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MODIFICATION OF THE KRASNOSEL'SKIĬ-MANN FIXED POINT ALGORITHM BY USING THREE-TERM CONJUGATE GRADIENTS

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Abstract. This paper considers the problem of finding a fixed point of a nonexpansive mapping on a real Hilbert space and modifies the Krasnosel'skiĭ-Mann algorithm by using a three-term conjugate gradient-like direction that is used to solve constrained optimization problems quickly. We prove that, under certain assumptions, the proposed algorithm converges to a fixed point of a nonexpansive mapping in the sense of the weak topology of a Hilbert space. We numerically compare the algorithm with the existing fixed point algorithms. The numerical results show that it reduces the running time and iterations needed to find a fixed point compared with those algorithms.

1. Introduction

This paper discusses the following fixed point problem [1, Chapter 4], [5, Chapter 3], [6, Chapter 1], [16, Chapter 3]:

Problem 1.1. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, and let $T \colon H \to H$ be nonexpansive; i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Then

$$\text{find } x^* \in \mathrm{F}(T) := \{x^* \in H \colon Tx^* = x^*\},$$

where one assumes $\mathrm{F}(T)$ is nonempty.

It is known that the Krasnosel'skiĭ-Mann algorithm [1, Subchapter 5.2], [2, Subchapter 1.2], [13,14] is a simple and useful fixed point algorithm for solving Problem 1.1. The algorithm is defined for all $n \in \mathbb{N}$ by

$$x_{n+1} := \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.1}$$

where $x_0 \in H$ is chosen arbitrarily and $\{\alpha_n\} \subset (0,1)$ satisfies $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$. It is guaranteed that Algorithm (1.1) weakly converges to some point in $\mathrm{F}(T)$ [1, Theorem 5.14]. Unfortunately, the algorithm converges slowly [4, Propositions 10 and 11]. Accordingly, modifications [7,10] have been developed in order to accelerate Algorithm (1.1).

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Let us consider the convergence of the Krasnosel'skiĭ-Mann algorithm (1.1) from the viewpoint of an unconstrained smooth convex optimization problem: given a Fréchet differentiable, convex function $f\colon H \to \mathbb{R}$ with the Lipschitz continuous gradient ∇f,

$$\text{minimize } f(x) \text{ subject to } x \in H. \tag{1.2}$$

Here, let us define a mapping $T_f\colon H \to H$ by $T_f := \mathrm{Id} - \lambda \nabla f$, where Id stands for the identity mapping on H, L is the Lipschitz constant of ∇f, and $\lambda \in [0, 2/L]$. Then, T_f satisfies the nonexpansivity condition and $\mathrm{F}(T_f) = \mathrm{argmin}_{x \in H} f(x)$ [8, Proposition 2.3]. The sequence $\{x_n\}$ generated by Algorithm (1.1) with T_f is

$$\begin{aligned} x_{n+1} &= x_n + (1-\alpha_n)(T_f x_n - x_n) \\ &= x_n - \lambda(1-\alpha_n)\nabla f(x_n), \end{aligned} \tag{1.3}$$

which means that Algorithm (1.1) with T_f is the steepest descent method for unconstrained smooth convex optimization and that it does not converge quickly. Hence, modified Krasnosel'skiĭ-Mann algorithms [7, 10] were developed to accelerate the search for fixed points of a nonexpansive mapping.

Hishinuma and Iiduka [7] presented the following algorithm.

$$\begin{aligned} d_n &:= (T x_n - x_n) + \beta_n d_{n-1}, \\ x_{n+1} &:= x_n + (1-\alpha_n) d_n. \end{aligned} \tag{1.4}$$

Algorithm (1.4) can be obtained by replacing the steepest descent direction $d_n := T_f x_n - x_n = -\lambda \nabla f(x_n)$ in (1.3) with the conjugate gradient-like direction [9, 12] $d_n = T_f x_n - x_n + \beta_n d_{n-1} = -\lambda \nabla f(x_n) + \beta_n d_{n-1}$, where $\beta_n \geq 0$. We can see that Algorithm (1.4) with $\beta_n := 0$ $(n \in \mathbb{N})$ coincides with Algorithm (1.1). Algorithm (1.4) weakly converges to a fixed point of T if $\sum_{n=0}^{\infty} \beta_n < \infty$ and if $\{T x_n - x_n\}$ is bounded [7, Theorem 3.3]. Iiduka [10] proposed line search fixed point algorithms that can determine a more adequate step size than one satisfying $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ at each iteration n so that the value of $\|x_n - T x_n\|$ decreases dramatically. The algorithms [10, Algorithm 2.1] use the conventional nonlinear conjugate gradient directions [15, Subchapter 5.2], such as the Hestenes-Stiefel, Fletcher-Reeves, Polak-Ribière-Polyak, and Dai-Yuan formulas, in contrast to the previous algorithm [7].

The main objective of this paper is to accelerate Algorithm (1.4). To reach this goal, we present an algorithm that can be obtained by replacing the conjugate gradient-like direction in (1.4) with the following *three-term conjugate gradient-like direction*:

$$d_{n+1} := (T x_n - x_n) + \beta_n d_n + \gamma_n \omega_n, \tag{1.5}$$

where $\gamma_n \geq 0$ and $\omega_n \in H$ $(n \in \mathbb{N})$. The three-term conjugate gradient-like direction was proposed in [9] to accelerate the previous fixed point optimization algorithms and the numerical results in [9, Section 4] showed that the algorithm in [9] converges to optimal solutions to concrete smooth convex optimization problems faster than the previous ones.

We prove that, under certain assumptions, the proposed algorithm with (1.5) weakly converges to a fixed point in $\mathrm{F}(T)$. Furthermore, we numerically compare it with Algorithms (1.1) and (1.4) and show that it performs better than them.

This paper is organized as follows. Section 2 devises the acceleration algorithm for solving Problem 1.1 and proves the weak convergence of the algorithm. Section 3 provides numerical examples. Section 4 concludes the paper.

2. Acceleration of the Krasnosel'skiĭ-Mann algorithm

Suppose that $T\colon H \to H$ is nonexpansive with $\mathrm{F}(T) \neq \emptyset$. The following is the proposed algorithm for solving Problem 1.1.

Algorithm 2.1.

Step 0: Choose $\alpha > 0$ and $x_0, \omega_0 \in H$ arbitrarily, and set $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,\infty)$, and $\{\gamma_n\} \subset [0,\infty)$. Compute $d_0 := (Tx_0 - x_0)/\alpha$.

Step 1: Compute $d_{n+1} \in H$ as

$$d_{n+1} := \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n + \gamma_n \omega_n.$$

Compute $x_{n+1} \in H$ as

$$\begin{cases} y_n := x_n + \alpha d_{n+1}, \\ x_{n+1} := \alpha_n x_n + (1-\alpha_n)\, y_n. \end{cases}$$

Put $n := n+1$, and go to Step 1.

The following is a convergence analysis of Algorithm 2.1.

Theorem 2.2. *Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy*

$$\text{(C1)} \sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty, \ \text{(C2)} \sum_{n=0}^{\infty} \beta_n < \infty, \ \text{(C3)} \sum_{n=0}^{\infty} \gamma_n < \infty.$$

Moreover, assume that

(C4) $\{Tx_n - x_n\}$ *is bounded and* (C5) $\{\omega_n\}$ *is a bounded sequence chosen arbitrarily* .

Then the sequence $\{x_n\}$ generated by Algorithm 2.1 weakly converges to a fixed point of T.

2.1. **Proof of Theorem 2.2.** We first prove the boundedness of $\{d_n\}$, $\{x_n\}$, and $\{y_n\}$.

Lemma 2.3. *Suppose that the assumptions in Theorem 2.2 hold. Then,*

(i) $\{d_n\}$ *is bounded;*
(ii) $\lim_{n\to\infty} \|x_n - u\|$ *exists for all $u \in \mathrm{F}(T)$. In particular, $\{x_n\}$ is bounded;*
(iii) $\{y_n\}$ *is bounded.*

Proof. (i) Conditions (C2) and (C3) ensure that $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \gamma_n = 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leq 1/3$ and $\gamma_n \leq 1/3$ for all $n \geq n_0$. Put $M_1 := \max\{\|d_{n_0}\|, (3/\alpha)\sup_{n\in\mathbb{N}} \|Tx_n - x_n\|\}$. Condition (C4) implies that $M_1 < \infty$. Moreover, put $M_2 := \max\{M_1, \sup_{n\in\mathbb{N}} \|\omega_n\|\}$. Condition (C5) and M_1 imply that

$M_2 < \infty$. We assume that $\|d_n\| \leq M_2$ for some $n \geq n_0$. From the triangle inequality, we find that

$$\begin{aligned}\|d_{n+1}\| &= \left\| \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n + \gamma_n \omega_n \right\| \\ &\leq \frac{1}{\alpha}\|Tx_n - x_n\| + \beta_n \|d_n\| + \gamma_n \|\omega_n\| \\ &\leq M_2.\end{aligned}$$

Induction shows that $\|d_n\| \leq M_2$ for all $n \geq n_0$; i.e., $\{d_n\}$ is bounded.

(ii) The definitions of y_n and d_n $(n \in \mathbb{N})$ imply that

$$\begin{aligned}y_n &= x_n + \alpha \left\{ \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n + \gamma_n \omega_n \right\} \\ &= Tx_n + \alpha(\beta_n d_n + \gamma_n \omega_n).\end{aligned} \tag{2.1}$$

The triangle inequality and (2.1) mean that, for all $u \in F(T)$ and for all $n \geq n_0$,

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n x_n + (1-\alpha_n)\{Tx_n + \alpha(\beta_n d_n + \gamma_n \omega_n)\} - u\| \\ &= \|\alpha_n (x_n - u) + (1-\alpha_n)\{Tx_n - u + \alpha(\beta_n d_n + \gamma_n \omega_n)\| \\ &\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\|Tx_n - u\| + \alpha(1-\alpha_n)\beta_n \|d_n\| \\ &\quad + \alpha(1-\alpha_n)\gamma_n \|\omega_n\|,\end{aligned}$$

which, together with the nonexpansivity of T, $1 - \alpha_n < 1$ $(n \in \mathbb{N})$, $\|d_n\| \leq M_2$, and $\|\omega_n\| \leq M_2$, implies that, for all $n \geq n_0$,

$$\|x_{n+1} - u\| \leq \|x_n - u\| + \alpha M_2 (\beta_n + \gamma_n).$$

Conditions (C2) and (C3) thus guarantee that $\lim_{n\to\infty} \|x_n - u\|$ exists for all $u \in F(T)$. This means $\{x_n\}$ is bounded.

(iii) The definition of $\{y_n\}$ $(n \in \mathbb{N})$ and the boundedness of $\{x_n\}$ and $\{d_n\}$ imply that $\{y_n\}$ is also bounded. □

Next, we prove the following lemma.

Lemma 2.4. *Suppose that the assumptions in Theorem 2.2 hold. Then,*

(i) $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$*;*

(ii) *There exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *which weakly converges to a fixed point of* T*.*

Proof. (i) Choose $u \in \mathrm{F}(T)$ arbitrarily. From the equality, $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ $(\alpha \in [0,1],\ x, y \in H)$, we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n) y_n - u\|^2 \\ &= \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 - \alpha_n(1-\alpha_n)\|x_n - y_n\|^2.\end{aligned}$$

From (2.1), the nonexpansivity of T, and the inequality, $\|x+y\|^2 \leq \|x\|^2 + 2\langle x+y, y\rangle$ $(x, y \in H)$, we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|y_n - u\|^2 &= \|(Tx_n - u) + \alpha(\beta_n d_n + \gamma_n \omega_n)\|^2 \\ &\leq \|Tx_n - u\|^2 + 2\alpha\beta_n \langle y_n - u, d_n\rangle + 2\alpha\gamma_n \langle y_n - u, \omega_n\rangle\end{aligned}$$

$$\leq \|x_n - u\|^2 + M_3\beta_n + M_4\gamma_n,$$

where $M_3 := \sup_{n\in\mathbb{N}} 2\alpha|\langle y_n - u, d_n\rangle| < \infty$ and $M_4 := \sup_{n\in\mathbb{N}} 2\alpha|\langle y_n - u, \omega_n\rangle| < \infty$. Hence, from $\|x_n - y_n\| = \alpha\|d_{n+1}\|$ $(n \in \mathbb{N})$, we find that, for all $n \in \mathbb{N}$,

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 + M_3\beta_n + M_4\gamma_n - \alpha^2\alpha_n(1-\alpha_n)\|d_{n+1}\|^2.$$

Therefore, for all $n \in \mathbb{N}$,

$$\alpha^2\alpha_n(1-\alpha_n)\|d_{n+1}\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + M_3\beta_n + M_4\gamma_n.$$

Summing up these inequalities from $n = 0$ to $n = N \in \mathbb{N}$ yields

$$\begin{aligned}\alpha^2\sum_{n=0}^{N}\alpha_n(1-\alpha_n)\|d_{n+1}\|^2 &\leq \|x_0 - u\|^2 - \|x_{N+1} - u\|^2 + M_3\sum_{n=0}^{N}\beta_n + M_4\sum_{n=0}^{N}\gamma_n\\ &\leq \|x_0 - u\|^2 + M_3\sum_{n=0}^{\infty}\beta_n + M_4\sum_{n=0}^{\infty}\gamma_n.\end{aligned}$$

Accordingly, (C2) and (C3) guarantee that

$$\alpha^2\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)\|d_{n+1}\|^2 < \infty.$$

Hence, (C1) means that

$$\liminf_{n\to\infty}\|d_{n+1}\| = 0. \tag{2.2}$$

From the definition of d_{n+1} $(n \in \mathbb{N})$, we have that, for all $n \geq n_0$,

$$\frac{1}{\alpha}\|Tx_n - x_n\| \leq \|d_{n+1}\| + \beta_n\|d_n\| + \gamma_n\|\omega_n\| \leq \|d_{n+1}\| + M_2(\beta_n + \gamma_n),$$

which, together with (2.2) and $\lim_{n\to\infty}(\beta_n + \gamma_n) = 0$ implies, that

$$\begin{aligned}\frac{1}{\alpha}\liminf_{n\to\infty}\|Tx_n - x_n\| &\leq \liminf_{n\to\infty}\{\|d_{n+1}\| + M_2(\beta_n + \gamma_n)\}\\ &= \liminf_{n\to\infty}\|d_{n+1}\| + M_2\lim_{n\to\infty}(\beta_n + \gamma_n)\\ &= 0.\end{aligned}$$

Thus, we find that

$$\liminf_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{2.3}$$

From (2.1), the nonexpansivity of T, and the triangle inequality, we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &= \|Tx_{n+1} - \alpha_n x_n - (1-\alpha_n)\{Tx_n + \alpha(\beta_n d_n + \gamma_n\omega_n)\}\|\\ &\leq \alpha_n\|Tx_{n+1} - x_n\| + (1-\alpha_n)\|x_{n+1} - x_n\|\\ &\quad + \alpha(1-\alpha_n)\beta_n\|d_n\| + \alpha(1-\alpha_n)\gamma_n\|\omega_n\|\end{aligned}$$

which, together with $\|d_n\| \leq M_2$, $\|\omega_n\| \leq M_2$ $(n \geq n_0)$, and the triangle inequality, implies that, for all $n \geq n_0$,

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &\leq \alpha_n\|Tx_{n+1} - x_n\| + (1-\alpha_n)\|x_{n+1} - x_n\|\\ &\quad + \alpha(1-\alpha_n)M_2(\beta_n + \gamma_n)\end{aligned}$$

$$\leq \alpha_n \|Tx_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ + \alpha (1 - \alpha_n) M_2 (\beta_n + \gamma_n) .$$

Hence, we find that, for all $n \geq n_0$,

$$\begin{aligned} &(1 - \alpha_n) \|Tx_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \alpha (1 - \alpha_n) M_2 (\beta_n + \gamma_n) \\ &= \|\alpha_n x_n + (1 - \alpha_n) \{Tx_n + \alpha (\beta_n d_n + \gamma_n \omega_n)\} - x_n\| + \alpha (1 - \alpha_n) M_2 (\beta_n + \gamma_n) \\ &= (1 - \alpha_n) \|Tx_n - x_n + \alpha (\beta_n d_n + \gamma_n \omega_n) \| + \alpha (1 - \alpha_n) M_2 (\beta_n + \gamma_n) \\ &\leq (1 - \alpha_n) \|Tx_n - x_n\| + 2\alpha (1 - \alpha_n) M_2 (\beta_n + \gamma_n) , \end{aligned}$$

which means that, for all $n \geq n_0$,

$$\|Tx_{n+1} - x_{n+1}\| \leq \|Tx_n - x_n\| + 2\alpha M_2 (\beta_n + \gamma_n) . \tag{2.4}$$

Therefore, (C2) and (C3) guarantee the existence of $\lim_{n\to\infty} \|Tx_n - x_n\|$. Equation (2.3) leads us to

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \liminf_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{2.5}$$

(ii) Since $\{x_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ which weakly converges to $z \in H$. Assume that $z \notin \mathrm{F}(T)$, i.e., $z \neq Tz$. Then Opial's condition, (2.5), and the nonexpansivity of T ensure that

$$\begin{aligned} \liminf_{i\to\infty} \|x_{n_i} - z\| &< \liminf_{i\to\infty} \|x_{n_i} - Tz\| \\ &= \liminf_{i\to\infty} \|x_{n_i} - Tx_{n_i} + Tx_{n_i} - Tz\| \\ &= \liminf_{i\to\infty} \|Tx_{n_i} - Tz\| \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - z\|. \end{aligned}$$

This is a contradiction. Hence, $z \in \mathrm{F}(T)$. This completes the proof. □

Now, we are in the position to prove Theorem 2.2.

Proof. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$. The boundedness of $\{x_n\}$ implies that $\{x_{n_j}\}$ weakly converges to $w \in \mathrm{F}(T)$. A similar discussion as in the proof of Lemma 2.4 (ii) leads us to $w \in \mathrm{F}(T)$. Assume that $z \neq w$. Then, Lemma 2.3 (ii) and Opial's condition mean that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - z\| = \lim_{i\to\infty} \|x_{n_i} - z\| &< \lim_{i\to\infty} \|x_{n_i} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\| = \lim_{j\to\infty} \|x_{n_j} - w\| \\ &< \lim_{j\to\infty} \|x_{n_j} - z\| = \lim_{n\to\infty} \|x_n - z\|. \end{aligned}$$

This is a contradiction. Hence, $z = w$. This guarantees that $\{x_n\}$ weakly converges to a fixed point of T. This completes the proof.

□

3. Numerical Examples

We applied Algorithm 2.1 and the existing algorithms, (1.1) and (1.4), to the following problem [3, 9, 11, 17].

Problem 3.1. Suppose that $C_0 \subset \mathbb{R}^N$ is a nonempty, bounded, closed convex set and $C_i \subset \mathbb{R}^N$ $(i = 1, 2, \dots, m)$ is a nonempty, closed convex set and $\Phi(x)$ is the mean square value of the distances from $x \in \mathbb{R}^N$ to C_i $(i = 1, 2, \dots, m)$, i.e.,

$$\Phi(x) := \frac{1}{m} \sum_{i=1}^{m} d(x, C_i)^2 = \frac{1}{m} \sum_{i=1}^{m} \left(\min_{y \in C_i} \|x - y\| \right)^2 \quad (x \in \mathbb{R}^n).$$

Then,

$$\text{find } x^* \in C_\Phi := \left\{ x^* \in C_0 : \Phi(x^*) = \min_{y \in C_0} \Phi(y) \right\}.$$

The set C_Φ is called the generalized convex feasible set and is a subset of C_0 whose elements are the closest to C_is in the sense of the mean square norm. The set C_Φ is well-defined even if $\cap_{i=0}^m C_i = \emptyset$. This is because it is the set of all minimizers of Φ over C_0. The boundedness and closedness of C_0 guarantee $C_\Phi \neq \emptyset$. Moreover, the condition $C_\Phi = \cap_{i=0}^m C_i$ holds when $\cap_{i=0}^m C_i \neq \emptyset$, which means C_Φ is a generalization of $\cap_{i=0}^m C_i$.

Here, we define a mapping $T \colon \mathbb{R}^N \to \mathbb{R}^N$ by

$$T := P_0 \left(\frac{1}{m} \sum_{i=1}^{m} P_i \right), \tag{3.1}$$

where $P_i := P_{C_i}$ $(i = 0, 1, \dots, m)$ stands for the metric projection onto C_i. Accordingly, Proposition 4.2 in [17] ensures that T defined by (3.1) is nonexpansive and

$$\mathrm{F}(T) = C_\Phi.$$

Therefore, Problem 3.1 coincides with Problem 1.1 with T defined as in (3.1).

The experiment used an Apple Macbook Air with a 1.3GHz Intel Core i5 CPU and 4GB DDR3 memory. Algorithms (1.1), (1.4), and 2.1 were written in C and complied by using gcc version 4.2.1. The operating system of the computer was Mac OSX version 10.8.5. We set $\alpha := 1$, $\alpha_n := 1/2$ $(n \in \mathbb{N})$, $\beta_n := 1/(n+1)^{1.001}$ $(n \in \mathbb{N})$, $\gamma_n := \beta_n$ $(n \in \mathbb{N})$ and $W_n := Tx_n - x_n$ $(n \in \mathbb{N})$. In the experiment, we chose C_i $(i = 0, 1, 2, \dots, m)$ to be a closed ball with center $c_i \in \mathbb{R}^N$ and radius $r_i > 0$. Thus, P_i $(i = 0, 1, \dots, m)$ can be computed with

$$P_i(x) := x + \frac{\|c_i - x\| - r_i}{\|c_i - x\|}(c_i - x) \text{ if } \|c_i - x\| > r_i,$$

or $P_i(x) := x$ if $\|c_i - x\| \leq r_i$.

3.1. Case of $\cap_{i=0}^m C_i \neq \emptyset$.

We set $N := 10^7$, $m := 2$, $C_0 := \{x \in \mathbb{R}^N \colon \|x\| \leq 800\}$, $C_1 := \{x \in \mathbb{R}^N \colon \|x - (0.3, 0.3, \dots, 0.3)^\top\| \leq 800\}$, $C_2 := \{x \in \mathbb{R}^N \colon \|x - (-0.2, -0.2, \dots, -0.2)^\top\| \leq 800\}$, in order to consider the case of $\cap_{i=0}^2 C_i \neq \emptyset$. We used a nonexpansive mapping T defined as in (3.1).

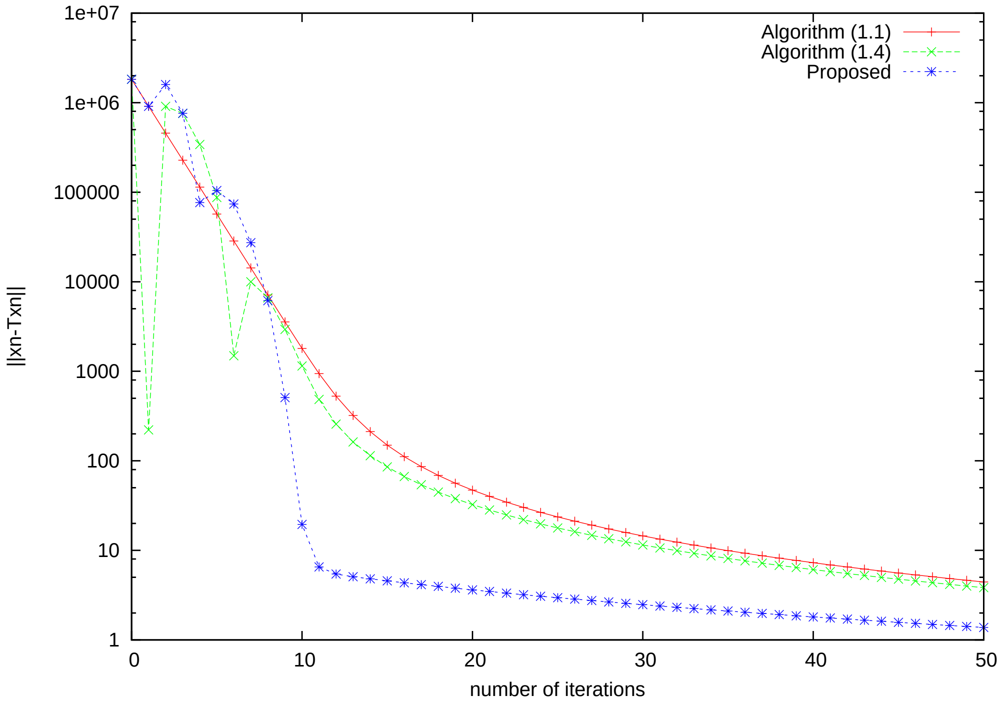

FIGURE 1. $\|x_n - Tx_n\|$ for Algorithms (1.1), (1.4), and 2.1, where $N := 10^7$ and $\cap_{i=0}^{2} C_i \neq \emptyset$

TABLE 1. $\|x_{50} - Tx_{50}\|$ for Algorithms (1.1), (1.4), and 2.1, where $N := 10^7$ and $\cap_{i=0}^{2} C_i \neq \emptyset$

Algorithm	$\|x_{50} - Tx_{50}\|$
Algorithm (1.1)	4.44171745
Algorithm (1.4)	3.83869077
Algorithm 2.1 (Proposed)	1.37881135

Figure 1 and Table 1 show $\|x_n - Tx_n\|$ for the three algorithms over the course of 50 iterations. We can see that $\|x_n - Tx_n\|$ of the proposed algorithm was shorter than those of the other algorithms. During the early iterations, the proposed algorithm converged faster than the other algorithms to a point in $\cap_{i=0}^{m} C_i$.

TABLE 2. Number of iterations to reach $\|x_n - Tx_n\| < 10^{-2}$

Algorithm	$\|x_n - Tx_n\|$	Iterations
Algorithm (1.1)	0.00998186	437
Algorithm (1.4)	0.00989580	432
Algorithm 2.1 (Proposed)	0.00996951	396

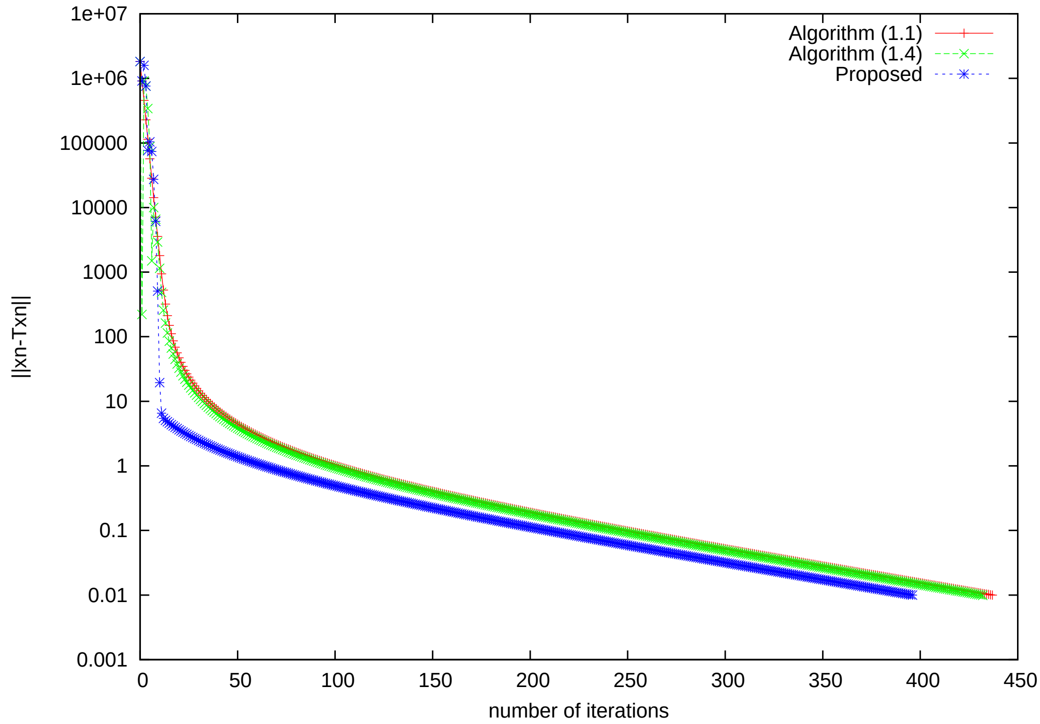

FIGURE 2. Number of iterations to reach $\|x_n - Tx_n\| < 10^{-2}$

Figure 2 and Table 2 show the number of iterations taken by the algorithms to satisfy $\|x_n - Tx_n\| < 10^{-2}$. The proposed algorithm took 396 iterations, the fewest number. Algorithm (1.1) and Algorithm (1.4) had almost the same results. Thus, we can see that the proposed algorithm was faster than the others.

TABLE 3. Number of seconds until $\|x_n - Tx_n\| < 10^{-2}$

Algorithm	$\|x_n - Tx_n\|$	Seconds
Algorithm (1.1)	0.0099819	425.162338
Algorithm (1.4)	0.0098958	419.604415
Algorithm 2.1 (Proposed)	0.0099695	384.775158

Figure 3 and Table 3 compare the times taken by the algorithms to satisfy $\|x_n - Tx_n\| < 10^{-2}$. It is clear that the proposed algorithm converged faster than the others to a point in $\cap_{i=0}^m C_i$.

We can conclude from the above that the proposed algorithm 2.1 performed better than Algorithm (1.1) or Algorithm (1.4).

3.2. **Case of $\cap_{i=0}^m C_i = \emptyset$.** We set $N := 10^7$, $m := 2$, $C_0 := \{x \in \mathbb{R}^N : \|x\| \leq 100\}$, $C_1 := \{x \in \mathbb{R}^N : \|x - (5.0, 5.0, \ldots, 5.0)^\top\| \leq 100\}$, $C_2 := \{x \in \mathbb{R}^N : \|x -$

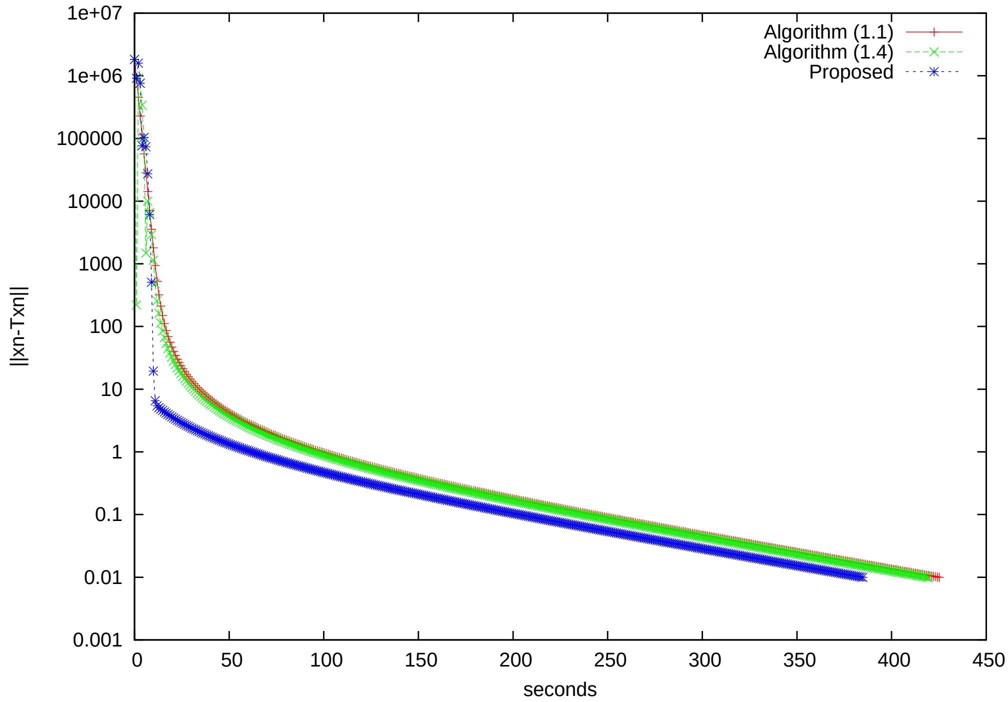

FIGURE 3. Number of seconds until $\|x_n - Tx_n\| < 10^{-2}$

TABLE 4. $\|x_{50} - Tx_{50}\|$ for Algorithms (1.1), (1.4), and 2.1, where $N := 10^7$ and $\cap_{i=0}^{2} C_i = \emptyset$

Algorithm	$\|x_{50} - Tx_{50}\|$
Algorithm (1.1)	0.000000001638652
Algorithm (1.4)	0.000000000009824
Algorithm 2.1 (Proposed)	0.000000000000899

$(-3.0, -3.0, \ldots, -3.0)^\top\| \leq 100\}$ in order to consider the case of $\cap_{i=0}^{2} C_i = \emptyset$. We used a nonexpansive mapping T defined as in (3.1).

Figure 4 and Table 4 show $\|x_n - Tx_n\|$ for the three algorithms over the course of 50 iterations. $\|x_n - Tx_n\|$ of the proposed algorithm was shorter than those of Algorithm (1.1) and Algorithm (1.4). During the early iterations, the proposed algorithm converged slightly faster than the other algorithms to a point in C_Φ.

Figure 5 and Table 5 show the number of iterations required by the algorithms to satisfy $\|x_n - Tx_n\| < 10^{-6}$. Although the proposed algorithm took the fewest iterations, 33, the results of the three algorithms were comparable.

Figure 6 and Table 6 show the time required by the algorithms to satisfy $\|x_n - Tx_n\| < 10^{-6}$. The proposed algorithm converged slightly faster than the others to a point in C_Φ.

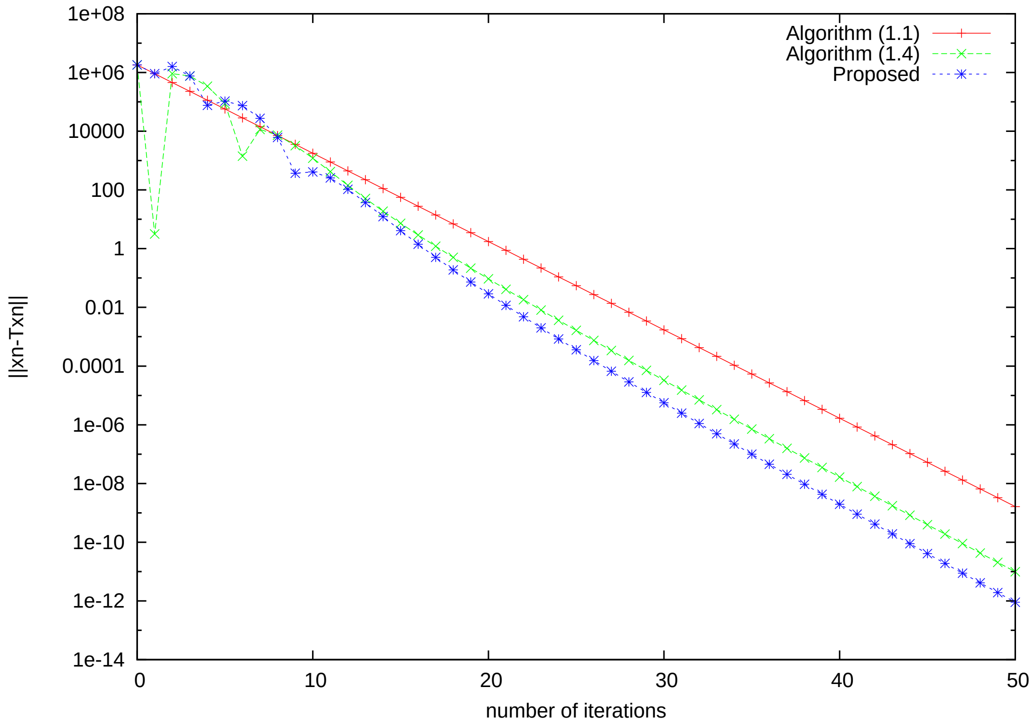

FIGURE 4. $\|x_n - Tx_n\|$ for Algorithms (1.1), (1.4), and 2.1, where $N := 10^7$ and $\cap_{i=0}^{2} C_i = \emptyset$

TABLE 5. Number of iterations taken to reach $\|x_n - Tx_n\| < 10^{-6}$

Algorithm	$\|x_n - Tx_n\|$	Iterations
Algorithm (1.1)	0.00000084	41
Algorithm (1.4)	0.00000072	35
Algorithm 2.1 (Proposed)	0.00000049	33

TABLE 6. Number of seconds until $\|x_n - Tx_n\| < 10^{-2}$

Algorithm	$\|x_n - Tx_n\|$	Seconds
Algorithm (1.1)	0.0000008	41.425032
Algorithm (1.4)	0.0000007	35.325500
Algorithm 2.1 (Proposed)	0.0000005	33.539761

We can conclude from these experiments that the proposed algorithm 2.1 outperformed Algorithm (1.1) and Algorithm (1.4).

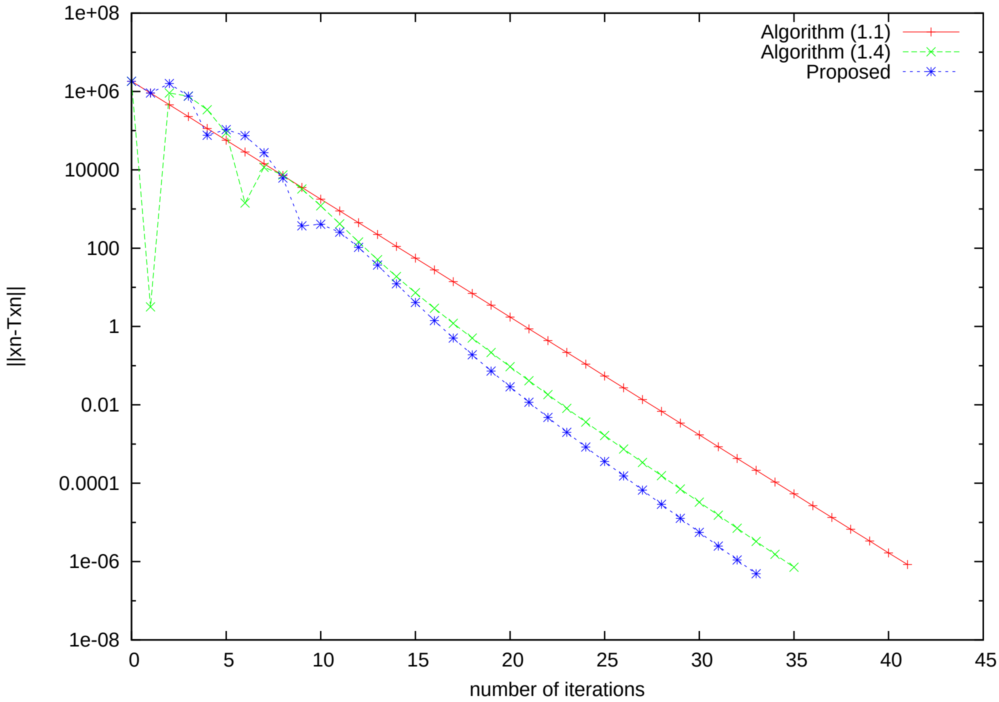

FIGURE 5. Number of iterations to reach $\|x_n - Tx_n\| < 10^{-6}$

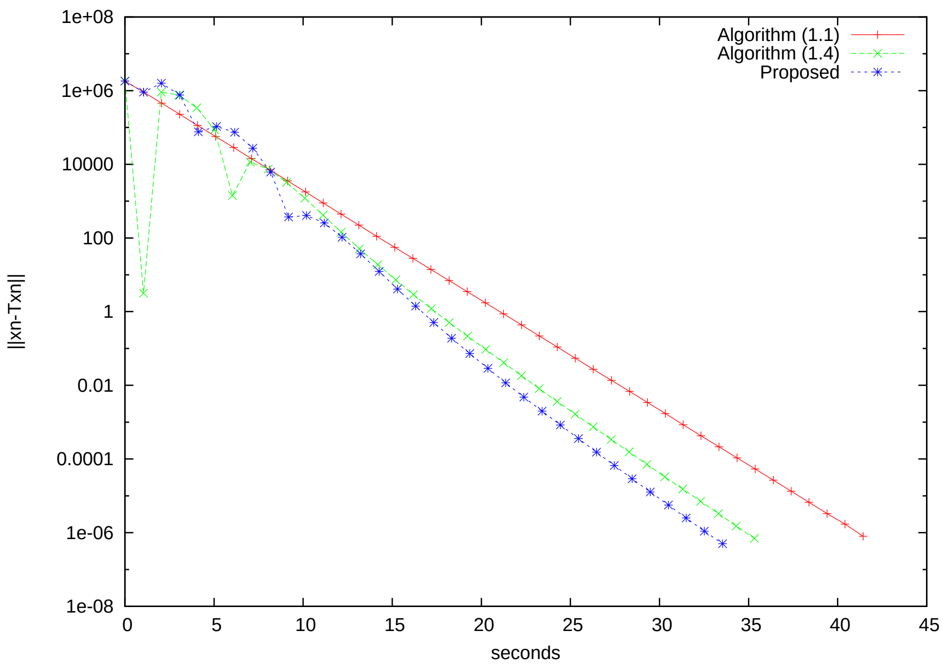

FIGURE 6. Number of seconds until $\|x_n - Tx_n\| < 10^{-6}$

4. Conclusion

This paper presented an algorithm to accelerate the Krasnosel'skiĭ-Mann algorithm for finding a fixed point of a nonexpansive mapping on a real Hilbert space and its convergence analysis. This convergence analysis guarantees that the proposed algorithm weakly converges to a fixed point of a nonexpansive mapping under certain assumptions. We also showed that the proposed algorithm outperformed the existing algorithms in experiments.

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