Linear and Nonlinear Analysis
Volume 3, Number 2, 2017, 251–260

Yokohama Publishers
ISSN 2188-8167

GENERALIZED KKM THEOREMS IN HADAMARD MANIFOLDS

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Abstract. The purpose of this article is to present a fixed point theorem for generalized KKM mappings in the Hadamard manifold settings. We derive the finite intersection property of this class of mappings. Besides, as an application of this property, we discuss the existence conditions on the generalized equilibrium problem and prove a manifold version of Fan minimax inequality.

1. Introduction

The celebrated KKM lemma was published by three Polish mathematicians, Knaster, Kuratowski and Mazurkiewicz, in 1929 [13]. This result concerns with coverings of a simplex by closed sets and is equivalent to the Brouwer fixed point theorem. The points of the intersection in the KKM lemma are of great interest primarily because of numerous applications in various fields of pure and applied mathematics. In 1961, Fan [6] extended the KKM lemma to topological vector spaces and gave some applications in the fixed point theory, minimax theory and game theory. Recently, further extensions have been developed with a variety of applications in topological vector spaces; see, e.g., [1–3, 17, 18] and the references therein.

It is worth noting that in the last decade many researchers have focused on extending some concepts and techniques from the topological vector space context to the geodesic metric space and Hadamard manifold settings (with no vector space structure). Several results in this nonlinear framework concerning the existence theorems for the fixed point problems, variational inequalities, equilibrium problems have been studied, see, e.g., [4, 7–11, 14, 15] and the references therein.

The purpose of this article is to present a fixed point theorem for the more general type of KKM mappings (see Definition 2.6) in the setting of Hadamard manifolds. To this end, we first recall and summarize some basic concepts and fundamental theorems in Riemannian Geometry; see Section 2. We also introduce the general class of generalized KKM mappings in our framework and derive the finite intersection property of this class of mappings. In Section 3, we use the

2010 *Mathematics Subject Classification.* Primary 58C30, 47H10, 90C33, 49K35.

Key words and phrases. Hadamard manifold; KKM mapping; generalized KKM mapping; generalized equilibrium problem; minimax inequality.

This research was supported by a grant MOST 105-2115-M-259-005 from the Ministry of Science and Technology of Taiwan.

results established in Section 2 not only to discuss the existence conditions on the generalized equilibrium problem (see (3.1)) but to prove a manifold version of Fan minimax inequality.

2. Preliminaries

Let M be a differentiable manifold with finite dimension n, T_xM the tangent space of M at x (T_xM is a linear space isomorphic to $\mathbb{R}^n$) and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M. When M is endowed with a Riemannian metric g and the corresponding norm denoted by $\|\cdot\|$, M is a Riemannian manifold. The inner product of two vectors $u, v \in T_xM$ is written as $\langle u, v\rangle_x = g_x(u, v)$, where g_x is the metric at a point x. The *norm* of a vector $v \in T_xM$ is set by $\|v\|_x = \sqrt{\langle v, v\rangle_x}$. If there is no confusion, we denote $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_x$ and $\|\cdot\| = \|\cdot\|_x$. The metrics can be used to define the length of a piecewise smooth curve $\gamma : [a, b] \to M$ joining $\gamma(a) = x$ to $\gamma(b) = y$ through

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt.$$

Minimizing this length functional over the set of all such curves, we obtain a Riemannian distance $d(x, y)$ which induces the original topology on M; see [5, Proposition 2.6, p.146].

Let ∇ be a Levi-Civita connection associated to $(M, \langle\cdot,\cdot\rangle)$. This connection defines a unique covariant derivative D/dt, where for each vector field V, along a smooth curve $\gamma : [a, b] \to M$, another vector field DV/dt is obtained, called the covariant derivative of V along γ. A curve $\gamma : [a, b] \to M$ is called a *geodesic* if $D\gamma'/dt = 0$ and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, then γ is said to be *normalized*. Note that for any $x \in M$ and any $v \in T_xM$, there exists a unique geodesic γ, starting at x with velocity v, defined on an interval. A geodesic joining x and y in M is said to be a *minimizing geodesic* (or a *minimal geodesic*) if its length is equal to $d(x, y)$. The Hopf-Rinow theorem [5, Theorem 2.8, p.146] asserts that if M is complete, then any pair of points in M can be joined by a minimal geodesic, and every bounded closed subset of M is compact. A subset $K \subset M$ is said to be convex if for any two points $x, y \in K$, the geodesic joining x to y is contained in K.

Let M be a complete Riemannian manifold and $x \in M$. The exponential map $\exp_x : T_xM \to M$ is defined as $\exp_x v = \gamma_v(1)$, where $\gamma_v : \mathbb{R} \to M$ is the geodesic starting at x with velocity v, i.e., $\gamma(0) = x$ and $\gamma'(0) = v$. Then for any $t \in \mathbb{R}$, we have $\exp_x tv = \gamma_v(t)$.

A Hadamard manifold is a complete simply connected Riemannian manifold with nonpositive sectional curvature. The following well-known result, as an application of Hopf-Rinow Theorem, can be found in [5, Theorem 3.1, p.149].

Theorem 2.1. *Let M be a Hadamard manifold. Then M is diffeomorphic to the Euclidian space $\mathbb{R}^n$, where $n = \dim M$; more precisely, $\exp_x : T_xM \to M$ is a diffeomorphism at any point $x \in M$. Moreover, for any two points $x, y \in M$ there exists a unique normalized geodesic joining x to y, which is a minimal geodesic.*

To discuss a special class of generalized KKM mappings, we need the following definition.

Definition 2.2. (Geodesic convex hull) Let X be a uniquely geodesic space and let $A \subset X$. Then, the (*geodesic*) *convex hull* of A, denoted co$A$, is the intersection of all geodesically convex subsets of X that contain A.

If X is a geodesically convex space, the geodesic convex hull of any nonempty subset A of X exists and is nonempty. Moreover, as an intersection of geodesically convex subsets, the geodesic convex hull is geodesically convex.

The following proposition gives a step-by-step construction of the geodesic convex hull; see [12, Lemma 3.3.1] and [16, Proposition 2.5.5].

Proposition 2.3. *Let X be a uniquely geodesic space and let $A \subset X$. We set $C_0 = A$ and for every integer $n \geq 0$, we let C_{n+1} be the union of all geodesics in X that join pairs of points in C_n. Then the geodesic convex hull of A is given by*

$$\text{co}A = \bigcup_{n\geq 0} C_n.$$

Let X and Y be topological spaces and let $S : X \to 2^Y$ be a multifunction. We denote $\langle X\rangle$ the class of all nonempty finite subsets of X. The closure of a subset E of X is denoted by $\overline{E}$. For $A \subset X$, the image of A under S is the set $S(A) = \bigcup\{Sx : x \in A\}$.

For any integer $n \geq 0$, we denote Δ_n the standard n-simplex of $\mathbb{R}^{n+1}$, that is,

$$\Delta_n = \left\{\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1} : \alpha_i \geq 0, \sum_{i=0}^{n} \alpha_i = 1\right\},$$

where $\{e_0, \dots, e_n\}$ is the set of the vertices of Δ_n.

The following KKM Lemma plays a significant role in fixed point theory; see [13].

Lemma 2.4. (KKM Lemma) *Suppose that $\{F_0, \dots, F_m\}$ are closed subsets of the n-simplex Δ_n in $\mathbb{R}^{n+1}$ such that $\text{co}\{e_i : i \in I\} \subset \bigcup_{i\in I} F_i$ for all $I \subset \{0, \dots, m\}$. Then $\bigcap_{i=0}^m F_i \neq \emptyset$.*

Let M be a Hadamard manifold and let $X \subset M$ be a convex set. A function $f : X \to \mathbb{R}$ is convex if for every geodesic $\gamma : [0, 1] \to X$ joining arbitrary two point $x, y \in X$, we have

$$f(\gamma(t)) \leq (1 - t)f(\gamma(0)) + tf(\gamma(1)), \quad \forall t \in [0, 1],$$

where $\gamma(0) = x$ and $\gamma(1) = y$. A function $f : X \to \mathbb{R}$ is concave if $-f$ is convex. An important implication of convexity is that if $f : X \to \mathbb{R}$ is convex, then the set $\{x \in X : f(x) \leq \alpha\}$ is convex, for any $\alpha \in \mathbb{R}$, that is, f is quasiconvex; see [19]. Similarly, if $f : X \to \mathbb{R}$ is concave, then the set $\{x \in X : f(x) \geq \alpha\}$ is convex, for any $\alpha \in \mathbb{R}$, that is, f is quasiconcave. It is well known that $f : X \to \mathbb{R}$ is quasiconvex if and only if for every geodesic $\gamma : [0, 1] \to X$ joining arbitrary two point $x, y \in X$, we have

$$f(\gamma(t)) \leq \max\{f(x), f(y)\}, \quad \forall t \in [0, 1];$$

see [20].

Proposition 2.5. *Let X be a convex subset of a Hadamard manifold, $f : X \to \mathbb{R}$ a function. Then $f : X \to \mathbb{R}$ is quasiconvex (resp., quasiconcave) if and only if for any $\{x_1, \dots, x_n\} \in \langle X \rangle$ we have*

$$f(x) \le \max_{1\le i\le n} f(x_i), \quad \forall x \in \mathrm{co}\{x_1, \dots, x_n\}, \tag{2.1}$$
$$(\textit{resp.}, \ f(x) \ge \min_{1\le i\le n} f(x_i), \quad \forall x \in \mathrm{co}\{x_1, \dots, x_n\}).$$

Proof. It suffices to show the only if part for the quasiconvexity case. Suppose that f is quasiconvex. For any $A = \{x_1, \dots, x_n\} \in \langle X \rangle$, it follows from Proposition 2.3 that

$$\mathrm{co}A = \bigcup_{k\ge 0} C_k,$$

where $C_0 = A$ and for every integer $k \ge 0$, C_{k+1} is the union of all geodesics in X that join pairs of points in C_k. We shall complete the proof of (2.1) for all $x \in C_k$ by induction on k. If $x \in C_1$, then x lies on some geodesic joining two points in A, say, x_{j_1} and x_{j_2}. Then

$$f(x) \le \max\{f(x_{j_1}), f(x_{j_2})\} \le \max_{1\le i\le n} f(x_i).$$

Proceeding inductively, suppose that the inequality (2.1) holds for $k = m$, that is,

$$f(x) \le \max_{1\le i\le n} f(x_i), \quad \forall x \in C_m.$$

Let $x \in C_{m+1}$. Since x lies on a geodesic joining two points y and z in C_m, it follows from the induction hypothesis and quasiconvexity of f that

$$f(x) \le \max\{f(y), f(z)\} \le \max_{1\le i\le n} f(x_i),$$

as required. □

Suppose that X and Y are convex subsets of a Hadamard manifold. A multifunction $T : X \to 2^Y$ is a KKM mapping if it satisfies $\mathrm{co}A \subset T(A)$, for any $A \in \langle X \rangle$. A more general class of generalized KKM mappings was first introduced by Chang and Zhang [3]. We now propose the analogous concept of this class of mappings in manifolds.

Definition 2.6. Let X be a nonempty set and Y a convex subset of a Hadamard manifold. A multifunction $T : X \to 2^Y$ is a *generalized KKM mapping* provided for any $\{x_1, \dots, x_n\} \in \langle X \rangle$, there is $\{y_1, \dots, y_n\} \in \langle Y \rangle$ such that

$$\mathrm{co}\{y_i : i \in I\} \subset \bigcup_{i\in I} T(x_i), \quad \forall I \subset \{1, \dots, n\}.$$

It is clear that a KKM mapping is a generalized KKM mapping, but the converse is not true, as is illustrated by the following example.

Example 2.7. Define a multifunction $T : [0, 1] \to 2^{[0,1]}$ by

$$T(x) = \begin{cases} [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{1}{4}, \frac{3}{4}] & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

To prove that T is a generalized KKM mapping, given any $\{x_1, \ldots, x_n\} \in \langle [0,1] \rangle$ choose $y_i = \frac{1}{2}x_i$, $i = 1, \ldots, n$. For any $I \subset \{1, \ldots, n\}$, let $A = \{x_i : i \in I\}$. There are three cases. First, if $A \subset [0, \frac{1}{2})$, then

$$\mathrm{co}\{y_i : i \in I\} \subset [0, \tfrac{1}{4}] \subset [0, \tfrac{1}{4}] \cup [\tfrac{3}{4}, 1] = T(A).$$

Second, if $A \subset [\frac{1}{2}, 1]$, then $T(A) = [\frac{1}{4}, \frac{3}{4}]$ and so $\mathrm{co}\{y_i : i \in I\} \subset [\frac{1}{4}, \frac{1}{2}] \subset T(A)$. Finally, if $A \cap [0, \frac{1}{2}) \neq \emptyset$ and $A \cap [\frac{1}{2}, 1] \neq \emptyset$, then $T(A) = [0, 1]$ and thus $\mathrm{co}\{y_i : i \in I\} \subset T(A)$. However, T is not a KKM mapping because we have $\mathrm{co}\{\frac{3}{8}, \frac{7}{16}\} \cap [T(\frac{3}{8}) \cup T(\frac{7}{16})] = \emptyset$.

We discuss the finite intersection property of generalized KKM mappings in the next result.

Lemma 2.8. *Let X be a nonempty set and Y a closed convex subset of a Hadamard manifold. If $T : X \to 2^Y$ is a generalized KKM mapping, then the collection $\{\overline{T(x)} : x \in X\}$ has the finite intersection property.*

Proof. Let $\{x_0, \ldots, x_n\} \in \langle X \rangle$ so that there is $\{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$\mathrm{co}\{y_i : i \in I\} \subset \bigcup_{i \in I} T(x_i), \quad \forall I \subset \{0, \ldots, n\}. \tag{2.2}$$

We shall construct inductively a compact subset $K(y_0, \ldots, y_n)$ of $\mathrm{co}\{y_0, \ldots, y_n\}$. First, take $K_0 = \{y_0\}$ and then, let $\gamma_1 : K_0 \times [0,1] \to Y$ be a geodesic joining from y_1 to y_0 defined by

$$\gamma_1(y_0, t) = \exp_{y_1} t \exp_{y_1}^{-1} y_0, \quad t \in [0, 1].$$

Then the image $K_1 = \gamma_1(K_0 \times [0, 1])$ of this geodesic is a compact set and $K_1 = \mathrm{co}\{y_0, y_1\}$. Now, consider a continuous mapping $\gamma_2 : K_1 \times [0,1] \to Y$ defined by

$$\gamma_2(z_1, t) = \exp_{y_2} t \exp_{y_2}^{-1} z_1, \quad z_1 \in K_1, t \in [0, 1].$$

Then its image $K_2 = \gamma_2(K_1 \times [0, 1])$, which is the set of the geodesics connecting from y_2 to points of K_1, is compact and $K_2 \subset \mathrm{co}\{y_0, y_1, y_2\}$. We proceed this process inductively to get a continuous mapping $\gamma_i : K_{i-1} \times [0,1] \to Y$, $3 \leq i \leq n$, defined by

$$\gamma_i(z_{i-1}, t) = \exp_{y_i} t \exp_{y_i}^{-1} z_{i-1}, \quad z_{i-1} \in K_{i-1}, t \in [0, 1],$$

so that its image $K_i = \gamma_i(K_{i-1} \times [0, 1])$ (which is the set of geodesics joining from y_i to points of K_{i-1}) is compact and

$$K_i \subset \mathrm{co}\{y_0, \ldots, y_i\}.$$

Consequently, the finite union of compact sets

$$K(y_0, \ldots, y_n) = \bigcup_{i=0}^{n} K_i$$

is a compact subset of $\mathrm{co}\{y_0, \ldots, y_n\}$ and hence closed.

To apply KKM Lemma 2.4, we need to define a continuous mapping $\varphi : \Delta_n \to K(y_0, \ldots, y_n)$ by induction from the i-simplex Δ_i with vertices $(e_0, \ldots, e_i)$ onto K_i

such that $\varphi(e_j) = y_j$, $j = 0, \dots, i$, as follows. First, if $u_1 \in \Delta_1$, we can write $u_1 = \alpha_0 e_0 + (1 - \alpha_0)e_1$ with $\alpha_0 \in [0, 1]$ and let

$$\varphi(u_1) = \gamma_1(y_0, \alpha_0).$$

Then $\varphi(e_i) = y_i$, $i = 0, 1$, and φ maps Δ_1 onto K_1, i.e., $\varphi(\Delta_1) = K_1$. Next, if $u_2 \in \Delta_2 \setminus \Delta_1$, then $u_2 = \alpha_1 u_1 + (1 - \alpha_1)e_2$ for some $u_1 \in \Delta_1$ and $\alpha_1 \in [0, 1]$. Let

$$\varphi(u_2) = \gamma_2(\varphi(u_1), \alpha_1)$$

so that $\varphi(e_2) = y_2$ and $\varphi(\Delta_2) = K_2$. Inductively, we construct a continuous mapping $\varphi : \Delta_i \to K_i$, for each $i = 3, \dots, n$, by

$$\varphi(u_i) = \gamma_i(\varphi(u_{i-1}), \alpha_{i-1}),$$

where $u_i = \alpha_{i-1}u_{i-1} + (1 - \alpha_{i-1})e_i$, for some $u_{i-1} \in \Delta_{i-1}$ and $\alpha_{i-1} \in [0, 1]$. Moreover, $\varphi(e_i) = y_i$ and $\varphi(\Delta_i) = K_i$. This establishes the continuity of the mapping $\varphi : \Delta_n \to K(y_0, \dots, y_n)$ whose image is $\varphi(\Delta_n) = K(y_0, \dots, y_n)$.

Let $E_i = \overline{T(x_i)} \cap K(y_0, \dots, y_n)$, $i = 0, \dots, n$, so that E_i is a closed subset of $K(y_0, \dots, y_n)$. For any $I = \{i_1, \dots, i_m\} \subset \{0, \dots, n\}$, let $u \in \mathrm{co}\{e_i : i \in I\}$ and write $u = \sum_{j=1}^m \alpha_{i_j} e_{i_j}$, where $\alpha_{i_j} \in [0, 1]$ and $\sum_{j=1}^m \alpha_{i_j} = 1$. By our construction we then have

$$\varphi(u) \in K(y_{i_1}, \dots, y_{i_m}) \subset \mathrm{co}\{y_{i_1}, \dots, y_{i_m}\},$$

which, together with (2.2), yields

$$\varphi(\mathrm{co}\{e_i : i \in I\}) \subset \mathrm{co}\{y_i : i \in I\} \cap K(y_0, \dots, y_n) \subset \bigcup_{i \in I} E_i.$$

Therefore $\mathrm{co}\{e_i : i \in I\} \subset \bigcup_{i \in I} \varphi^{-1}(E_i)$. According to KKM Lemma 2.4, we obtain $\bigcap_{i=0}^n \varphi^{-1}(E_i) \neq \emptyset$; hence

$$\varphi^{-1}\left(\bigcap_{i=0}^n \overline{T(x_i)}\right) = \bigcap_{i=0}^n \varphi^{-1}\left(\overline{T(x_i)}\right) \neq \emptyset.$$

We conclude that $\bigcap_{i=0}^n \overline{T(x_i)} \neq \emptyset$. □

Theorem 2.9. *Let X be a nonempty set, Y a convex subset of a Hadamard manifold M and let $T : X \to 2^Y$ be a closed-valued multifunction such that $T(x_0)$ is bounded for some x_0. Then T is a generalized KKM mapping if and only if $\bigcap_{x \in X} T(x) \neq \emptyset$.*

Proof. Suppose that T is a generalized KKM mapping. Then Lemma 2.8 guarantees that the collection $\{T(x) : x \in X\}$ of closed sets has the finite intersection property. The Hopf-Rinow Theorem asserts that a Hadamard manifold has the Heine-Borel property; hence the closed and bounded subset $T(x_0)$ of M is compact. It follows that $\bigcap_{x \in X} T(x) \neq \emptyset$. Conversely, if $\bigcap_{x \in X} T(x) \neq \emptyset$, then the collection $\{T(x) : x \in X\}$ of closed sets has the finite intersection property. Hence for any $\{x_1, \dots, x_n\} \in \langle X \rangle$, there is $y \in \bigcap_{i=1}^n T(x_i)$. Choose $y_i = y$ for $i = 1, \dots, n$. We get

$$\mathrm{co}\{y_i : i \in I\} = \mathrm{co}\{y\} \subset \bigcup_{i \in I} T(x_i), \quad \forall I \subset \{1, \dots, n\};$$

therefore T is a generalized KKM mapping. □

Theorem 2.9 immediately implies the following; cf. also [4, Lemma 3.1].

Corollary 2.10. *Let X be a convex subset of a Hadamard manifold and let $T : X \to 2^X$ be a closed-valued multifunction such that $T(x_0)$ is bounded for some x_0. If T is a KKM mapping, then $\bigcap_{x\in X} T(x) \neq \emptyset$.*

3. Main Theorems

In this section, we shall use the results established in Section 2 to consider a more general equilibrium problem and Ky Fan's type minimax inequality on a Hadamard manifold.

Let X be a convex subset of a Hadamard manifold, Y a nonempty set and $F, \psi : X \times Y \to \mathbb{R}$ two real-valued bifunctions. The generalized equilibrium problem (for short, $(\mathrm{EP})_\psi$) is formulated as follows:

$$\text{find } x_0 \in X \text{ such that } F(x_0, y) + \psi(x_0, y) \geq 0, \quad \forall y \in Y. \tag{3.1}$$

In particular, when $X = Y$, the $(\mathrm{EP})_\psi$ problem is reduced to the classical equilibrium problem provided $\psi \equiv 0$; it is the mixed equilibrium problem if $\psi(x, y) = \varphi(x) - \varphi(y)$, for some function $\varphi : X \to \mathbb{R}$.

We recall that an extended real-valued function $f : X \to [-\infty, \infty]$ on a topological space X is lower semicontinuous (l.s.c.) if $\{x \in X : f(x) > c\}$ is open for each $c \in \mathbb{R}$; it is upper semicontinuous (u.s.c.) if $\{x \in X : f(x) < c\}$ is open for each $c \in \mathbb{R}$.

The next result describes the existence of solutions to the $(\mathrm{EP})_\psi$ problem.

Theorem 3.1. *Let X be a convex subset of a Hadamard manifold and Y a nonempty set. Suppose that $F, \psi : X \times Y \to \mathbb{R}$ and $\varphi : X \times X \to \mathbb{R}$ are real-valued bifunctions satisfying the following conditions:*

(i) *$\varphi(x, x) \geq 0$, $\forall x \in X$;*
(ii) *for each $y \in Y$, there is $z \in X$ such that $F(x, y) + \psi(x, y) \geq \varphi(x, z)$, $\forall x \in X$;*
(iii) *for each $x \in X$, the mapping $\varphi(x, \cdot)$ is l.s.c. and quasiconvex;*
(iv) *there exists a bounded subset K of X and a point $(\bar{x}, \bar{y}) \in X \times Y$ such that*

$$F(\bar{x}, \bar{y}) + \psi(\bar{x}, \bar{y}) < \varphi(\bar{x}, z), \quad \forall z \in X \setminus K.$$

Then there exists $x_0 \in X$ such that

$$F(x_0, y) + \psi(x_0, y) \geq 0, \quad \forall y \in X.$$

Proof. Define $T : Y \to 2^X$ by

$$T(y) = \{z \in X : F(x, y) + \psi(x, y) \geq \varphi(x, z),\ \forall x \in X\}.$$

Fix any $(x, y) \in X \times Y$. Let

$$A(x, y) = \{z \in X : F(x, y) + \psi(x, y) \geq \varphi(x, z)\}$$

so that Condition (iii) asserts that $A(x, y)$ is a closed set. Hence $T(y) = \bigcap_{x\in X} A(x, y)$ is closed for each $y \in Y$.

To prove that T is a generalized KKM mapping, let $\{y_1, \ldots, y_n\} \in \langle Y \rangle$. According to Condition (ii), we can choose $\{z_1, \ldots, z_n\} \in \langle X \rangle$ such that

$$F(x, y_i) + \psi(x, y_i) \geq \varphi(x, z_i), \quad \forall x \in X,\ \ i = 1, \ldots, n. \tag{3.2}$$

Let $x \in X$ and $I \subset \{1, \ldots, n\}$. Since the mapping $z \mapsto \varphi(x, z)$ is quasiconvex, Proposition 2.5 states that

$$\varphi(x, z) \leq \max_{i \in I} \varphi(x, z_i), \quad \forall z \in \operatorname{co}\{z_i : i \in I\}.$$

This means that for any $I \subset \{1, \ldots, n\}$ and $z \in \operatorname{co}\{z_i : i \in I\}$, there is $i_k \in I$ such that $\varphi(x, z) \leq \varphi(x, z_{i_k})$, so combined with (3.2), we obtain that

$$F(x, y_{i_k}) + \psi(x, y_{i_k}) \geq \varphi(x, z);$$

therefore $z \in T(y_{i_k})$. This shows that T is a generalized KKM mapping. In particular, it follows from Condition (iv) that $T(\bar{y})$ is a subset of K and so is bounded. By Theorem 2.9 there exists a point $x_0 \in \bigcap_{y \in Y} T(y)$. Consequently, we conclude that

$$F(x_0, y) + \psi(x_0, y) \geq \varphi(x_0, x_0) \geq 0, \quad \forall y \in X,$$

and the proof is finished. □

The following result is an immediate consequence of Theorem 3.1 because a convex function must be quasiconvex.

Corollary 3.2. *Let X be a compact convex subset of a Hadamard manifold and Y a nonempty set. Suppose that $F, \psi : X \times Y \to \mathbb{R}$ and $\varphi : X \times X \to \mathbb{R}$ are real-valued bifunctions satisfying the following conditions:*

(i) $\varphi(x, x) \geq 0$, $\forall x \in X$;
(ii) *for each $y \in Y$, there is $z \in X$ such that $F(x, y) + \psi(x, y) \geq \varphi(x, z)$, $\forall x \in X$;*
(iii) *for each $x \in X$, the mapping $\varphi(x, \cdot)$ is l.s.c. and convex.*

Then there exists $x_0 \in X$ such that

$$F(x_0, y) + \psi(x_0, y) \geq 0, \quad \forall y \in X.$$

We shall consider two functions to derive a two-function theorem whose proof is analogous to that of Theorem 3.1.

Theorem 3.3. *Let X be a convex subset of a Hadamard manifold, Y a nonempty set and $\lambda \in \mathbb{R}$. Suppose that $f : X \times Y \to \mathbb{R}$ and $\varphi : X \times X \to \mathbb{R}$ are real-valued bifunctions satisfying the following conditions:*

(i) $\varphi(x, x) \leq \lambda$, $\forall x \in X$;
(ii) *for each $y \in Y$, there is $z \in X$ such that $f(x, y) \leq \varphi(x, z)$, $\forall x \in X$;*
(iii) *for each $x \in X$, the mapping $\varphi(x, \cdot)$ is u.s.c. and quasiconcave;*
(iv) *there exists a bounded subset K of X and a point $(\bar{x}, \bar{y}) \in X \times Y$ such that*

$$f(\bar{x}, \bar{y}) > \varphi(\bar{x}, z), \quad \forall z \in X \setminus K.$$

Then there exists $x_0 \in X$ such that

$$\sup_{y \in Y} f(x_0, y) \leq \lambda.$$

Proof. Define a multifunction $T : Y \to 2^X$ by

$$T(y) = \{z \in X : f(x, y) \leq \varphi(x, z), \ \forall x \in X\}.$$

We can modify the argument of Theorem 3.1, (setting $\psi \equiv 0$, using $-f$ and $-\varphi$ in place of F and φ, respectively), to prove that T is a generalized KKM mapping with closed values and $T(\bar{y})$ is a bounded set, and so is compact. According to Theorem 2.9, we have some point $x_0 \in \bigcap_{y\in Y} T(y)$, which implies that

$$f(x_0, y) \le \varphi(x_0, x_0) \le \lambda, \quad \forall y \in X.$$

In other words, $\sup_{y\in Y} f(x_0, y) \le \lambda$, as desired. □

The following Fan minimax inequality in a Hadamard manifold version is simply a consequence of Theorem 3.3 with $\lambda = \sup_{x\in X} \varphi(x, x)$.

Corollary 3.4. *Let X be a convex subset of a Hadamard manifold and Y a nonempty set. Suppose that $f : X \times Y \to \mathbb{R}$ and $\varphi : X \times X \to \mathbb{R}$ are real-valued bifunctions satisfying the following conditions:*

(i) *for each $y \in Y$, there is $z \in X$ such that $f(x, y) \le \varphi(x, z)$, $\forall x \in X$;*
(ii) *for each $x \in X$, the mapping $\varphi(x, \cdot)$ is u.s.c. and quasiconcave;*
(iii) *there exists a bounded subset K of X and a point $(\bar{x}, \bar{y}) \in X \times X$ such that*

$$f(\bar{x}, \bar{y}) > \varphi(\bar{x}, z), \quad \forall z \in X \setminus K.$$

Then

$$\inf_{x\in X} \sup_{y\in Y} f(x, y) \le \sup_{x\in X} \varphi(x, x).$$

Acknowledgements

The author is very grateful to Professor Wataru Takahashi and the anonymous referee(s) for valuable remarks.

References

[1] M. Balaj, *A common fixed point theorem with application to vector equilibrium problems*, Appl. Math. Lett. **23** (2010), 241–245.
[2] T. H. Chang and C. L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl. **203** (1996), 224–235.
[3] S. S. Chang and Y. Zhang, *Generalized KKM theorem and variational inequalities*, J. Math. Anal. Appl. **159** (1991), 208–223.
[4] V. Colao, G. López, G. Marino and V. Martín-Márquez, *Equilibrium problems in Hadamard manifolds*, J. Math. Anal. Appl. **388** (2012), 61–77.
[5] M. P. do Carmo, *Riemannian Geometry*, Bikhausen, Boston, 1992.
[6] K. Fan, *A Generalization of Tychonoff's fixed-point theorem*, Math. Ann. **142** (1961), 305–310.
[7] S. Huang, *The Δ-convergence of iterations for nonexpansive mappings in CAT(κ) spaces*, J. Nonlinear Convex Anal. **13** (2012), 541–554.
[8] S. Huang, *Nonexpansive semigroups in CAT(κ) spaces*, Fixed Point Theory Appl. **2014**: 2014; 44.
[9] S. Huang, *Approximations with weak contractions in Hadamard manifolds*, Linear Nonlinear Anal. **1** (2015), 317–328.
[10] S. Huang, *Viscosity approximations with weak contractions in geodesic metric spaces of non-positive curvature*, J. Nonlinear Convex Anal. **17** (2016), 77–91.
[11] S. Huang and Y. Kimura, *A projection method for approximating fixed points of quasinonexpansive mappings in Hadamard spaces*, Fixed Point Theory Appl. **2016**: 2016; 36. DOI 10.1186/s13663-016-0523-6.
[12] J. Jost, *Nonpositive Curvature: Geometric and Analytic Aspects*, Birkhäuser Verlag, Basel, 1997.

[13] B. Knaster, C. Kuratowski, and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale simplexe*, Fund. Math. **14** (1929), 132–137.
[14] A. Kristály, *Nash-type equilibria on Riemannian manifolds: A variational approach*, J. Math. Pures Appl. **101** (2014), 660–688.
[15] S. Z. Németh, *Variational inequalities on Hadamard manifolds*, Nonlinear Anal. **52** (2003), 1491–1498.
[16] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature*, European Mathematical Society, 2nd edition, Zürich, 2014.
[17] S. Park, *Generalizations of Ky Fan's matching theorems and their applications*, J. Math. Anal. Appl. **141** (1989), 164–176.
[18] S. Park, *New generalizations of basic theorems in the KKM theory*, Nonlinear Anal. **74** (2011), 3000–3010.
[19] T. Rapcsák, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$*, Journal of Optimization Theory and Applications*, Kluwer Academic Publishers, Dordrecht, 1997.
[20] C. Udriste, *Convex Function and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publishers, 1994.

Manuscript received 30 June 2017

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