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UNIFORMLY NONEXPANSIVE SEQUENCES

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ABSTRACT. The aim of this paper is to introduce a uniformly nonexpansive sequence and to give some properties and a characterization of the sequence.

1. Introduction

The aim of this paper is to introduce and study a notion of a sequence of nonexpansive mappings in a Banach space, which is called a uniformly nonexpansive sequence.

This paper is organized as follows: In the third section we see that a uniformly nonexpansive sequence is similar to a strongly nonexpansive sequence discussed in [1, 2]. Indeed, we show that the class of uniformly nonexpansive sequences is properly contained in that of strongly nonexpansive sequences; the uniform nonexpansiveness is preserved under the composition; every sequence of firmly nonexpansive mappings is a uniformly nonexpansive sequence. In the forth section we provide a characterization of a uniformly nonexpansive sequence.

2. Preliminaries

Throughout this paper, E denotes a real Banach space with norm $\|\cdot\|$, C a nonempty subset of E, and $\mathbb{N}$ the set of positive integers.

A mapping $T\colon C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T\colon C \to E$ is said to be firmly nonexpansive [3] if

$$\|Tx - Ty\| \leq \|r(x - y) + (1 - r)(Tx - Ty)\| \tag{2.1}$$

for all $x, y \in C$ and $r > 0$. A mapping $T\colon C \to E$ is said to be strongly nonexpansive [5] if it is nonexpansive and

$$\lim_{n\to\infty} \|x_n - y_n - (Tx_n - Ty_n)\| = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$. It is clear that the identity mapping on E is firmly nonexpansive and strongly nonexpansive; it is also clear that every firmly nonexpansive mapping is nonexpansive. It is known that every firmly nonexpansive mapping is strongly nonexpansive if E is uniformly convex; see [5, Proposition 2.1].

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Taking [4] into account, we provide an equivalent condition for strong nonexpansiveness as follows (see also Corollary 4.5):

Lemma 2.1. *Let T be a mapping of C into E. Then the following are equivalent:*

(1) *T is strongly nonexpansive;*
(2) *for each $M > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that*

$$u, v \in C,\ \|u - v\| \leq M,\ \|u - v\| - \|Tu - Tv\| < \delta \\ \Rightarrow \|u - v - (Tu - Tv)\| < \epsilon. \tag{2.2}$$

Proof. We first show that (2) implies (1). Let $x, y \in C$. If $x - y = Tx - Ty$, then $\|Tx - Ty\| \leq \|x - y\|$ holds clearly. Thus we assume that $x - y \neq Tx - Ty$ and set $M = \|x - y\|$ and $\epsilon = \|x - y - (Tx - Ty)\|$. Then it is obvious that $M > 0$ and $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that (2.2) holds. Thus $\|x - y\| - \|Tx - Ty\| \geq \delta$ and hence $\|Tx - Ty\| \leq \|x - y\|$. Therefore we know that T is nonexpansive. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$. Set $M = \sup_n \|x_n - y_n\|$. Without loss of generality, we may assume that $M > 0$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that (2.2) holds. Since $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - y_n\| - \|Tx_n - Ty_n\| < \delta$ for all $n \geq N$ and hence $\|x_n - y_n - (Tx_n - Ty_n)\| < \epsilon$ for all $n \geq N$. This means that $\|x_n - y_n - (Tx_n - Ty_n)\| \to 0$. Consequently, we conclude that T is strongly nonexpansive.

We next show that (1) implies (2). Suppose that T is strongly nonexpansive and (2) does not hold. Then there exist $M > 0$, $\epsilon > 0$, and sequences $\{x_n\}$ and $\{y_n\}$ in C such that

$$\|x_n - y_n\| \leq M,\ \|x_n - y_n\| - \|Tx_n - Ty_n\| < 1/n, \\ \text{and } \|x_n - y_n - (Tx_n - Ty_n)\| \geq \epsilon.$$

Since T is strongly nonexpansive, it follows that $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$ and $\|x_n - y_n - (Tx_n - Ty_n)\| \to 0$, which is a contradiction. □

A sequence $\{T_n\}$ of mappings of C into E is said to be a strongly nonexpansive sequence [1, 2] if each T_n is nonexpansive and

$$\lim_{n\to\infty} \|x_n - y_n - (T_n x_n - T_n y_n)\| = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$.

We need the following lemma:

Lemma 2.2 ([2, Lemma 2.1])**.** *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a uniformly convex Banach space E and $\{\lambda_n\}$ a sequence in $[0, 1]$. Suppose that $\liminf_{n\to\infty} \lambda_n > 0$ and*

$$\lambda_n \|x_n\|^2 + (1 - \lambda_n) \|y_n\|^2 - \|\lambda_n x_n + (1 - \lambda_n) y_n\|^2 \to 0.$$

Then $(1 - \lambda_n) \|x_n - y_n\| \to 0$.

The rest of this section, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and D a nonempty subset of H. A mapping $A\colon D \to H$ is said to be inverse-strongly-monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in D$. In this case, A is called an α-inverse-strongly-monotone mapping. Let α be a positive real number, $A\colon D \to H$ an α-inverse-strongly-monotone mapping, and I the identity mapping on H. It is known that

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 - \lambda(2\alpha - \lambda) \|Ax - Ay\|^2 \tag{2.3}$$

holds for all $x, y \in D$ and $\lambda \geq 0$; see, for example, [7]. Thus it follows from (2.3) that $I - \lambda A$ is nonexpansive for $\lambda \in [0, 2\alpha]$.

3. Uniformly nonexpansive sequences

Throughout this section, let C be a nonempty subset of a Banach space E. Inspired by [1, 2, 4], we introduce a uniformly nonexpansive sequence as follows: A sequence $\{T_n\}$ of mappings of C into E is said to be a *uniformly nonexpansive sequence* if for each $M > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} n \in \mathbb{N},\ x, y \in C,\ \|x - y\| \leq M,\ \|x - y\| - \|T_n x - T_n y\| < \delta \\ \Rightarrow \|x - y - (T_n x - T_n y)\| < \epsilon. \end{aligned} \tag{3.1}$$

Remark 3.1. It is clear from Lemma 2.1 that if $\{T_n\}$ is a uniformly nonexpansive sequence, then each T_n is (strongly) nonexpansive.

A uniformly nonexpansive sequence is an example of a strongly nonexpansive sequence:

Lemma 3.2. *Let $\{T_n\}$ be a sequence of mappings of C into E. Suppose that $\{T_n\}$ is a uniformly nonexpansive sequence. Then $\{T_n\}$ is a strongly nonexpansive sequence.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0. \tag{3.2}$$

Set $M = \sup_n \|x_n - y_n\| + 1$ and let $\epsilon > 0$ be given. Since $\{T_n\}$ is a uniformly nonexpansive sequence, there exists $\delta > 0$ such that (3.1) holds. From (3.2), we see that there exists $N \in \mathbb{N}$ such that $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| < \delta$ for all $n \geq N$. Thus it follows from (3.1) that $\|x_n - y_n - (T_n x_n - T_n y_n)\| < \epsilon$ for all $n \geq N$, and hence $\|x_n - y_n - (T_n x_n - T_n y_n)\| \to 0$. Since each T_n is nonexpansive by Remark 3.1, we conclude that $\{T_n\}$ is a strongly nonexpansive sequence. ☐

We deal with some examples of uniformly nonexpansive sequences and strongly nonexpansive sequences.

Example 3.3. Let $S\colon C \to E$ be a strongly nonexpansive mapping. Set $T_n = S$ for $n \in \mathbb{N}$. Then $\{T_n\}$ is a uniformly nonexpansive sequence by Lemma 2.1.

The following example shows that the converse of Lemma 3.2 is not true.

Example 3.4. Let $T: E \to E$ be a mapping defined by $Tx = -x$ for $x \in E$. Set $S_1 = T$ and $S_n = I$ for $n \geq 2$, where I is the identity mapping on E. Then it is easy to check that $\{S_n\}$ is a strongly nonexpansive sequence and is not a uniformly nonexpansive sequence.

A sequence of firmly nonexpansive mappings is a typical example of a uniformly nonexpansive sequence.

Lemma 3.5. *Let $\{T_n\}$ be a sequence of firmly nonexpansive mappings of C into a uniformly convex Banach space E. Then $\{T_n\}$ is a uniformly nonexpansive sequence.*

Proof. Suppose that $\{T_n\}$ is not a uniformly nonexpansive sequence. Then there exist $M > 0$ and $\epsilon > 0$ such that for each $m \in \mathbb{N}$ there exist $x_m, y_m \in C$ and $n_m \in \mathbb{N}$ such that

$$\|x_m - y_m\| \leq M,\ \|x_m - y_m\| - \|T_{n_m}x_m - T_{n_m}y_m\| < 1/m, \\ \text{and } \|x_m - y_m - (T_{n_m}x_m - T_{n_m}y_m)\| \geq \epsilon. \tag{3.3}$$

Let $r \in (0,1)$ be fixed. Since each T_{n_m} is firmly nonexpansive, it follows from (2.1) that

$$\begin{aligned} r\|x_m - y_m\|^2 &+ (1-r)\|T_{n_m}x_m - T_{n_m}y_m\|^2 \\ &- \|r(x_m - y_m) + (1-r)(T_{n_m}x_m - T_{n_m}y_m)\|^2 \\ &\leq r\left(\|x_m - y_m\|^2 - \|T_{n_m}x_m - T_{n_m}y_m\|^2\right) \\ &= r(\|x_m - y_m\| + \|T_{n_m}x_m - T_{n_m}y_m\|)(\|x_m - y_m\| - \|T_{n_m}x_m - T_{n_m}y_m\|) \\ &\leq \frac{2rM}{m} \to 0 \end{aligned}$$

as $m \to \infty$. Thus Lemma 2.2 implies that $\|x_m - y_m - (T_{n_m}x_m - T_{n_m}y_m)\| \to 0$, which is a contradiction. Therefore, $\{T_n\}$ is a uniformly nonexpansive sequence. $\square$

The following is an example using a monotone mapping in a Hilbert space.

Example 3.6. Let α be a positive real number, D a nonempty subset of a real Hilbert space H, A an α-inverse-strongly-monotone mapping of D into H, I the identity mapping on H, and $\{\lambda_n\}$ a sequence in $[0, 2\alpha)$ such that $0 < \sup_n \lambda_n < 2\alpha$. Set $T_n = I - \lambda_n A$ for $n \in \mathbb{N}$. Then $\{T_n\}$ is a uniformly nonexpansive sequence. Indeed, (2.3) and the nonexpansiveness of T_n imply that

$$\begin{aligned} \|(I - T_n)x - (I - T_n)y\|^2 &= \lambda_n^2 \|Ax - Ay\|^2 \\ &\leq \frac{\lambda_n}{2\alpha - \lambda_n}\left(\|x - y\|^2 - \|T_n x - T_n y\|^2\right) \\ &\leq 2\|x - y\| \frac{\sup_n \lambda_n}{2\alpha - \sup_n \lambda_n}(\|x - y\| - \|T_n x - T_n y\|) \end{aligned}$$

for all $x, y \in D$. This shows that $\{T_n\}$ is a uniformly nonexpansive sequence.

We know that the composition of two strongly nonexpansive sequences is a strongly nonexpansive sequence; see [1, Theorem 3.4] and [2, Theorem 3.2]. Uniformly nonexpansive sequences have a similar property as follows:

Theorem 3.7. *Let C and D be two nonempty subsets of a Banach space E. Let $\{S_n\}$ be a sequence of mappings of D into E and $\{T_n\}$ a sequence of mappings of C into E. Suppose that both $\{S_n\}$ and $\{T_n\}$ are uniformly nonexpansive sequences and $T_n(C) \subset D$ for each $n \in \mathbb{N}$. Then $\{S_nT_n\}$ is a uniformly nonexpansive sequence.*

Proof. Let $M > 0$ and $\epsilon > 0$ be given. By assumption, there exist $\delta > 0$ such that

$$n \in \mathbb{N},\ x, y \in D,\ \|x - y\| \le M,\ \|x - y\| - \|S_nx - S_ny\| < \delta \\ \Rightarrow \|x - y - (S_nx - S_ny)\| < \epsilon/2$$

and

$$n \in \mathbb{N},\ x, y \in C,\ \|x - y\| \le M,\ \|x - y\| - \|T_nx - T_ny\| < \delta \\ \Rightarrow \|x - y - (T_nx - T_ny)\| < \epsilon/2.$$

Suppose that $u, v \in C$, $\|u - v\| \le M$, and $\|u - v\| - \|S_nT_nu - S_nT_nv\| < \delta$. Since S_n and T_n are nonexpansive, we have

$$\|u - v\| - \|T_nu - T_nu\| < \delta,\ \|T_nu - T_nv\| \le \|u - v\| \le M, \\ \text{and } \|T_nu - T_nv\| - \|S_nT_nu - S_nT_nu\| < \delta.$$

Therefore we have

$$\|u - v - (S_nT_nu - S_nT_nv)\| \\ \le \|u - v - (T_nu - T_nv)\| + \|T_nu - T_nv - (S_nT_nu - S_nT_nv)\| < \epsilon.$$

This shows that $\{S_nT_n\}$ is a uniformly nonexpansive sequence. □

Example 3.8. Let α, D, H, A, I, $\{\lambda_n\}$, and $\{T_n\}$ be the same as in Example 3.6. Suppose that D is closed and convex. Set $S_n = P_D$ for $n \in \mathbb{N}$, where P_D is the metric projection of D onto H. Since P_D is firmly nonexpansive (see [6]), P_D is strongly nonexpansive. Thus Example 3.3 shows that $\{S_n\}$ is a uniformly nonexpansive sequence. Therefore Example 3.6 and Theorem 3.7 imply that $\{S_nT_n\} = \{P_D(I - \lambda_nA)\}$ is a uniformly nonexpansive sequence.

A uniformly nonexpansive sequence can be generated by a sequence of nonexpansive mappings as follows:

Lemma 3.9. *Let $\{\alpha_n\}$ be a sequence in $(0, 1]$, $\{S_n\}$ a sequence of nonexpansive mappings of C into E, and I the identity mapping on E. Suppose that $\inf_n \alpha_n > 0$ and E is uniformly convex. Set $T_n = \alpha_nI + (1 - \alpha_n)S_n$ for $n \in \mathbb{N}$. Then $\{T_n\}$ is a uniformly nonexpansive sequence.*

Remark 3.10. Under the assumptions of Lemma 3.9, it is clear that if $\alpha_n \ne 1$, then the fixed-point set of S_n is equal to that of T_n.

Lemma 3.9 is a direct consequence of the following theorem; see [1, Theorem 3.11] and [2, Theorem 3.7]:

Theorem 3.11. *Let $\{\alpha_n\}$ be a sequence in $(0, 1]$, $\{S_n\}$ a sequence of nonexpansive mappings of C into E, and $\{U_n\}$ a sequence of mappings of C into E. Suppose that $\inf_n \alpha_n > 0$, E is uniformly convex, and $\{U_n\}$ is a uniformly nonexpansive sequence.*

Set $T_n = \alpha_n U_n + (1-\alpha_n) S_n$ for $n \in \mathbb{N}$. Then $\{T_n\}$ is a uniformly nonexpansive sequence.

Proof. Suppose that $\{T_n\}$ is not a uniformly nonexpansive sequence. Then there exist $M > 0$ and $\epsilon > 0$ such that for each $m \in \mathbb{N}$ there exist $x_m, y_m \in C$ and $n_m \in \mathbb{N}$ such that (3.3) holds. Since S_n and U_n are nonexpansive, it is clear from the definition of T_n that

$$0 \le \alpha_n \left(\|x-y\| - \|U_n x - U_n y\| \right) \le \|x-y\| - \|T_n x - T_n y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Taking into account $\inf_n \alpha_n > 0$ and (3.3), we know that $\|x_m - y_m\| - \|U_{n_m} x_m - U_{n_m} y_m\| \to 0$, and hence

$$\|x_m - y_m - (U_{n_m} x_m - U_{n_m} y_m)\| \to 0 \tag{3.4}$$

because $\{U_n\}$ is a uniformly nonexpansive sequence. On the other hand, since S_n, U_n, and T_n are nonexpansive,

$$\begin{aligned} &\alpha_n \|U_n x - U_n y\|^2 + (1-\alpha_n) \|S_n x - S_n y\|^2 - \|T_n x - T_n y\|^2 \\ &\quad \le \|x-y\|^2 - \|T_n x - T_n y\|^2 \\ &\quad \le 2\|x-y\| \left(\|x-y\| - \|T_n x - T_n y\| \right) \end{aligned}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Thus, (3.3) and Lemma 2.2 imply that

$$(1-\alpha_{n_m}) \|U_{n_m} x_m - U_{n_m} y_m - (S_{n_m} x_m - S_{n_m} y_m)\| \to 0. \tag{3.5}$$

Using (3.4) and (3.5), we conclude that

$$\begin{aligned} &\|x_m - y_m - (T_{n_m} x_m - T_{n_m} y_m)\| \\ &\quad \le \|x_m - y_m - (U_{n_m} x_m - U_{n_m} y_m)\| \\ &\qquad + \|U_{n_m} x_m - U_{n_m} y_m - (T_{n_m} x_m - T_{n_m} y_m)\| \\ &\quad = \|x_m - y_m - (U_{n_m} x_m - U_{n_m} y_m)\| \\ &\qquad + (1-\alpha_{n_m}) \|U_{n_m} x_m - U_{n_m} y_m - (S_{n_m} x_m - S_{n_m} y_m)\| \to 0 \end{aligned}$$

as $m \to \infty$, which contradicts (3.3). Therefore, $\{T_n\}$ is a uniformly nonexpansive sequence. □

Proof of Lemma 3.9. Set $U_n = I$ for $n \in \mathbb{N}$. Since I is strongly nonexpansive, Example 3.3 shows that $\{U_n\}$ is a uniformly nonexpansive sequence. Thus Theorem 3.11 implies the conclusion. □

4. A characterization for a uniformly nonexpansive sequence

In [5], Bruck and Reich provided an equivalent condition for strong nonexpansiveness; see Corollary 4.5 below. Inspired by the condition, we prove the following:

Theorem 4.1. *Let C be a nonempty subset of a Banach space E and $\{T_n\}$ a sequence of mappings of C into E. Then the following are equivalent:*

(1) *$\{T_n\}$ is a uniformly nonexpansive sequence;*

(2) *for any* $M > 0$ *there exists a nondecreasing function* $\gamma\colon [0, 2M] \to [0, M]$ *such that* $\gamma(t) > 0$ *for all* $t \in (0, 2M]$ *and*

$$\gamma(\|x - y - (T_n x - T_n y)\|) \le \|x - y\| - \|T_n x - T_n y\| \tag{4.1}$$

for all $n \in \mathbb{N}$ *and* $x, y \in C$ *with* $\|x - y\| \le M$.

Before proving Theorem 4.1, we show some lemmas. The rest of this section, let C be a nonempty subset of a Banach space E and $\{T_n\}$ a sequence of nonexpansive mappings of C into E.

The following lemma is clear from the nonexpansiveness of T_n.

Lemma 4.2. *Suppose that* $M > 0$ *and set*

$$L_M = \sup\{\|x - y - (T_n x - T_n y)\| : x, y \in C,\ \|x - y\| \le M,\ n \in \mathbb{N}\}. \tag{4.2}$$

Then $0 \le L_M \le 2M$.

Lemma 4.3. *Suppose that* $M > 0$ *and* $L_M = 0$*, where* L_M *is defined by* (4.2). *Let* $\gamma\colon [0, 2M] \to [0, M]$ *be a function defined by* $\gamma(t) = t/2$ *for* $t \in [0, 2M]$. *Then* (4.1) *holds for all* $n \in \mathbb{N}$ *and* $x, y \in C$ *with* $\|x - y\| \le M$.

Proof. Let $x, y \in C$ with $\|x - y\| \le M$. Then $\|x - y - (T_n x - T_n y)\| \le L_M = 0$. Since T_n is nonexpansive, it follows that $\gamma(\|x - y - (T_n x - T_n y)\|) = 0 \le \|x - y\| - \|T_n x - T_n y\|$ for all $n \in \mathbb{N}$. □

Lemma 4.4. *Suppose that* $M > 0$, $\{T_n\}$ *is a uniformly nonexpansive sequence, and* $L_M > 0$*, where* L_M *is defined by* (4.2). *Let* $\gamma\colon [0, L_M) \to [0, M]$ *be a function defined by*

$$\gamma(t) = \inf\{\|u - v\| - \|T_n u - T_n v\| : n \in \mathbb{N},\ u, v \in C,\ \|u - v\| \le M,\ \|u - v - (T_n u - T_n v)\| \ge t\} \tag{4.3}$$

for $t \in [0, L_M)$. *Then the following hold:*

(1) $0 \le \gamma(t) \le M$ *for all* $t \in [0, L_M)$;
(2) γ *is nondecreasing;*
(3) (4.1) *holds if* $n \in \mathbb{N}$, $x, y \in C$, $\|x - y\| \le M$, *and* $\|x - y - (T_n x - T_n y)\| < L_M$;
(4) $\gamma(0) = 0$ *and* $\gamma(t) > 0$ *if* $t \in (0, L_M)$.

Proof. Let $t \in [0, L_M)$ be fixed. Then, by the definition of L_M, there exist $x, y \in C$ and $n \in \mathbb{N}$ such that $\|x - y\| \le M$ and $\|x - y - (T_n x - T_n y)\| > t$. Thus $\gamma(t) < \infty$. Since T_n is nonexpansive, it is clear that

$$0 \le \gamma(t) \le \|x - y\| - \|T_n x - T_n y\| \le \|x - y\| \le M.$$

Therefore, (1) holds.

(2) and (3) follow from the definition of γ.

Lastly, we show (4). It is easy to verify that $\gamma(0) = 0$. Suppose that there exists $t \in (0, L_M)$ with $\gamma(t) = 0$. By the definition of γ, for each $m \in \mathbb{N}$, there exist $u_m, v_m \in C$ and $n_m \in \mathbb{N}$ such that

$$\|u_m - v_m\| \le M,\ \|u_m - v_m - (T_{n_m} u_m - T_{n_m} v_m)\| \ge t, \quad \text{and} \quad \|u_m - v_m\| - \|T_{n_m} u_m - T_{n_m} v_m\| < 1/m.$$

Since $\{T_n\}$ is a uniformly nonexpansive sequence, there exists $\delta > 0$ such that

$$n \in \mathbb{N}, \|x-y\| \le M, \ \|x-y\| - \|T_n x - T_n y\| < \delta \\ \Rightarrow \|x - y - (T_n x - T_n y)\| < t/2.$$

Choosing $m \in \mathbb{N}$ satisfying that $1/m < \delta$, we have $\|u_m - v_m\| \le M$, $\|u_m - v_m\| - \|T_{n_m} u_m - T_{n_m} v_m\| < \delta$, and

$$t \le \|u_m - v_m - (T_{n_m} u_m - T_{n_m} v_m)\| < t/2,$$

which is a contradiction. Therefore, (4) holds. □

Using lemmas above, we can prove Theorem 4.1.

Proof of Theorem 4.1. We first show that (1) implies (2). Let M be a positive real number and L_M a real number defined by (4.2). Lemma 4.2 shows that $0 \le L_M \le 2M$. If $L_M = 0$, then the assertion follows from Lemma 4.3. Suppose that $L_M > 0$ and γ is a function defined by (4.3) for $t \in [0, L_M)$. Lemma 4.4 implies that γ can be extended to a function defined on $[0, 2M]$ as follows:

$$\gamma(t) = \sup\{\gamma(s) : 0 \le s < L_M\} \text{ for } t \in [L_M, 2M].$$

Using Lemma 4.4, we know that the extended γ is the desired function.

We next show that (2) implies (1). Suppose that $\{T_n\}$ is not a uniformly nonexpansive sequence. Then there exist $M > 0$ and $\epsilon > 0$ such that for each $m \in \mathbb{N}$ there exist $x_m, y_m \in C$ and $n_m \in \mathbb{N}$ such that (3.3) holds. By (2), there exists a nondecreasing function $\gamma\colon [0, 2M] \to [0, M]$ such that $\gamma(t) > 0$ for all $t \in (0, 2M]$ and (4.1) holds for all $n \in \mathbb{N}$ and $x, y \in C$ with $\|x - y\| \le M$. Thus it follows that

$$\begin{aligned} 0 < \gamma(\epsilon) &\le \gamma(\|x_m - y_m - (T_{n_m} x_m - T_{n_m} y_m)\|) \\ &\le \|x_m - y_m\| - \|T_{n_m} x_m - T_{n_m} y_m\| < 1/m \to 0 \end{aligned}$$

as $m \to \infty$, which is a contradiction. Therefore, $\{T_n\}$ is a uniformly nonexpansive sequence. □

Using Theorem 4.1, we obtain the following:

Corollary 4.5 ([5]). *Let C be a nonempty subset of a Banach space E and T a mapping of C into E. Then the following are equivalent:*

(1) *T is strongly nonexpansive;*
(2) *for any $M > 0$, there exists a nondecreasing function $\gamma\colon [0, 2M] \to [0, M]$ such that $\gamma(t) > 0$ for all $t \in (0, 2M]$ and*

$$\gamma(\|x - y - (Tx - Ty)\|) \le \|x - y\| - \|Tx - Ty\|$$

for all $x, y \in C$ with $\|x - y\| \le M$.

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