be a positive real sequence such that $\liminf_n r_n > 0$. For a given point $u \in H$, generate a sequence $\{x_n\}$ by $x_1 = x \in H$, $C_1 = H$, and*

$$\begin{cases} y_n = J_{r_n} x_n, \\ C_{n+1} = \{z \in H : \langle y_n - z, x_n - y_n \rangle \geq 0\} \cap C_n, \\ x_{n+1} \in \{z \in H : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}^2\} \cap C_{n+1} \end{cases}$$

for all $n \in \mathbb{N}$. Then

$$\limsup_{n \to \infty} \|x_n - y_n\| \leq \delta_0.$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0} u$.

Proof. We fist show by induction that $A^{-1}0 \subset C_n$ for each $n \in \mathbb{N}$. It is clear that $A^{-1}0 \subset H = C_1$. Suppose that $A^{-1}0 \subset C_k$ for some $k \in \mathbb{N}$. Since $A^{-1}0$ is nonempty, we have from (2.2) that for each $(z, 0) \in A$

$$\left\langle y_k - z, \frac{x_k - y_k}{r_k} - 0 \right\rangle \geq 0.$$

So, we get $\langle y_k - z, x_k - y_k \rangle \geq 0$. Hence we have that $A^{-1}0 \subset C_{k+1}$. This implies that $A^{-1}0 \subset C_n$ for all $n \in \mathbb{N}$.

Since C_n includes $A^{-1}0$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a sequence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $p_n = P_{C_n} u$ for all $n \in \mathbb{N}$. Then, by Theorem 2.1, we have that $\{p_n\}$ converges strongly to $p_0 = P_{C_0} u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$ and hence $\{p_n\}$ is bounded. Since $x_n \in C_n$ and $d(u, C_n) = \|u - p_n\|$, we have that

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n^2$$

for every $n \in \mathbb{N} \setminus \{1\}$. Therefore, we have that $\{x_n\}$ is bounded. Then, by (2.1), we have that for any $\lambda \in]0, 1[$

$$\begin{aligned} \|p_n - u\|^2 &\leq \|\lambda p_n + (1 - \lambda) x_n - u\|^2 \\ &= \lambda \|p_n - u\|^2 + (1 - \lambda) \|x_n - u\|^2 - \lambda(1 - \lambda) \|p_n - x_n\|^2 \end{aligned}$$

and thus

$$\lambda\|p_n - x_n\|^2 \le \|x_n - u\|^2 - \|p_n - u\|^2 \le \delta_n^2.$$

Tending $\lambda \to 1$, we have that $\|p_n - x_n\|^2 \le \delta_n^2$ and thus $\|p_n - x_n\| \le \delta_n$. Using the definition of p_n, we have that $p_{n+1} \in C_{n+1}$. Thus we have

$$\langle y_n - p_{n+1}, x_n - y_n\rangle \ge 0$$

and hence

$$\langle x_n - p_{n+1}, x_n - y_n\rangle \ge \|x_n - y_n\|^2.$$

Then we obtain that

$$\|x_n - y_n\| \le \|x_n - p_{n+1}\| \le \|x_n - p_n\| + \|p_n - p_{n+1}\| \le \delta_n + \|p_n - p_{n+1}\|$$

for every $n \in \mathbb{N} \setminus \{1\}$. From $\lim_n p_n = p_0$ and $\limsup_n \delta_n = \delta_0$, we have that

$$\limsup_{n\to\infty} \|x_n - y_n\| \le \delta_0.$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then we have that

$$\limsup_{n\to\infty} \|x_n - y_n\| \le 0$$

and

$$\limsup_{n\to\infty} \|x_n - p_n\| \le \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - p_n\| = 0$$

and hence

$$\lim_{n\to\infty} x_n = p_0 \text{ and } \lim_{n\to\infty} y_n = p_0.$$

From $\liminf_n r_n > 0$, we obtain that

$$\lim_{n\to\infty} \left\| \frac{x_n - y_n}{r_n} \right\| = 0.$$

For each $(u, v) \in A$, we obtain from (2.2) that

$$\left\langle u - y_n, v - \frac{x_n - y_n}{r_n} \right\rangle \geqq 0.$$

for each $n \in \mathbb{N} \setminus \{1\}$. Tending $n \to \infty$, we have that

$$\langle u - p_0, v - 0\rangle \geqq 0.$$

By the maximality of A, this implies that $p_0 \in A^{-1}0$. Since $A^{-1}0 \subset C_0$, we get that $p_0 = P_{C_0}u = P_{A^{-1}0}u$, which completes the proof. □

4. Applications

In this section, using Theorem 3.1, we give three applications. We first consider the convex minimization problem: Let H be a Hilbert space and let $f : H \to]-\infty, \infty]$ be a proper lower semicontinuous convex function. Then, the convex minimization problem is to find $x_0 \in H$ such that

$$f(x_0) = \min_{z \in H} f(z).$$

The subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle y - x, z \rangle \le f(y),\ \forall y \in H\}$$

for all $x \in H$. Then we know that the subdifferential $\partial f \subset H \times H$ of f is maximal monotone. It is easy to see that $(\partial f)^{-1}0 = \operatorname{argmin}\{f(x) : x \in H\}$. If J_r is the resolvent of ∂f for $r > 0$, then we also know that

$$J_r x = \operatorname*{argmin}_{z \in H} \left\{ f(z) + \frac{1}{2r} \|z - x\|^2 \right\}$$

for all $x \in H$ (see [14, 15, 21] for more details).

As a direct consequence of Theorem 3.1, we can show the following result.

Corollary 4.1. *Let H be a Hilbert space and let $f : H \to]-\infty, \infty]$ be a proper lower semicontinuous convex function with $(\partial f)^{-1}0 \neq \emptyset$. Let $\{\delta_n\}$ be a bounced nonnegative real sequence such that $\delta_0 = \limsup_n \delta_n$ and let $\{r_n\}$ be a positive real sequence such that $\liminf_n r_n > 0$. For a given point $u \in H$, generate a sequence $\{x_n\}$ by $x_1 = x \in H$, $C_1 = H$, and*

$$\begin{cases} y_n = \operatorname*{argmin}_{z \in H} \left\{ f(z) + \dfrac{1}{2r_n} \|z - x_n\|^2 \right\}, \\ C_{n+1} = \{z \in H : \langle y_n - z, x_n - y_n \rangle \ge 0\} \cap C_n, \\ x_{n+1} \in \{z \in H : \|u - z\|^2 \le d(u, C_{n+1})^2 + \delta_{n+1}^2\} \cap C_{n+1} \end{cases}$$

for all $n \in \mathbb{N}$. Then

$$\limsup_{n \to \infty} \|x_n - y_n\| \le \delta_0.$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{(\partial f)^{-1}0} u$.

Next, we consider the variational inequality problem: Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to H$ be a single-valued mapping. Then, the variational inequality problem is to find $x_0 \in C$ such that

$$\langle y - x_0, Tx_0 \rangle \ge 0 \tag{4.1}$$

for each $y \in C$. The set of such solutions is denoted by $VI(C, T)$.

A single-valued mapping T is said to be hemicontinuous if T is continuous from each line segment of C to H with the weak topology. Let T be a single-valued, monotone and hemicontinuous operator of C to H and let $N_C(x)$ be the normal cone to C at $x \in H$, that is,

$$N_C(x) := \{z \in H : \langle x - y, z \rangle \ge 0, \quad \forall y \in C\}.$$

We define the multi-valued operator $A \subset H \times H$ by

$$Az := \begin{cases} Tz + N_C(z), & z \in C, \\ \emptyset, & z \notin C. \end{cases}$$

Then, A is maximal monotone and $A^{-1}0 = VI(C,T)$ (see [16, 21]). We also know that, for each $r > 0$ and $x \in H$,

$$J_r x = (I + rA)^{-1}x = VI\left(C, T_{r,x}\right),$$

where $T_{r,x}z := Tz + (z - x)/r$ for each $z \in C$ (see [21] for more details) .

As a direct consequence of Theorem 3.1, we can show the following result.

Corollary 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a single-valued, monotone and hemicontinuous operator of C to H. Let $\{\delta_n\}$ be a bounded nonnegative real sequence such that $\delta_0 = \limsup_n \delta_n$ and let $\{r_n\}$ be a positive real sequence such that $\liminf_n r_n > 0$. For a given point $u \in H$, generate a sequence $\{x_n\}$ by $x_1 = x \in H$, $C_1 = H$, and*

$$\begin{cases} y_n = VI(C, T_{r_n,x_n}), \\ C_{n+1} = \{z \in H : \langle y_n - z, x_n - y_n\rangle \geq 0\} \cap C_n, \\ x_{n+1} \in \{z \in H : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}^2\} \cap C_{n+1} \end{cases}$$

for all $n \in \mathbb{N}$. If $VI(C,T)$ is nonempty, then

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \delta_0.$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{VI(C,T)}u$.

Finally, we consider the equilibrium problem: Let C be a nonempty closed convex subset of a Hilbert space H and let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Then, the equilibrium problem for f is to find $x_0 \in C$ such that

$$f(x_0, y) \geq 0$$

for all $y \in C$. The set of such solutions is denoted by $EP(f)$ (see [2, 4, 19] for more details). For solving the equilibrium problem, let us assume that a bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y);$$

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. We define the $F_r x$ by

$$F_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}.$$

Then, $F_r x$ is consists of at most one point. That is, $F_r : H \to C$ is single-valued mapping. Such a F_r is called the resolvent of f for r (see [2, 3, 19] for more details).

Let $A_f \subset H \times H$ be a multi-valued operator define by

$$A_f x = \begin{cases} \{z \in H : f(x,y) \geq \langle y - x, z \rangle, \ \forall y \in C\} & x \in C, \\ \emptyset & x \notin C. \end{cases}$$

We know that $EP(f) = A_f^{-1}0$ and A_f is maximal monotone. We also know that the resolvent F_r of f coincides with the resolvent $(I + rA_f)^{-1}$ of A_f for each $r > 0$, that is, $F_r = (I + rA_f)^{-1}$ (see [1,20] for more details) .

As a direct consequence of Theorem 3.1, we can show the following result.

Corollary 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $\{\delta_n\}$ be a bounded nonnegative real sequence such that $\delta_0 = \limsup_n \delta_n$ and let $\{r_n\}$ be a positive real sequence such that $\liminf_n r_n > 0$. For a given point $u \in H$, generate a sequence $\{x_n\}$ by $x_1 = x \in H$, $C_1 = H$, and*

$$\begin{cases} y_n = F_{r_n} x_n, \\ C_{n+1} = \{z \in H : \langle y_n - z, x_n - y_n \rangle \geq 0\} \cap C_n, \\ x_{n+1} \in \{z \in H : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}^2\} \cap C_{n+1} \end{cases}$$

for all $n \in \mathbb{N}$. If $EP(f)$ is nonempty, then

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \delta_0.$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{EP(f)}u$.

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