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ATTRACTIVE POINT AND ERGODIC THEOREMS FOR TWO NONLINEAR MAPPINGS IN HILBERT SPACES

MAYUMI HOJO, S. TAKAHASHI, AND WATARU TAKAHASHI*

Abstract. In this paper, using means, we study attractive points of nonlinear mappings in Hilbert spaces. Then we obtain attractive point and fixed point theorems for commutative 2-generalized hybrid mappings in Hilbert spaces. Using this result, we prove a nonlinear mean convergence theorem for commutative 2-generalized hybrid mappings in Hilbert spaces.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [15] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|,\ \forall x \in C\}$.

We know from [15] that $A(T)$ is closed and convex. This property is important for proving our main theorem. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [2] we know the first nonlinear ergodic theorem in a Hilbert space: Let C be a bounded, closed and convex subset of H and let $T : C \to C$ be nonexpansive. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$. In 2010, Kocourek, Takahashi and Yao [4] defined a broad class of nonlinear mappings in a Hilbert space: Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [4] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

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for all $x, y \in C$. Such a mapping T is called (α, β)-*generalized hybrid*. We also know the following mapping: For $\lambda \in \mathbb{R}$, a mapping $U : C \to H$ is called λ-*hybrid* [1] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is *nonspreading* [6,7] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [14] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [3]. We also know that λ-hybrid mappings are in the class of generalized hybrid mappings. The nonlinear ergodic theorem by Baillon [2] for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [4]. Recently, Kohsaka [5] also proved the following theorem:

Theorem 1.1 ([5]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative λ and μ-hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

On the other hand, Takahashi and Takeuchi [15] proved the following attractive point and mean convergence theorem without convexity in a Hilbert space.

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Assume that $\{T^n z\}$ for some $z \in C$ is bounded and define*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n} T^k x$$

for all $x \in C$ and $n \in \mathbb{N}$. Then $\{S_n x\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)} T^n x$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.

Maruyama, Takahashi and Yao [10] also defined a more broad class of nonlinear mappings called 2-generalized hybrid which contains generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H. A mapping $T : C \to H$ is 2-*generalized hybrid* [10] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned} \tag{1.3}$$

for all $x, y \in C$.

In this paper, using means, we study attractive points of nonlinear mappings in Hilbert spaces. Then we obtain attractive point and fixed point theorems for commutative 2-generalized hybrid mappings in Hilbert spaces. Using this result, we prove a nonlinear mean convergence theorem for commutative 2-generalized hybrid mappings in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [13]. Furthermore, in a Hilbert space, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$. Indeed, we have that

$$\begin{aligned} 2\langle x-y, z-w\rangle &= 2\langle x,z\rangle - 2\langle x,w\rangle - 2\langle y,z\rangle + 2\langle y,w\rangle \\ &= (-\|x\|^2 + 2\langle x,z\rangle - \|z\|^2) + (\|x\|^2 - 2\langle x,w\rangle + \|w\|^2) \\ &\quad + (\|y\|^2 - 2\langle y,z\rangle + \|z\|^2) + (-\|y\|^2 + 2\langle y,w\rangle - \|w\|^2) \\ &= \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2. \end{aligned}$$

From (2.2), we have that

$$\langle (x-y)+(x-w), y-w\rangle = \|x-w\|^2 - \|x-y\|^2 \tag{2.3}$$

for all $x, y, w \in H$. Indeed, we have that

$$\begin{aligned} &2\langle (x-y)+(x-w), y-w\rangle = 2\langle (x-w)-(y-x), (y-w)-0\rangle \\ &\quad = \|x-w-0\|^2 + \|y-x-(y-w)\|^2 - \|x-w-(y-w)\|^2 - \|y-x-0\|^2 \\ &\quad = 2\|x-w\|^2 - 2\|y-x\|^2 \end{aligned}$$

and hence $\langle (x-y)+(x-w), y-w\rangle = \|x-w\|^2 - \|x-y\|^2$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (a_1, a_2, a_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(a_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1,1,1,\dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(a_{n+1}) = \mu_n(a_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (a_1, a_2, a_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n.$$

In particular, if $f = (a_1, a_2, a_3, \dots) \in l^\infty$ and $a_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(a_n) = a$. See [12] for the proof of existence of a Banach limit and its other elementary properties.

Using a mean, we obtain the following result; see [9, 11]: Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H. \tag{2.4}$$

We call such a unique $z_0 \in H$ the *mean vector* of $\{x_n\}$ for μ. For the sake of completeness, we give the proof. Since $\{x_n\}$ is bounded, we have that for any $y \in H$, $\{\langle x_n, y\rangle\}$ is in l^∞. Since μ is a mean on l^∞, we can define a real valued function g as follows:

$$g(y) = \mu_n\langle x_n, y\rangle, \quad \forall y \in H.$$

We have that for any $y, z \in H$ and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} g(\alpha y + \beta z) &= \mu_n\langle x_n, \alpha y + \beta z\rangle = \alpha\mu_n\langle x_n, y\rangle + \beta\mu_n\langle x_n, z\rangle \\ &= \alpha g(y) + \beta g(z). \end{aligned}$$

Then g is a linear functional of H into $\mathbb{R}$. Furthermore we have that for any $y \in H$,

$$\begin{aligned} |g(y)| = |\mu_n\langle x_n, y\rangle| &\le \|\mu_n\| \sup_{n\in\mathbb{N}} |\langle x_n, y\rangle| \\ &\le \|\mu_n\| \sup_{n\in\mathbb{N}} \|x_n\| \, \|y\| = (\sup_{n\in\mathbb{N}} \|x_n\|) \, \|y\|. \end{aligned}$$

Put $K = \sup_{n\in\mathbb{N}} \|x_n\|$. We have that

$$|g(y)| \le K\|y\|, \quad \forall y \in H.$$

Then g is bounded. By the Riesz theorem, there exists $z_0 \in H$ such that

$$g(y) = \langle z_0, y\rangle, \quad \forall y \in H. \tag{2.5}$$

It is obvious that such $z_0 \in H$ is unique. Furthermore we have $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$. In fact, if $z_0 \notin \overline{co}\{x_n : n \in \mathbb{N}\}$, then there exists $y_0 \in H$ from the separation theorem such that

$$\langle z_0, y_0\rangle < \inf\left\{\langle z, y_0\rangle : z \in \overline{co}\{x_n : n \in \mathbb{N}\}\right\}.$$

Using the property of a mean, we have that

$$\begin{aligned} \langle z_0, y_0\rangle &< \inf\left\{\langle z, y_0\rangle : z \in \overline{co}\{x_n : n \in \mathbb{N}\}\right\} \\ &\le \inf\{\langle x_n, y_0\rangle : n \in \mathbb{N}\} \le \mu_n\langle x_n, y_0\rangle = \langle z_0, y_0\rangle. \end{aligned}$$

This is a contradiction. Thus we have $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$.

3. Attractive point theorems

Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is 2-*generalized hybrid* [10] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ &\qquad \le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned} \tag{3.1}$$

for all $x, y \in C$. We call such a mapping $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-*generalized hybrid*. We know that the class of the mappings above covers well-known mappings. For example, the class of $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mappings is the class of (α_2, β_2)-generalized hybrid mappings in the sense of Kocourek, Takahashi and Yao [4]. If $x = Tx$ in (3.1), then for any $y \in C$,

$$\alpha_1\|x-Ty\|^2 + \alpha_2\|x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2$$
$$\leq \beta_1\|x - y\|^2 + \beta_2\|x - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2.$$

Hence we have that

$$\|x - Ty\| \leq \|x - y\|, \quad \forall x \in F(T),\ y \in C. \tag{3.2}$$

Thus, a 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Now, we prove an attractive point theorem for commutative 2-generalized hybrid mappings in a Hilbert space. Before proving the theorem, we show the following lemma.

Lemma 3.1. *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be mappings of C into itself. Suppose that there exist a mean μ on l^∞ and a sequence $\{x_n\} \subset H$ such that $\{x_n\}$ is bounded and*

$$\mu_n\|x_n - Sy\|^2 \leq \mu_n\|x_n - y\|^2 \text{ and } \mu_n\|x_n - Ty\|^2 \leq \mu_n\|x_n - y\|^2, \quad \forall y \in C.$$

Then $A(S) \cap A(T)$ is nonempty. In particular, the mean vector $z_0 \in H$ of $\{x_n\}$ for μ is an element of $A(S) \cap A(T)$. Additionally, if C is closed and convex and $\{x_n\} \subset C$, then $F(S) \cap F(T)$ is nonempty.

Proof. Since $\{x_n\}$ is bounded, we have from (2.5) that there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H. \tag{3.3}$$

Using this z_0, we have from (2.3) and the assumption of T that for any $v \in C$,

$$\begin{aligned}\langle (z_0 - v) + (z_0 - Sv), v - Sv\rangle &= \mu_n\langle (x_n - v) + (x_n - Sv), v - Sv\rangle \\ &= \mu_n(\|x_n - Sv\|^2 - \|x_n - v\|^2) \\ &= \mu_n\|x_n - Sv\|^2 - \mu_n\|x_n - v\|^2 \\ &\leq 0.\end{aligned}$$

Using (2.3) again, we have that

$$\langle (z_0 - v) + (z_0 - Sv), v - Sv\rangle = \|z_0 - Sv\|^2 - \|z_0 - v\|^2.$$

Thus we have that

$$\|z_0 - Sv\|^2 - \|z_0 - v\|^2 \leq 0, \quad \forall v \in C$$

and hence

$$\|z_0 - Sv\| \leq \|z_0 - v\|, \quad \forall v \in C.$$

Therefore we have $z_0 \in A(S)$. Similarly, we have that

$$\|z_0 - Tv\| \leq \|z_0 - v\|, \quad \forall v \in C.$$

and hence $z_0 \in A(T)$. Therefore we have $z_0 \in A(S) \cap A(T)$. Additionally, if C is closed and convex and $\{x_n\} \subset C$, we have that

$$z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\} \subset C.$$

Since $z_0 \in A(S) \cap A(T)$ and $z_0 \in C$, we have that

$$\|Sz_0 - z_0\| \le \|z_0 - z_0\| = 0 \quad \text{and} \quad \|Tz_0 - z_0\| \le \|z_0 - z_0\| = 0$$

and hence $z_0 \in F(S) \cap F(T)$. This completes the proof. $\square$

Using Lemma 3.1, we can prove an attractive point theorem for commutative 2-generalized hybrid mappings in a Hilbert space.

Theorem 3.2. *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be commutative 2-generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

Proof. Since a mapping S is 2-generalized hybrid, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|S^2x - Sy\|^2 &+ \alpha_2\|Sx - Sy\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Sy\|^2 \\ &\le \beta_1\|S^2x - y\|^2 + \beta_2\|Sx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. Take $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then for any $y \in C$ and $k, l \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}\alpha_1&\|S^{k+2}T^lz - Sy\|^2 + \alpha_2\|S^{k+1}T^lz - Sy\|^2 + (1 - \alpha_1 - \alpha_2)\|S^kT^lz - Sy\|^2 \\ &\le \beta_1\|S^{k+2}T^lz - y\|^2 + \beta_2\|S^{k+1}T^lz - y\|^2 + (1 - \beta_1 - \beta_2)\|S^kT^lz - y\|^2 \\ &= \beta_1\{\|S^{k+2}T^lz - Sy\|^2 + \|Sy - y\|^2 + 2\langle S^{k+2}T^lz - Sy, Sy - y\rangle\} \\ &\quad + \beta_2\{\|S^{k+1}T^lz - Sy\|^2 + \|Sy - y\|^2 + 2\langle S^{k+1}T^lz - Sy, Sy - y\rangle\} \\ &\quad + (1 - \beta_1 - \beta_2)\{\|S^kT^lz - Sy\|^2 + \|Sy - y\|^2 + 2\langle S^kT^lz - Sy, Sy - y\rangle\}.\end{aligned}$$

This implies that

$$\begin{aligned}0 \le &(\beta_1 - \alpha_1)\{\|S^{k+2}T^lz - Sy\|^2 - \|S^kT^lz - Sy\|^2\} \\ &+ (\beta_2 - \alpha_2)\{\|S^{k+1}T^lz - Sy\|^2 - \|S^kT^lz - Sy\|^2\} + \|Sy - y\|^2 \\ &+ 2\langle \beta_1S^{k+2}T^lz + \beta_2S^{k+1}T^lz + (1 - \beta_1 - \beta_2)S^kT^lz - Sy, Sy - y\rangle.\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n$, we have

$$\begin{aligned}0 \le &(\beta_1 - \alpha_1)\{\|S^{n+2}T^lz - Sy\|^2 + \|S^{n+1}T^lz - Sy\|^2 \\ &\qquad - \|ST^lz - Sy\|^2 - \|T^lz - Sy\|^2\} \\ &+ (\beta_2 - \alpha_2)\{\|S^{n+1}T^lz - Sy\|^2 - \|T^lz - Sy\|^2\} + (n + 1)\|Sy - y\|^2 \\ &+ 2\Big\langle \sum_{k=0}^{n} S^kT^lz + \beta_1(S^{n+2}T^lz + S^{n+1}T^lz - ST^lz - T^lz) \\ &\qquad + \beta_2(S^{n+1}T^lz - T^lz) - (n + 1)Sy, Sy - y\Big\rangle.\end{aligned}$$

Furthermore, summing up these inequalities with respect to $l = 0, 1, \dots, n$, we have

$$0 \le (\beta_1 - \alpha_1)\sum_{l=0}^{n}\{\|S^{n+2}T^lz - Sy\|^2 + \|S^{n+1}T^lz - Sy\|^2$$

$$
\begin{aligned}
&- \|ST^l z - Sy\|^2 - \|T^l z - Sy\|^2\} \\
&+ (\beta_2 - \alpha_2) \sum_{l=0}^{n} \{\|S^{n+1}T^l z - Sy\|^2 - \|T^l z - Sy\|^2\} + (n+1)^2 \|Sy - y\|^2 \\
&+ 2\Big\langle \sum_{l=0}^{n} \sum_{k=0}^{n} S^k T^l z + \beta_1 \sum_{l=0}^{n} (S^{n+2}T^l z + S^{n+1}T^l z - ST^l z - T^l z) \\
&+ \beta_2 \sum_{l=0}^{n} (S^{n+1}T^l z - T^l z) - (n+1)^2 Sy, Sy - y \Big\rangle.
\end{aligned}
$$

Dividing by $(n+1)^2$, we have

$$
\begin{aligned}
0 \leq\ & (\beta_1 - \alpha_1) \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\|S^{n+2}T^l z - Sy\|^2 + \|S^{n+1}T^l z - Sy\|^2 \\
&- \|ST^l z - Sy\|^2 - \|T^l z - Sy\|^2\} \\
&+ (\beta_2 - \alpha_2) \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\|S^{n+1}T^l z - Sy\|^2 - \|T^l z - Sy\|^2\} + \|Sy - y\|^2 \\
&+ 2\Big\langle S_n z + \beta_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+2}T^l z + S^{n+1}T^l z - ST^l z - T^l z) \\
&+ \beta_2 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+1}T^l z - T^l z) - Sy, Sy - y \Big\rangle,
\end{aligned}
$$

where $S_n z = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l z$. Since $\{S^k T^l z\}$ is bounded by assumption, $\{S_n z\}$ is bounded. Taking a Banach limit μ to both sides of this inequality, we have that

$$
\begin{aligned}
0 \leq\ & (\beta_1 - \alpha_1) \mu_n \Big(\frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\|S^{n+2}T^l z - Sy\|^2 + \|S^{n+1}T^l z - Sy\|^2 \\
&- \|ST^l z - Sy\|^2 - \|T^l z - Sy\|^2\} \Big) \\
&+ (\beta_2 - \alpha_2) \mu_n \Big(\frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\|S^{n+1}T^l z - Sy\|^2 - \|T^l z - Sy\|^2\} \Big) + \|Sy - y\|^2 \\
&+ 2\mu_n \Big\langle S_n z + \beta_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+2}T^l z + S^{n+1}T^l z - ST^l z - T^l z) \\
&+ \beta_2 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+1}T^l z - T^l z) - Sy, Sy - y \Big\rangle
\end{aligned}
$$

and hence

$$
0 \leq \|Sy - y\|^2 + 2\mu_n \langle S_n z - Sy, Sy - y \rangle.
$$

We obtain from (2.2) that

$$
0 \leq \|Sy - y\|^2 + 2\mu_n \langle S_n z - Sy, Sy - y \rangle
$$

$$= \|Sy - y\|^2 + \mu_n\|S_nz - y\|^2 + \|Sy - Sy\|^2 - \mu_n\|S_nz - Sy\|^2 - \|Sy - y\|^2$$
$$= \mu_n\|S_nz - y\|^2 - \mu_n\|S_nz - Sy\|^2$$

and hence

$$\mu_n\|S_nz - Sy\|^2 \leq \mu_n\|S_nz - y\|^2.$$

Similarly, since a mapping T is 2-generalized hybrid, there exist $\alpha_1', \alpha_2', \beta_1', \beta_2' \in \mathbb{R}$ such that

$$\alpha_1'\|T^2x - Ty\|^2 + \alpha_2'\|Tx - Ty\|^2 + (1 - \alpha_1' - \alpha_2')\|x - Ty\|^2$$
$$\leq \beta_1'\|T^2x - y\|^2 + \beta_2'\|Tx - y\|^2 + (1 - \beta_1' - \beta_2')\|x - y\|^2$$

for all $x, y \in C$. Replacing S and T by T and S for the above proof, respectively, we have

$$\mu_n\|S_nz - Ty\|^2 \leq \mu_n\|S_nz - y\|^2.$$

By Lemma 3.1, we have that $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then we have from Lemma 3.1 that $F(S) \cap F(T)$ is nonempty. This completes the proof. □

Using Theorem 3.2, we have the following theorem for commutative generalized hybrid mappings in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be commutative generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

Proof. The generalized hybrid mappings S and T of C into itself are 2-generalized hybrid mappings. That is, an (α, β)-generalized hybrid mapping S is a $(0, \alpha, 0, \beta)$-generalized hybrid mapping and an (α', β')-generalized hybrid mapping S is a $(0, \alpha', 0, \beta')$-generalized hybrid mapping. Thus we have the desired result from Theorem 3.2. □

Using Theorem 3.2, we also have the attractive point theorem by Takahashi and Takeuchi [15] for generalized hybrid mappings in a Hilbert space.

Theorem 3.4. *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a generalized hybrid mapping of C into itself. Suppose that there exists an element $z \in C$ such that $\{T^nz\}$ is bounded. Then $A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(T)$ is nonempty.*

4. Nonlinear erdodic theorems

In this section, we prove a mean convergence theorem for commutative 2-generalized hybrid mappings without convexity in a Hilbert space.

Let $D = \{(k, l) : k, l \in \mathbb{N} \cup \{0\}\}$. Then D is a directed set by the binary relation:

$$(k, l) \leq (i, j) \quad \text{if } k \leq i \text{ and } l \leq j.$$

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let P be the metric projection of H onto $A(S) \cap A(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l)\in D} PS^kT^lx$. In particular, if C is closed and convex, $\{S_n x\}$ converges weakly to an element q of $F(S) \cap F(T)$.

Proof. Let $S : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping. Since $A(S) \cap A(T)$ is nonempty, closed and convex, there exists the metric projection P of H onto $A(S) \cap A(T)$. We have that

$$0 \leq \langle v - Pv, Pv - u \rangle, \quad \forall u \in A(S) \cap A(T),\ v \in C.$$

Adding up $\|Pv - u\|^2$ to both sides of this inequality, we have from (2.2) that

$$\begin{aligned} \|Pv - u\|^2 &\leq \|Pv - u\|^2 + 2 \langle v - Pv, Pv - u \rangle \\ &= \|Pv - u\|^2 + \|v - u\|^2 + \|Pv - Pv\|^2 - \|v - Pv\|^2 - \|Pv - u\|^2 \\ &= \|v - u\|^2 - \|v - Pv\|^2. \end{aligned} \tag{4.1}$$

Since $\|Sz - u\| \leq \|z - u\|$ and $\|Tz - u\| \leq \|z - u\|$ for any $u \in A(S) \cap A(T)$ and $z \in C$, it follows that for any $(k,l), (i,j) \in D$ with $(k,l) \leq (i,j)$,

$$\begin{aligned} \|S^iT^jx - PS^iT^jx\| &\leq \|S^iT^jx - PS^kT^lx\| \\ &\leq \|S^kT^lx - PS^kT^lx\|. \end{aligned}$$

Hence the sequence $\|S^kT^lx - PS^kT^lx\|$ is nonincreasing and then there exists $\lim_{(k,l)\in D} \|S^kT^lx - PS^kT^lx\|$. Putting $u = PS^kT^lx$ and $v = S^iT^jx$ with $(k,l) \leq (i,j)$ in (4.1), we have that

$$\begin{aligned} \|PS^iT^jx - PS^kT^lx\|^2 &\leq \|S^iT^jx - PS^kT^lx\|^2 - \|S^iT^jx - PS^iT^jx\|^2 \\ &\leq \|S^kT^lx - PS^kT^lx\|^2 - \|S^iT^jx - PS^iT^jx\|^2 \end{aligned}$$

and hence $\{PS^kT^lx\}$ is a Cauchy net; see [8, 16]. Therefore $\{PS^kT^lx\}$ converges strongly to a point $q \in A(S) \cap A(T)$ since $A(S) \cap A(T)$ is closed. Next, consider an arbitrary subsequence $\{S_{n_i}x\}$ of $\{S_nx\}$ convergent weakly to a point v. From the proof of Theorem 3.2, we know that

$$\begin{aligned} 0 \leq (\beta_1 - \alpha_1) &\frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\|S^{n+2}T^lx - Sy\|^2 + \|S^{n+1}T^lx - Sy\|^2 \\ &- \|ST^lx - Sy\|^2 - \|T^lx - Sy\|^2\} \\ &+ (\beta_2 - \alpha_2) \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\|S^{n+1}T^lx - Sy\|^2 - \|T^lx - Sy\|^2\} + \|Sy - y\|^2 \\ &+ 2\Big\langle S_nx + \beta_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+2}T^lx + S^{n+1}T^lx - ST^lx - T^lx) \end{aligned}$$

$$+ \beta_2 \frac{1}{(n+1)^2} \sum_{l=0}^{n} (S^{n+1}T^l x - T^l x) - Sy, Sy - y \Big\rangle,$$

where $S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$. Letting $n_i \to \infty$, we obtain

$$0 \leq \|Sy - y\|^2 + 2\langle v - Sy, Sy - y\rangle.$$

Then, we obtain

$$\begin{aligned} 0 &\leq \|Sy - y\|^2 + 2\langle v - Sy, Sy - y\rangle \\ &= \|Sy - y\|^2 + \|v - y\|^2 + \|Sy - Sy\|^2 - \|v - Sy\|^2 - \|Sy - y\|^2 \\ &= \|v - y\|^2 - \|v - Sy\|^2 \end{aligned}$$

and hence $\|v - Sy\| \leq \|v - y\|$. This implies that $v \in A(S)$. Similarly, let $T : C \to C$ be an $(\alpha_1', \alpha_2', \beta_1', \beta_2')$-generalized hybrid mapping. Replacing S and T by T and S in the proof, respectively, we have $v \in A(T)$. Therefore $v \in A(S) \cap A(T)$. Rewriting the characterization of the metric projection P, we have that for any $u \in A(S) \cap A(T)$,

$$0 \leq \Big\langle S^k T^l x - PS^k T^l x, PS^k T^l x - u \Big\rangle$$

and hence

$$\begin{aligned} \Big\langle S^k T^l x - PS^k T^l x, u - q \Big\rangle &\leq \Big\langle S^k T^l x - PS^k T^l x, PS^k T^l x - q \Big\rangle \\ &\leq \|S^k T^l x - PS^k T^l x\| \cdot \|PS^k T^l x - q\| \\ &\leq K\|PS^k T^l x - q\|, \end{aligned}$$

where K is an upper bound for $\|S^k T^l x - PS^k T^l x\|$. Summing up these inequalities for $k = 0, 1, \ldots, n$ and $l = 0, 1, \ldots, n$ and dividing by $(n + 1)^2$, we arrive to

$$\Big\langle S_n x - \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} PS^k T^l x, u - q \Big\rangle \leq K \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \|PS^k T^l x - q\|.$$

Letting $n_i \to \infty$, we get

$$\langle v - q, u - q\rangle \leq 0.$$

This holds for any $u \in A(S) \cap A(T)$. Therefore we have $Pv = q$. But because $v \in A(S) \cap A(T)$, we have $v = q$. Thus the sequence $\{S_n x\}$ converges weakly to the point $q \in A(S) \cap A(T)$. In particular, if C is closed and convex, $\{S_n x\}$ converges weakly to an element q of $F(S) \cap F(T)$. □

Using Theorem 4.1, we get the nonlinear ergodic theorem (Theorem 1.1) by Kohsaka [5]. Furthermore, we can prove the following nonlinear ergodic theorem by Lin and Takahashi [9] for 2-generalized hybrid mappings in a Hilbert space.

Theorem 4.2 ([9])**.** *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a 2-generalized hybrid mapping of C into itself such that $A(T)$ is nonempty. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in A(T)$*, where* $z_0 = \lim_{n\to\infty} P_{A(T)}T^n x$*. Additionally, if* C *is closed and convex, then* $\{S_n x\}$ *converges weakly to* $z_0 \in F(T)$*, where* $z_0 = \lim_{n\to\infty} P_{F(T)}T^n x$*.*

Using Theorem 4.1, we also have the following nonlinear ergodic theorem by Takahashi and Takeuchi [15].

Theorem 4.3 ([15])**.** *Let* H *be a Hilbert space, let* C *be a nonempty subset of* H *and let* T *be a generalized hybrid mapping of* C *into itself such that* $A(T)$ *is nonempty. Then for any* $x \in C$*,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in A(T)$*, where* $z_0 = \lim_{n\to\infty} P_{A(T)}T^n x$*. Additionally, if* C *is closed and convex, then* $\{S_n x\}$ *converges weakly to* $z_0 \in F(T)$*, where* $z_0 = \lim_{n\to\infty} P_{F(T)}T^n x$*.*

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Mayumi Hojo
Shibaura Institute of Technology, Tokyo 135-8548, Japan
E-mail address: mayumi-h@shibaura-it.ac.jp

Satoru Takahashi
Yokohama Publishers, 101, 6–27, Satsukigaoka, Aoba-ku, Yokohama 227-0053, Japan
E-mail address: info@ybook.co.jp

Wataru Takahashi
Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net