Linear and Nonlinear Analysis
Volume 3, Number 2, 2017, 261–274

Yokohama Publishers
ISSN 2188-8167

FIXED POINT THEOREMS FOR CONTRACTIVELY WIDELY MORE GENERALIZED HYBRID MAPPINGS IN METRIC SPACES

KEN HASEGAWA, TOSHIHARU KAWASAKI, AND TETSUO KOBAYASHI

Abstract. In this paper we consider a broad class of mappings containing Kannan mappings and contratively generalized hybrid mappings. Then we deal with fixed point theorems for such a mapping. Using these results, we show directly well-known fixed point theorems in complete metric spaces.

1. Introduction

Let (X,d) be a metric space. A mapping T from X into itself is said to be contractive if there exists k with $k \in [0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y)$$

for any $x,y \in X$. Such a mapping is called a k-contractive mapping. A mapping T from X into itself is said to be Kannan [4] if there exists k with $k \in \left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty) \leq k(d(x,Tx)+d(y,Ty))$$

for any $x,y \in X$. A mapping T from X into itself is said to be contractively nonspreading [1,3,8] if there exists k with $k \in \left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty) \leq k(d(x,Ty)+d(y,Tx))$$

for any $x,y \in X$. A mapping T from X into itself is said to be contractively hybrid [2] if there exists k with $k \in \left[0,\frac{1}{3}\right)$ such that

$$d(Tx,Ty) \leq k(d(Tx,y)+d(Ty,x)+d(x,y))$$

for any $x,y \in X$. Recently, Hasegawa, Komiya and Takahashi [2] introduced the concept of contratively generalized hybrid mappings on metric spaces and studied fixed point theorems for such mappings on complete metric spaces. A mapping T from X into itself is said to be contratively generalized hybrid if there exist $\alpha,\beta,r \in \mathbb{R}$ with $r \in [0,1)$ such that

$$\alpha d(Tx,Ty)+(1-\alpha)d(x,Ty) \leq r(\beta d(Tx,y)+(1-\beta)d(x,y))$$

2010 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Fixed point theorem, contractive mapping, Kannan mapping, contractively nonspreading mapping, contractively hybrid mapping, contractively generalized hybrid mapping, contractively widely more generalized hybrid mapping, metric space.

for any $x, y \in X$. Such a mapping is called an (α, β, r)-contratively generalized hybrid mapping; see also Kocourek, Takahashi and Yao [6] for such a mapping in Hilbert spaces. For instance, if $\alpha = 1$ and $\beta = 0$, then an (α, β, r)-contratively generalized hybrid mapping is contractive; if $\alpha = 1 + r$ and $\beta = 1$, then an (α, β, r)-contratively generalized hybrid mapping is contractively nonspreading; if $\alpha = 1 + \frac{r}{2}$ and $\beta = \frac{1}{2}$, then an (α, β, r)-contratively generalized hybrid mapping is contractively hybrid; see Hasegawa, Komiya and Takahashi [2].

In this paper, motivated by Hasegawa, Komiya and Takahashi [2], we consider a broad class of mappings containing Kannan mappings and contratively generalized hybrid mappings. Then we deal with fixed point theorems for such a mapping. Using these results, we show directly well-known fixed point theorems in complete metric spaces.

2. Preliminaries

We know the following Caristi's fixed point theorem which was generalized by Takahashi [7].

Theorem 2.1. *Let (X, d) be a complete metric space, let ψ be a proper, bounded below, and lower semicontinuous mapping from X into $(-\infty, \infty]$, and let T be a mapping from X into itself. Suppose that*

$$d(x, Tx) + \psi(Tx) \leq \psi(x)$$

for any $x \in X$. Then T has a fixed point.

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$, which is the dual space of ℓ^∞. Then we denote by $\mu(x)$ the value of μ at $x = (x_1, x_2, \ldots) \in \ell^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(x)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, \ldots)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then

$$\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n$$

holds for any $x = (x_1, x_2, \ldots) \in \ell^\infty$. In particular, if $x = (x_1, x_2, \ldots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we obtain $\mu_n(x_n) = a$. See [7] for the proof of existence of a Banach limit and its other elementary properties.

Moreover we use the following lemma and theorem showed by Hasegawa, Komiya and Takahashi [2].

Lemma 2.2. *Let (X, d) be a metric space, let $\{x_n\}$ be a bounded sequence in X, let μ be a mean on ℓ^∞ and let g be a mapping from X into $\mathbb{R}$ defined by*

$$g(x) = \mu_n d(x_n, x)$$

for any $x \in X$. Then g is continuous.

Theorem 2.3. *Let (X, d) be a complete metric space, let μ be a mean on ℓ^∞ and let T be a mapping from X into itself. Suppose that there exist a real number r with $0 \leq r < 1$ and $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and*

$$\mu_n d(T^n z, Tx) \leq r \mu_n d(T^n z, x)$$

for any $x \in X$. Then the following hold:

(i) T *has a unique fixed point* $u \in X$*;*

(ii) $u = \lim_{n\to\infty} T^n x$ *for any* $x \in X$.

3. Fixed point theorems

In this section we consider an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from a metric space X into itself; see also Kawasaki and Takahashi [5] for such a mapping in Hilbert spaces.

Definition 3.1. Let (X, d) be a metric space and let T be a mapping from X into itself. We say that T is contractively widely more generalized hybrid if T satisfies the following condition: there exist real numbers α, β, γ, δ, ε and ζ such that

$$\begin{aligned} \alpha d(Tx, Ty) + \beta d(x, Ty) + \gamma d(Tx, y) + \delta d(x, y) \\ +\varepsilon d(x, Tx) + \zeta d(y, Ty) \leq 0 \end{aligned} \tag{3.1}$$

for any $x, y \in X$. Such a mapping T is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping.

Firstly we consider criteria for an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping T from a metric space X into itself such that $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

Lemma 3.2. *Let (X, d) be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B1)*,* (B2) *or* (B3)*:*

(B1) $\alpha + \beta + \zeta \geq 0$ *and* $\alpha + 2\min\{\beta, 0\} + \delta + \varepsilon + \zeta > 0$*;*

(B2) $\alpha + \gamma + \varepsilon \geq 0$ *and* $\alpha + 2\min\{\gamma, 0\} + \delta + \varepsilon + \zeta > 0$*;*

(B3) $2\alpha + \beta + \gamma + \varepsilon + \zeta \geq 0$ *and* $\alpha + \min\{\beta + \gamma, 0\} + \delta + \varepsilon + \zeta > 0$.

Then $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

Proof. In the case of (B1), replacing x and y by $T^{n-1}x$ and $T^n x$, respectively, we obtain

$$\begin{aligned} &\alpha d(T^n x, T^{n+1}x) + \beta d(T^{n-1}x, T^{n+1}x) + \gamma d(T^n x, T^n x) + \delta d(T^{n-1}x, T^n x) \\ &\qquad +\varepsilon d(T^{n-1}x, T^n x) + \zeta d(T^n x, T^{n+1}x) \\ &= (\alpha + \zeta) d(T^{n+1}x, T^n x) + \beta d(T^{n+1}x, T^{n-1}x) + (\delta + \varepsilon) d(T^n x, T^{n-1}x) \\ &\leq 0. \end{aligned}$$

Since

$$\begin{aligned} d(T^{n+1}x, T^n x) - d(T^n x, T^{n-1}x) &\leq d(T^{n+1}x, T^{n-1}x) \\ &\leq d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x), \end{aligned}$$

we obtain

$$\beta d(T^{n+1}x, T^n x) - |\beta| d(T^n x, T^{n-1}x) \leq \beta d(T^{n+1}x, T^{n-1}x)$$

and hence

$$(\alpha + \beta + \zeta) d(T^{n+1}x, T^n x) + (-|\beta| + \delta + \varepsilon) d(T^n x, T^{n-1}x) \leq 0.$$

If $\alpha+\beta+\zeta=0$, then by $\alpha+2\min\{\beta,0\}+\delta+\varepsilon+\zeta>0$ we obtain

$$\begin{aligned} -|\beta|+\delta+\varepsilon &= 2\min\{\beta,0\}-\beta+\delta+\varepsilon \\ &> -(\alpha+\beta+\zeta) \\ &= 0. \end{aligned}$$

Then we obtain

$$d(T^n x, T^{n-1}x) \leq 0$$

for any $n \in \mathbb{N}$, that is, $\{T^n x \mid n \in \mathbb{N}\cup\{0\}\} = \{x\}$, and hence it is a Cauchy sequence. If $\alpha+\beta+\zeta>0$, we obtain

$$\begin{aligned} d(T^{n+1}x, T^n x) &\leq -\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta} d(T^n x, T^{n-1}x) \\ &\leq \max\left\{-\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta}, 0\right\} d(T^n x, T^{n-1}x) \\ &\leq \left(\max\left\{-\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta}, 0\right\}\right)^n d(Tx, x). \end{aligned}$$

By $\alpha+2\min\{\beta,0\}+\delta+\varepsilon+\zeta>0$ we obtain $-\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta}<1$. Therefore we obtain

$$\begin{aligned} d(T^m x, T^n x) &\leq \sum_{i=n+1}^{m} d(T^i x, T^{i-1}x) \\ &\leq \sum_{i=n+1}^{m} \left(\max\left\{-\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta}, 0\right\}\right)^{i-1} d(Tx, x) \\ &\leq \sum_{i=n+1}^{\infty} \left(\max\left\{-\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta}, 0\right\}\right)^{i-1} d(Tx, x) \\ &= \frac{\left(\max\left\{-\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta}, 0\right\}\right)^n}{1-\max\left\{-\frac{-|\beta|+\delta+\varepsilon}{\alpha+\beta+\zeta}, 0\right\}} d(Tx, x) \end{aligned}$$

for any $m, n \in \mathbb{N}\cup\{0\}$ with $m \geq n$, and hence $\{T^n x \mid n \in \mathbb{N}\cup\{0\}\}$ is a Cauchy sequence.

In the case of (B2), replacing the variables x and y in (3.1), we obtain

$$\begin{aligned} \alpha d(Tx,Ty)+\gamma d(x,Ty)+\beta d(Tx,y)+\delta d(x,y) \\ +\zeta d(x,Tx)+\varepsilon d(y,Ty) \leq 0. \end{aligned} \tag{3.2}$$

Therefore we obtain the desired result by (B1).

In the case of (B3), adding (3.1) and (3.2), we obtain

$$\begin{aligned} 2\alpha d(Tx,Ty)+(\beta+\gamma)d(x,Ty)+(\beta+\gamma)d(Tx,y)+2\delta d(x,y) \\ +(\varepsilon+\zeta)d(x,Tx)+(\varepsilon+\zeta)d(y,Ty) \leq 0. \end{aligned}$$

Therefore we obtain the desired result by (B1). □

Using Lemma 3.2, we obtain directly the following theorem.

Theorem 3.3. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B1)*,* (B2) *or* (B3)*. Then for any $x \in X$ there exists* $\lim_{n\to\infty} T^n x$.

Remark 3.4. Let (X, d) be a metric space and let $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ be a Cauchy sequence in X. Then $\{x_n \mid n \in \mathbb{N}\cup\{0\}\}$ is bounded. Indeed, since $\{x_n \mid n \in \mathbb{N}\cup\{0\}\}$ is a Cauchy sequence, for any positive number ρ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \rho$ for any $m, n \geq N$. Put $M = \max\{d(x_0, x_N), \dots, d(x_{N-1}, x_N), \rho\}$. Then $d(x_n, x_N) \leq M$ for any $n \in \mathbb{N} \cup \{0\}$.

Using Theorem 2.1, we show the following fixed point theorem.

Theorem 3.5. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (C1)*,* (C2) *or* (C3)*:*

(C1) $\zeta > 0$*,* $\alpha + \beta \geq 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} \geq 0$*;*
(C2) $\varepsilon > 0$*,* $\alpha + \gamma \geq 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\zeta, 0\} \geq 0$*;*
(C3) $\varepsilon + \zeta > 0$*,* $2\alpha + \beta + \gamma \geq 0$ *and* $\alpha + \beta + \gamma + \delta \geq 0$*.*

Then T has a fixed point if and only if there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Proof. If T has a fixed point u, then $\{T^n u \mid n \in \mathbb{N} \cup \{0\}\} = \{u\}$ is bounded.

Conversely suppose that there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In the case of (C1), replacing x and y by $T^n z$ and x, respectively, we obtain

$$\begin{aligned}\alpha d(T^{n+1}z, Tx) + \beta d(T^n z, Tx) + \gamma d(T^{n+1}z, x) + \delta d(T^n z, x)\\ + \varepsilon d(T^n z, T^{n+1}z) + \zeta d(x, Tx) \leq 0.\end{aligned}$$

Since $\min\{\varepsilon, 0\} \leq \varepsilon$, we obtain

$$\begin{aligned}\alpha d(T^{n+1}z, Tx) + \beta d(T^n z, Tx) + \gamma d(T^{n+1}z, x) + \delta d(T^n z, x)\\ + \min\{\varepsilon, 0\} d(T^n z, T^{n+1}z) + \zeta d(x, Tx) \leq 0.\end{aligned}$$

Since $\min\{\varepsilon, 0\} \leq 0$ and $d(T^n z, T^{n+1}z) \leq d(T^n z, x) + d(x, T^{n+1}z)$, we obtain

$$\begin{aligned}\alpha d(T^{n+1}z, Tx) + \beta d(T^n z, Tx) + (\gamma + \min\{\varepsilon, 0\}) d(T^{n+1}z, x)\\ + (\delta + \min\{\varepsilon, 0\}) d(T^n z, x) + \zeta d(x, Tx) \leq 0.\end{aligned}$$

Since $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded, we can apply a Banach limit μ to the inequality above. Then we obtain

$$(\alpha + \beta)\mu_n d(T^n z, Tx) + (\gamma + \delta + 2\min\{\varepsilon, 0\})\mu_n d(T^n z, x) + \zeta d(x, Tx) \leq 0.$$

By $\zeta > 0$ we obtain

$$d(x, Tx) + \frac{\alpha + \beta}{\zeta}\mu_n d(T^n z, Tx) \leq -\frac{\gamma + \delta + 2\min\{\varepsilon, 0\}}{\zeta}\mu_n d(T^n z, x).$$

By $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} \geq 0$ we obtain $-(\gamma + \delta + 2\min\{\varepsilon, 0\}) \leq \alpha + \beta$, and hence

$$d(x, Tx) + \frac{\alpha + \beta}{\zeta}\mu_n d(T^n z, Tx) \leq \frac{\alpha + \beta}{\zeta}\mu_n d(T^n z, x).$$

By $\zeta > 0$, $\alpha + \beta \geq 0$ and Lemma 2.2, $\frac{\alpha+\beta}{\zeta}\mu_n d(T^n z, \cdot)$ is proper, bounded below, and continuous. Therefore by Theorem 2.1 T has a fixed point.

Moreover suppose that $\alpha + \beta + \gamma + \delta > 0$ holds. Let u and v be fixed points of T. Then we obtain

$$\begin{aligned}&\alpha d(Tu, Tv) + \beta d(u, Tv) + \gamma d(Tu, v) + \delta d(u, v) + \varepsilon d(u, Tu) + \zeta d(v, Tv)\\ &\quad = (\alpha + \beta + \gamma + \delta) d(u, v) \leq 0.\end{aligned}$$

By $\alpha + \beta + \gamma + \delta > 0$ we obtain $d(u, v) \leq 0$ and hence $u = v$. Therefore T has a unique fixed point.

In the cases of (C2) and (C3), we obtain the desired results similarly to latter parts of the proof of Lemma 3.2. □

Using Lemma 3.2, Remark 3.4 and Theorem 3.5, we obtain the following fixed point theorem.

Theorem 3.6. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying the following:*

(B) *one of* (B1), (B2) *and* (B3) *holds;*
(C) *one of* (C1), (C2) *and* (C3) *holds.*

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Using Theorem 2.3, we show the following fixed point theorem.

Theorem 3.7. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (H1), (H2) *or* (H3)*:*

(H1) $\alpha + \beta + \zeta > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} + 2\min\{\zeta, 0\} > 0$*;*
(H2) $\alpha + \gamma + \varepsilon > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} + 2\min\{\zeta, 0\} > 0$*;*
(H3) $2\alpha + \beta + \gamma + \varepsilon + \zeta > 0$ *and* $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon + \zeta, 0\} > 0$.

Then T has a fixed point if and only if there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Moreover the following hold:

(i) *T has a unique fixed point $u \in X$;*
(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Proof. If T has a fixed point u, then $\{T^n u \mid n \in \mathbb{N} \cup \{0\}\} = \{u\}$ is bounded.

Conversely suppose that there exists $z \in X$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In the case of (H1), replacing x and y by $T^n z$ and x, respectively, we obtain

$$\begin{aligned}&\alpha d(T^{n+1}z, Tx) + \beta d(T^n z, Tx) + \gamma d(T^{n+1}z, x) + \delta d(T^n z, x)\\ &\qquad + \varepsilon d(T^n z, T^{n+1}z) + \zeta d(x, Tx) \leq 0.\end{aligned}$$

Since $\min\{\varepsilon, 0\} \leq \varepsilon$, we obtain

$$\begin{aligned}&\alpha d(T^{n+1}z, Tx) + \beta d(T^n z, Tx) + \gamma d(T^{n+1}z, x) + \delta d(T^n z, x)\\ &\qquad + \min\{\varepsilon, 0\} d(T^n z, T^{n+1}z) + \zeta d(x, Tx) \leq 0.\end{aligned}$$

Since $\min\{\varepsilon, 0\} \le 0$ and $d(T^n z, T^{n+1}z) \le d(T^n z, x) + d(x, T^{n+1}z)$, we obtain

$$\alpha d(T^{n+1}z, Tx) + \beta d(T^n z, Tx) + (\gamma + \min\{\varepsilon, 0\}) d(T^{n+1}z, x) \\ + (\delta + \min\{\varepsilon, 0\}) d(T^n z, x) + \zeta d(x, Tx) \le 0.$$

Since

$$d(T^n z, Tx) - d(T^n z, x) \le d(x, Tx) \le d(T^n z, Tx) + d(T^n z, x),$$

we obtain

$$\zeta d(T^n z, Tx) - |\zeta| d(T^n z, x) \le \zeta d(x, Tx)$$

and hence

$$\alpha d(T^{n+1}z, Tx) + (\beta + \zeta) d(T^n z, Tx) + (\gamma + \min\{\varepsilon, 0\}) d(T^{n+1}z, x) \\ + (\delta + \min\{\varepsilon, 0\} - |\zeta|) d(T^n z, x) \le 0.$$

Since $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded, we can apply a Banach limit μ to the inequality above. Then we obtain

$$(\alpha + \beta + \zeta) \mu_n d(T^n z, Tx) \\ + (\gamma + \delta + 2\min\{\varepsilon, 0\} - |\zeta|) \mu_n d(T^n z, x) \le 0.$$

By $\alpha + \beta + \zeta > 0$ we obtain

$$\begin{aligned} \mu_n d(T^n z, Tx) &\le -\frac{\gamma + \delta + 2\min\{\varepsilon, 0\} - |\zeta|}{\alpha + \beta + \zeta} \mu_n d(T^n z, x) \\ &\le \max\left\{ -\frac{\gamma + \delta + 2\min\{\varepsilon, 0\} - |\zeta|}{\alpha + \beta + \zeta}, 0 \right\} \mu_n d(T^n z, x). \end{aligned}$$

By $\alpha + \beta + \gamma + \delta + 2\min\{\varepsilon, 0\} + 2\min\{\zeta, 0\} > 0$ we obtain $-\frac{\gamma+\delta+2\min\{\varepsilon,0\}-|\zeta|}{\alpha+\beta+\zeta} < 1$. Therefore by Theorem 2.3 T has a unique fixed point $u \in X$ and $u = \lim_{n\to\infty} T^n x$ for any $x \in X$.

In the cases of (H2) and (H3), we obtain the desired results similarly to latter parts of the proof of Lemma 3.2. □

Using Lemma 3.2, Remark 3.4 and Theorem 3.7, we obtain the following fixed point theorem.

Theorem 3.8. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying the following:*

(B) *one of* (B1), (B2) *and* (B3) *holds;*
(H) *one of* (H1), (H2) *and* (H3) *holds.*

Then the following hold:

(i) *T has a unique fixed point $u \in X$;*
(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Moreover, if (B) is satisfied, we also show the following fixed point theorem.

Theorem 3.9. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-contractively widely more generalized hybrid mapping from X into itself satisfying* (B), *and one of* (M1), (M2) *and* (M3)*:*

(M1) $\alpha + \beta + \zeta > 0$;
(M2) $\alpha + \gamma + \varepsilon > 0$;
(M3) $2\alpha + \beta + \gamma + \varepsilon + \zeta > 0$.

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then the following hold:

(i) *T has a unique fixed point $u \in X$;*
(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Proof. By Theorem 3.3 there exists $u \in X$ such that $u = \lim_{n\to\infty} T^n x$. In the case of (M1), replacing x and y by $T^n x$ and u, respectively, we obtain

$$\begin{aligned}\alpha d(T^{n+1}x, Tu) + \beta d(T^n x, Tu) + \gamma d(T^{n+1}x, u) + \delta d(T^n x, u)\\ + \varepsilon d(T^n x, T^{n+1}x) + \zeta d(u, Tu) \leq 0.\end{aligned}$$

Since $u = \lim_{n\to\infty} T^n x$, we obtain

$$(\alpha + \beta + \zeta) d(u, Tu) \leq 0.$$

By $\alpha + \beta + \zeta > 0$ we obtain $d(u, Tu) \leq 0$ and hence u is a fixed point of T.

Moreover suppose that $\alpha + \beta + \gamma + \delta > 0$ holds. Let u and v be fixed points of T. Then we obtain

$$\begin{aligned}&\alpha d(Tu, Tv) + \beta d(u, Tv) + \gamma d(Tu, v) + \delta d(u, v) + \varepsilon d(u, Tu) + \zeta d(v, Tv)\\ &= (\alpha + \beta + \gamma + \delta) d(u, v) \leq 0.\end{aligned}$$

By $\alpha + \beta + \gamma + \delta > 0$ we obtain $d(u, v) \leq 0$ and hence $u = v$. Therefore T has a unique fixed point.

In the cases of (M2) and (M3), we obtain the desired results similarly to latter parts of the proof of Lemma 3.2. □

4. Applications

Theorem 4.1. *Let (X, d) be a complete metric space and let T be a contractively generalized hybrid mapping form X into itself, that is, there exist $\alpha, \beta, r \in \mathbb{R}$ with $0 \leq r < 1$ such that*

$$\alpha d(Tx, Ty) + (1 - \alpha) d(x, Ty) \leq r(\beta d(Tx, y) + (1 - \beta) d(x, y))$$

for any $x, y \in X$. Suppose that $\alpha > r(1 + |\beta|)$. Then the following hold:

(i) *T has a unique fixed point $u \in X$;*
(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Proof. T is $(\alpha, 1 - \alpha, -r\beta, -r(1 - \beta), 0, 0)$-contractively widely more generalized hybrid. Since

$$\begin{aligned}&\alpha + (1 - \alpha) + 0 = 1 > 0,\\ &\alpha + (1 - \alpha) + (-r\beta) + (-r(1 - \beta)) + 2 \cdot 0 + 2 \cdot 0 = 1 - r > 0,\end{aligned}$$

T satisfies (H1). If $\beta \geq 0$ and $\alpha > r(1 + \beta)$, then we obtain

$$\begin{aligned}&\alpha + (-r\beta) + 0 = \alpha - r\beta > r \geq 0,\\ &\alpha + 2\min\{-r\beta, 0\} + (-r(1 - \beta)) + 0 + 0 = \alpha - r(1 + \beta) > 0;\end{aligned}$$

if $\beta < 0$ and $\alpha > r(1-\beta)$, then we obtain

$$\alpha + (-r\beta) + 0 = \alpha - r\beta > r(1-2\beta) \geq 0,$$
$$\alpha + 2\min\{-r\beta, 0\} + (-r(1-\beta)) + 0 + 0 = \alpha - r(1-\beta) > 0.$$

In both cases T satisfies (B2). Therefore by Theorem 3.8 T has a unique fixed point $u \in X$ and $u = \lim_{n\to\infty} T^n x$ for any $x \in X$. ☐

Theorem 4.2. *Let (X, d) be a complete metric space and let T be a mapping form X into itself satisfying there exist $\varepsilon, \zeta \in \mathbb{R}$ such that $\varepsilon + \zeta < 1$ and*

$$d(Tx, Ty) \leq \varepsilon d(x, Tx) + \zeta d(y, Ty)$$

for any $x, y \in X$. Then the following hold:

(i) *T has a unique fixed point $u \in X$;*
(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Proof. T is $(1, 0, 0, 0, -\varepsilon, -\zeta)$-contractively widely more generalized hybrid. Since

$$2 \cdot 1 + 0 + (-\varepsilon) + (-\zeta) = 2 - (\varepsilon + \zeta) > 0,$$
$$1 + 0 + 0 + (-\varepsilon) + (-\zeta) = 1 - (\varepsilon + \zeta) > 0,$$

T satisfies (B3) and (M3). Moreover

$$1 + 0 + 0 + 0 > 0$$

holds. Therefore by Theorem 3.9 T has a unique fixed point $u \in X$ and $u = \lim_{n\to\infty} T^n x$ for any $x \in X$. ☐

In the remaining part of this section we discuss a special case of contractively widely more generalized hybrid mapping in metric spaces, which gives us a good vision for some applications. This mapping is defined as follows.

Definition 4.3. Let (X, d) be a metric space and let T be a mapping from X into itself. We say that T is a comprehensive contraction if T satisfies the following condition: there exist $\beta, \gamma, \delta, \varepsilon$ and ζ with $\beta, \gamma, \delta, \varepsilon, \zeta \geq 0$ and $\beta + \gamma + \delta + \varepsilon + \zeta < 1$ such that

$$d(Tx, Ty) \leq \beta d(x, Ty) + \gamma d(Tx, y) + \delta d(x, y) + \varepsilon d(x, Tx) + \zeta d(y, Ty)$$

for any $x, y \in X$. Such a mapping T is called a $(\beta, \gamma, \delta, \varepsilon, \zeta)$-comprehensive contraction.

The following theorem is derived easily from Theorem 3.9.

Theorem 4.4. *Let (X, d) be a complete metric space and let T be a $(\beta, \gamma, \delta, \varepsilon, \zeta)$-comprehensive contraction from X into itself. Then the following hold:*

(i) *T has a unique fixed point $u \in X$;*
(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

Proof. Any $(\beta, \gamma, \delta, \varepsilon, \zeta)$-comprehensive contraction is a $(1, -\beta, -\gamma, -\delta, -\varepsilon, -\zeta)$-contractively widely more generalized hybrid mapping. Since

$$2 \cdot 1 - \beta - \gamma - \varepsilon - \zeta > 1 + \delta > 0,$$
$$1 + \min\{-\beta - \gamma, 0\} - \delta - \varepsilon - \zeta = 1 - \beta - \gamma - \delta - \varepsilon - \zeta > 0,$$

T satisfies (B3) and (M3). Moreover T satisfies

$$1 - \beta - \gamma - \delta > \varepsilon + \zeta \geq 0.$$

Therefore by Theorem 3.9 T has a unique fixed point $u \in X$ and $u = \lim_{n\to\infty} T^n x$ for any $x \in X$. $\square$

Let (X, d) be a metric space. A mapping T from X into itself is said to be (1) contractive, (2) Kannan, (3) contractively nonspreading, and (4) contractively hybrid if there exists $k \in [0, 1)$ such that

(1) $d(Tx, Ty) \leq kd(x, y)$;
(2) $d(Tx, Ty) \leq \frac{k}{2}(d(x, Tx) + d(y, Ty))$;
(3) $d(Tx, Ty) \leq \frac{k}{2}(d(x, Ty) + d(Tx, y))$;
(4) $d(Tx, Ty) \leq \frac{k}{3}(d(x, Ty) + d(Tx, y) + d(x, y))$;

for any $x, y \in X$, respectively. It holds that

(1) any contractive mapping T is a $(0, 0, k, 0, 0)$-comprehensive contraction;
(2) any Kannan mapping T is a $(0, 0, 0, \frac{k}{2}, \frac{k}{2})$-comprehensive contraction;
(3) any contractively nonspreading mapping T is a $(\frac{k}{2}, \frac{k}{2}, 0, 0, 0)$-comprehensive contraction;
(4) any contractively hybrid mapping T is a $(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}, 0, 0)$-comprehensive contraction.

For each mapping by Theorem 4.4 the following hold:

(i) T has a unique fixed point $u \in X$;
(ii) $u = \lim_{n\to\infty} T^n x$ for any $x \in X$.

In 2011, Hasegawa, Komiya and Takahashi proved the following theorem.

Theorem 4.5. *Let E be a Banach space, let C be a nonempty closed convex subset of E, let α, β, γ be real numbers with $0 < \gamma < 1$, and let T be an (α, β, γ)- contractively generalized hybrid mapping form C into itself such that the set of fixed points of T is nonempty. Take $x_0, x_1 \in C$ and define $x_{n+2} = T(\gamma x_{n+1} + (1 - \gamma)x_n)$ for any $n \in \mathbb{N}$. Then, $\{x_n\}$ is convergent to a unique fixed point of T.*

Theorem 4.5 says that, when we think that $\{x_n\}$ is a sequence of time-series vectors, the 2-period moving average sequence $\{x_n\}$ of vectors is convergent to a unique fixed point of T. We extend the convergence of a 2-period moving average sequence of vectors to that of any m-period moving average sequence of vectors constructing by a comprehensive contraction T. For instance, let T be a comprehensive contraction, put $m = 3$ and let $\{x_n\}$ be a sequence in a Banach space E as follows: take $x_1, x_2, x_3 \in E$ and define $x_{n+3} = T\left(\frac{1}{2}x_{n+2} + \frac{1}{3}x_{n+1} + \frac{1}{6}x_n\right)$. Then $\{x_n\}$ converges to a fixed point $u \in C$.

Lemma 4.6. *Let (X, d) be a complete metric space and let T be a $(\beta, \gamma, \delta, \varepsilon, \zeta)$-comprehensive contraction from X into itself. Then T is a quasi-contractive mapping, that is, there exists $k \in [0, 1)$ such that*

$$d(Tx, u) \leq kd(x, u)$$

for any $x \in X$ and for any $u \in F(T)$.

Proof. In the case of $\varepsilon \geq \zeta$ we obtain

$$\begin{aligned} d(Tu,Ty) &= d(u,Ty) \\ &\leq \beta d(u,Ty) + \gamma d(u,y) + \delta d(u,y) + \varepsilon d(u,u) + \zeta d(y,Ty) \\ &= \beta d(u,Ty) + (\gamma+\delta) d(u,y) + \zeta d(y,Ty) \\ &\leq \beta d(u,Ty) + (\gamma+\delta) d(u,y) + \zeta (d(u,Ty) + d(u,y)) \\ &= (\beta+\zeta) d(u,Ty) + (\gamma+\delta+\zeta) d(u,y). \end{aligned}$$

Therefore we obtain

$$(1-\beta-\zeta) d(u,Ty) \leq (\gamma+\delta+\zeta) d(u,y).$$

Since $1-\beta-\zeta > \gamma+\delta+\varepsilon \geq 0$, we obtain

$$d(u,Ty) \leq \frac{\gamma+\delta+\zeta}{1-\beta-\zeta} d(u,y).$$

Since $\varepsilon \geq \zeta$, we obtain $1-\beta-\zeta > \gamma+\delta+\varepsilon \geq \gamma+\delta+\zeta \geq 0$ and hence

$$0 \leq \frac{\gamma+\delta+\zeta}{1-\beta-\zeta} < 1.$$

In the case of $\varepsilon < \zeta$ we obtain

$$\begin{aligned} d(Tx,Tu) &= d(Tx,u) \\ &\leq \beta d(x,u) + \gamma d(Tx,u) + \delta d(x,u) + \varepsilon d(x,Tx) + \zeta d(u,u) \\ &= (\beta+\delta) d(x,u) + \gamma d(Tx,u) + \varepsilon d(x,Tx) \\ &\leq (\beta+\delta) d(x,u) + \gamma d(Tx,u) + \varepsilon (d(x,u) + d(u,Tx)) \\ &= (\beta+\delta+\varepsilon) d(x,u) + (\gamma+\varepsilon) d(Tx,u). \end{aligned}$$

Therefore we obtain

$$(1-\gamma-\varepsilon) d(Tx,u) \leq (\beta+\delta+\varepsilon) d(x,u).$$

Since $1-\gamma-\varepsilon > \beta+\delta+\zeta \geq 0$, we obtain

$$d(Tx,u) \leq \frac{\beta+\delta+\varepsilon}{1-\gamma-\varepsilon} d(x,u).$$

Since $\varepsilon < \zeta$, we obtain $1-\gamma-\varepsilon > \beta+\delta+\zeta > \beta+\delta+\varepsilon \geq 0$ and hence

$$0 \leq \frac{\beta+\delta+\varepsilon}{1-\gamma-\varepsilon} < 1.$$

□

Lemma 4.7. *Let $k, a_1, \ldots, a_m$ be real numbers with $0 \leq k < 1$, $\sum_{i=1}^m a_i = 1$ and $0 < a_i < 1$ for any $i = 1, \ldots, m$. Take $P_1, \ldots, P_m \in \mathbb{R}$ and define*

$$P_{m+\ell} = k \sum_{i=1}^{m} a_i P_{m+\ell-i}$$

for any $\ell \in \mathbb{N}$. Then

$$\lim_{n\to\infty} P_n = 0.$$

Proof. Put $P = \max\{|P_1|, \ldots, |P_m|\}$. Let us show by induction that $|P_{\ell m+i}| \leq k^\ell P$ for any $i = 1, \ldots, m$ and for any $\ell \in \mathbb{N}$. Put $\ell = 1$. If $i = 1$, then

$$\begin{aligned} |P_{m+1}| &= \left| k \sum_{i=1}^{m} a_i P_{m+1-i} \right| \\ &\leq k \sum_{i=1}^{m} a_i |P_{m+1-i}| \\ &\leq kP \sum_{i=1}^{m} a_i \\ &= kP. \end{aligned}$$

Moreover, since $0 \leq k < 1$, $|P_{m+1}| \leq P$ holds. If $i = 2$, then

$$\begin{aligned} |P_{m+2}| &= \left| k \sum_{i=1}^{m} a_i P_{m+2-i} \right| \\ &\leq k \sum_{i=1}^{m} a_i |P_{m+2-i}| \\ &\leq kP \sum_{i=1}^{m} a_i \\ &= kP. \end{aligned}$$

Moreover, since $0 \leq k < 1$, $|P_{m+2}| \leq P$ holds. Proceeding with this way until $i = m$, we obtain

$$|P_{m+i}| \leq kP$$

for any $i = 1, \ldots, m$. Next, take $\ell \in \mathbb{N}$ and suppose that $|P_{\ell m+i}| \leq k^\ell P$ for any $i = 1, \ldots, m$. If $i = 1$, then

$$\begin{aligned} |P_{(\ell+1)m+1}| &= \left| k \sum_{i=1}^{m} a_i P_{(\ell+1)m+1-i} \right| \\ &\leq k \sum_{i=1}^{m} a_i |P_{\ell m+m+1-i}| \\ &\leq k^{\ell+1} P \sum_{i=1}^{m} a_i \\ &= k^{\ell+1} P. \end{aligned}$$

Moreover, since $0 \leq k < 1$, $|P_{(\ell+1)m+1}| \leq k^\ell P$ holds. If $i = 2$, then

$$\begin{aligned} |P_{(\ell+1)m+2}| &= \left| k \sum_{i=1}^{m} a_i P_{(\ell+1)m+2-i} \right| \\ &\leq k \sum_{i=1}^{m} a_i |P_{\ell m+m+2-i}| \end{aligned}$$

$$\begin{aligned} &\leq k^{\ell+1} P \sum_{i=1}^{m} a_i \\ &= k^{\ell+1} P. \end{aligned}$$

Moreover, since $0 \leq k < 1$, $|P_{(\ell+1)m+2}| \leq k^{\ell} P$ holds. Proceeding with this way until $i = m$, we obtain

$$|P_{(\ell+1)m+i}| \leq k^{\ell+1} P$$

for any $i = 1, \ldots, m$. Since $0 \leq k < 1$, we obtain

$$\lim_{n\to\infty} |P_n| \leq \lim_{\ell\to\infty} k^{\ell} P = 0$$

and hence

$$\lim_{n\to\infty} P_n = 0.$$

□

Theorem 4.8. *Let E be a Banach space, let C be a nonempty closed convex subset of E, let T be a $(\beta, \gamma, \delta, \varepsilon, \zeta)$-comprehensive contraction from C into itself, and let $a_1, \ldots, a_m$ be real numbers such that $0 < a_i < 1$ for any $i = 1, \ldots, m$ and $\sum_{i=1}^{m} a_i = 1$. Take $x_1, \ldots, x_m \in C$ and define*

$$x_{m+\ell} = T\left(\sum_{i=1}^{m} a_i x_{m+\ell-i}\right)$$

for any $\ell \in \mathbb{N}$. Then $\{x_n\}$ is convergent to a unique fixed point of T.

Proof. By Theorem 4.4 T has a unique fixed point $u \in C$. Since by Lemma 4.6 T is quasi-contractive, there exists $k \in [0, 1)$ such that

$$\|Tx - u\| \leq k\|x - u\|$$

for any $x \in X$. Put $P_i = \|x_i - u\|$ for any $i = 1, \ldots, m$ and define

$$P_{m+\ell} = k \sum_{i=1}^{m} a_i P_{m+\ell-i}$$

for any $\ell \in \mathbb{N}$. Let us show by induction that $\|x_n - u\| \leq P_n$ for any $n \in \mathbb{N}$. By definition we obtain $\|x_i - u\| = P_i$ for any $i = 1, \ldots, m$. Take $\ell \in \mathbb{N}$ and suppose that $\|x_n - u\| \leq P_n$ for any n with $n < m + \ell$. Then we obtain

$$\begin{aligned} \|x_{m+\ell} - u\| &= \left\| T\left(\sum_{i=1}^{m} a_i x_{m+\ell-i}\right) - u \right\| \\ &\leq k \left\| \sum_{i=1}^{m} a_i x_{m+\ell-i} - u \right\| \\ &\leq k \sum_{i=1}^{m} a_i \|x_{m+\ell-i} - u\| \\ &\leq k \sum_{i=1}^{m} a_i P_{m+\ell-i} = P_{m+\ell}. \end{aligned}$$

On the other hand, by Lemma 4.7 we obtain

$$\lim_{n\to\infty} P_n = 0.$$

Therefore we obtain

$$\lim_{n\to\infty} \|x_n - u\| = 0.$$

$\square$

Acknowledgements. The authors are grateful to Professor Wataru Takahashi for his suggestions and comments.

References

[1] S. K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.
[2] K. Hasegawa, T. Komiya and W. Takahashi, *Fixed point theorems for generalized contractive mappings in metric spaces and estimating expressions*, Scientiae Mathematicae Japonicae **74** (2011), 15–27.
[3] S. Iemoto, W. Takahashi and H. Yingtaweesittikul, *Nonlinear operators, fixed points and completeness of metric spaces*, Fixed Point Theory and its Applications (ed. L. J. Lin, A. Petrusel and H. K. Xu), Yokohama Publishers, Yokohama, 2010, pp. 93–101.
[4] R. Kannan, *Some results on fixed points*, Amer. Math. Monthly **76** (1969), 405–408.
[5] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 71–87.
[6] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese Journal of Mathematics **14** (2010), 2497–2511.
[7] W. Takahashi, *Nonlinear Functional Analysis. Fixed Points Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.
[8] T. Zamfirescu, *Fixed point theorems in metric spaces*, Arch. Math. (Basel) **23** (1972), 292–298.

Manuscript received 7 June 2015
revised 3 November 2016

K. Hasegawa
3-2-1 Numama, Zushi-shi, Kanagawa 249–0004, Japan
E-mail address: ken-hase@major.ocn.ne.jp

T. Kawasaki
College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail address: toshiharu.kawasaki@nifty.ne.jp

T. Kobayashi
College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail address: tekoba@ge.ce.nihon-u.ac.jp