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QUASICONVEXITY OF SUM OF QUASICONVEX FUNCTIONS

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ABSTRACT. In this paper, we study quasiconvexity of sum of quasiconvex functions. At first, we show characterizations of quasiconvexity via upper semicontinuous functions in terms of Q-subdifferential. We introduce a set of quasiconvex functions which is closed under addition and multiplication by positive scalars. As applications, we investigate Lagrange duality and surrogate duality for quasiconvex programming.

1. Introduction

In functional analysis, whether a set of functions is closed under some operations or not is an essential and important research aspect. Especially, the set of all convex functions is closed under addition and multiplication by positive scalars. The fact implies various important results in optimization, for example, Lagrange dual problem for convex programming is also a convex programming problem, the set of all difference of convex functions is a vector space, and so on. In quasiconvex analysis, although the set of all quasiconvex function is closed under multiplication by positive scalars, the set of all quasiconvex functions is not closed under addition. The lack of this property can cause theoretical difficulties in applications. Actually, Lagrange dual problem for quasiconvex programming is not a quasiconvex programming problem in general. Some researchers assume that the objective function of Lagrange dual problem is quasiconvex without sufficient conditions, see [6, 12]. Hence, it is expected to study quasiconvexity of sum of quasiconvex functions. In [3, 5], quasiconvexity of a separable function has been studied. However, it is not enough to study quasiconvexity of sum of quasiconvex functions in general setting.

In convex analysis, various researchers characterize some types of convexity in terms of subdifferentials. It is well known that a real-valued function on the n-dimensional Euclidean space $\mathbb{R}^n$ is convex if and only if $\partial f(x)$, the subdifferential of f at $x \in \mathbb{R}^n$, is non-empty for each x. For further results, see [1, 2, 10, 11, 24] and references therein. In quasiconvex analysis, there are so many subdifferentials for quasiconvex functions, for example, quasi-subdifferential, infradifferential, lower subdifferential, and so on, see [1, 4, 7–9, 13, 15–23, 26–28, 30, 31, 33]. Most of them are special cases of Moreau's generalized conjugation in [19]. However, as far as

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we know, there are not many characterizations of quasiconvexity in terms of such subdifferentials.

In this paper, we study quasiconvexity of sum of quasiconvex functions. At first, we show characterizations of quasiconvexity via upper semicontinuous functions in terms of Q-subdifferential. We introduce a set of quasiconvex functions which is closed under addition and multiplication by positive scalars. As applications, we investigate Lagrange duality and surrogate duality for quasiconvex programming.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we show characterizations of quasiconvexity via upper semicontinuous functions in terms of Q-subdifferential. In Section 4, we study quasiconvexity of sum of quasiconvex functions. In Section 5, we introduce some applications.

2. Preliminaries

Let $\langle v, x \rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, +\infty]$. We denote the domain of f by dom$f$, that is, dom$f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$. The epigraph of f is defined as epi$f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if epi$f$ is convex. The subdifferential of f at x is defined as $\partial f(x) = \{v \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle v, y - x \rangle\}$. Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \alpha) := \{x \in \mathbb{R}^n \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \overline{\mathbb{R}}$. A function f is said to be quasiconvex if for all $\alpha \in \mathbb{R}$, $L(f, \leq, \alpha)$ is a convex set. Any convex function is quasiconvex, but the opposite is not always true. A subset A of $\mathbb{R}^n$ is said to be evenly convex if it is the intersection of some family of open halfspaces. Note that the whole space and the empty set are evenly convex. Clearly, every evenly convex set is convex. Furthermore, any open convex set and any closed convex set are evenly convex. A function f is said to be evenly quasiconvex (strictly evenly quasiconvex) if $L(f, \leq, \alpha)$ ($L(f, <, \alpha)$, respectively) is evenly convex for each $\alpha \in \mathbb{R}$. It is clear that every evenly quasiconvex function is quasiconvex. It is easy to show that every strictly evenly quasiconvex function is evenly quasiconvex, see the proof of Theorem 3.1. However, converse implications are not generally true, in detail, see [22,25]. In addition, every lower semicontinuous (lsc) quasiconvex function is evenly quasiconvex, and every upper semicontinuous (usc) quasiconvex function is strictly evenly quasiconvex.

In quasiconvex analysis, various types of conjugates and subdifferentials have been introduced. In this paper, the following subdifferentials and Q-conjugate play important roles. In [7], Greenberg and Pierskalla introduced the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) = \{v \in \mathbb{R}^n \mid \langle v, x \rangle \geq \langle v, x_0 \rangle \text{ implies } f(x) \geq f(x_0)\}.$$

The Greenberg-Pierskalla subdifferential is closely related to λ-quasiconjugate, in detail, see [7, 17, 19]. Next, we introduce the following conjugate function and subdifferential. The Q-conjugate of f, $f^Q : \mathbb{R}^{n+1} \to \overline{\mathbb{R}}$, is defined as follows: for each $(v, t) \in \mathbb{R}^{n+1}$,

$$f^Q(v, t) = -\inf\{f(x) \mid \langle v, x \rangle \geq t\}.$$

In addition, the Q-conjugate of $g : \mathbb{R}^{n+1} \to \overline{\mathbb{R}}$ is the function $g^{\underline{Q}} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that for each $x \in \mathbb{R}^n$

$$g^{\underline{Q}}(x) := -\inf\{g(v,t) \mid \langle v, x\rangle \geq t\},$$

and the Q-biconjugate of f is the function $f^{QQ} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that for each $x \in \mathbb{R}^n$

$$f^{QQ}(x) := (f^Q)^{\underline{Q}}(x) = -\inf\{f^Q(v,t) \mid \langle v, x\rangle \geq t\}.$$

Q-subdifferential of f at $x \in \mathbb{R}^n$ is defined as follows:

$$\partial^Q f(x) := \{(v,t) \in \mathbb{R}^{n+1} \mid \inf\{f(y) \mid \langle v, y\rangle \geq t\} \geq f(x), \langle v, x\rangle \geq t\}.$$

Greenberg-Pierskalla subdifferential, Q-conjugate and Q-subdifferential are special cases of Moreau's generalized conjugation, in detail, see [17,19,21,31]. We introduce the following previous results.

Theorem 2.1. *[17,19,21,31] Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Then, the following statements hold:*

(i) $f \geq f^{QQ}$.
(ii) $f = f^{QQ}$ *if and only if f is evenly quasiconvex.*
(iii) *If f is usc quasiconvex then $\partial f^Q(x) \neq \emptyset$ for each $x \in \mathbb{R}^n$.*
(iv) *Let $x_0 \in \mathbb{R}^n$. The following statements are equivalent:*
  (1) *x_0 is a global minimizer of f in $\mathbb{R}^n$,*
  (2) $0 \in \partial^{GP} f(x_0)$,
  (3) $\partial^{GP} f(x_0) = \mathbb{R}^n$,
  (4) $(0,0) \in \partial^Q f(x_0)$.
(v) $\partial^Q f(x) = \partial^Q(\lambda f)(x)$ *for each $x \in \mathbb{R}^n$ and $\lambda > 0$.*
(vi) $\partial f(x) \subset \partial^{GP} f(x) = \{v \in \mathbb{R}^n \mid (v, \langle v, x\rangle) \in \partial^Q f(x)\}$ *for each $x \in \mathbb{R}^n$.*

The following theorem is well known in convex analysis. The proof is easy and will be omitted. Motivated by the result, we study similar characterizations of quasiconvexity in the next section.

Theorem 2.2. *Let f be a real-valued function on $\mathbb{R}^n$. Then, f is convex if and only if $\partial f(x) \neq \emptyset$ for each $x \in \mathbb{R}^n$.*

3. Characterizations of quasiconvexity in terms of Q-subdifferential

In the following theorem, we show characterizations of quasiconvexity via usc functions in terms of Q-subdifferential.

Theorem 3.1. *Let f be an usc function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Then, the following statements are equivalent:*

(i) *f is quasiconvex,*
(ii) *f is evenly quasiconvex,*
(iii) *f is strictly evenly quasiconvex,*
(iv) $f = f^{QQ}$,
(v) *for each $x \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^n$ such that $(v, \langle v, x\rangle) \in \partial^Q f(x)$,*
(vi) *for each $x \in \mathbb{R}^n$, $\partial^Q f(x) \neq \emptyset$.*

Proof. We show the following implications:

1. (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii),

2. (ii) $\iff$ (iv),
3. (iii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (iv).

1. Assume that (iii) holds. Then, we can easily show that for each $\alpha \in \mathbb{R}$,

$$L(f, \leq, \alpha) = \bigcap_{\varepsilon > 0} L(f, <, \alpha + \varepsilon).$$

Since $L(f, <, \alpha + \varepsilon)$ is evenly convex for each $\varepsilon > 0$, $L(f, \leq, \alpha)$ is the intersection of some family of open halfspaces. This means that (iii) implies (ii). It is clear that (ii) implies (i). Since f is usc, if (i) holds then $L(f, <, \alpha)$ is open convex for each $\alpha \in \mathbb{R}$. This means that $L(f, <, \alpha)$ is evenly convex. Hence f is strictly evenly quasiconvex.

2. By Theorem 2.1, (ii) and (iv) are equivalent.

3. Assume that (iii) holds and let $x \in \mathbb{R}^n$. If $L(f, <, f(x)) = \emptyset$, then x is a global minimizer of f in $\mathbb{R}^n$. By Theorem 2.1, $(0, \langle 0, x\rangle) = (0, 0) \in \partial^Q f(x)$. Assume that $L(f, <, f(x)) \neq \emptyset$, then $f(x) > -\infty$. If $f(x) \in \mathbb{R}$, by the assumption, $L(f, <, f(x))$ is a nonempty evenly convex set and $x \notin L(f, <, f(x))$. Hence, there exist $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that for each $y \in L(f, <, f(x))$,

$$\langle v, x\rangle \geq \alpha > \langle v, y\rangle .$$

For each $y \in \mathbb{R}^n$,

$$\langle v, y\rangle \geq \langle v, x\rangle \Longrightarrow y \notin L(f, <, f(x)) \iff f(y) \geq f(x),$$

that is, $\inf\{f(y) \mid \langle v, y\rangle \geq \langle v, x\rangle\} \geq f(x)$. This shows that $(v, \langle v, x\rangle) \in \partial^Q f(x)$. Assume that $f(x) = \infty$, then $x \notin \mathrm{dom} f$. Since f is usc and strictly evenly quasiconvex, $\mathrm{dom} f$ is open convex. Hence, there exist $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that for each $y \in \mathrm{dom} f$,

$$\langle v, x\rangle \geq \alpha > \langle v, y\rangle .$$

For each $y \in \mathbb{R}^n$,

$$\langle v, y\rangle \geq \langle v, x\rangle \Longrightarrow y \notin \mathrm{dom} f \iff f(y) = \infty = f(x),$$

that is, $\inf\{f(y) \mid \langle v, y\rangle \geq \langle v, x\rangle\} \geq f(x)$. This shows that $(v, \langle v, x\rangle) \in \partial^Q f(x)$. Hence (iii) implies (v). Clearly, (v) implies (vi). Finally, we prove that (vi) implies (iv). Assume that (vi) holds and let $x \in \mathbb{R}^n$. Then there exists $(v, t) \in \partial f^Q(x)$. By Theorem 2.1 and the definition of Q-subdifferential,

$$f(x) \leq \inf\{f(y) \mid \langle v, y\rangle \geq t\} = -f^Q(v, t) \leq f^{QQ}(x) \leq f(x).$$

This shows that $f(x) = f^{QQ}(x)$. This completes the proof. □

4. Quasiconvexity of sum of quasiconvex functions

In this section, we study quasiconvexity of sum of quasiconvex functions. Especially, we introduce a set of quasiconvex functions which is closed under addition and multiplication by positive scalars. For the sake of simplicity, we assume that functions are real-valued.

By Theorem 3.1, we show the following characterization of quasiconvexity of sum of real-valued usc functions in terms of Q-subdifferential.

Theorem 4.1. *Let f and g be real-valued usc functions. Then, $f+g$ is quasiconvex if and only if for each $x \in \mathbb{R}^n$, $\partial^Q(f+g)(x) \neq \emptyset$.*

Next, we show sufficient conditions for quasiconvexity of sum of quasiconvex functions. The following lemma is important.

Lemma 4.2. *Let f and g be real-valued functions on $\mathbb{R}^n$. Then, for each $x \in \mathbb{R}^n$,*

$$\partial^Q(f+g)(x) \supset \partial^Q f(x) \cap \partial^Q g(x).$$

Proof. Let $x \in \mathbb{R}^n$ and $(v,t) \in \partial^Q f(x) \cap \partial^Q g(x)$. Then,

$$\begin{aligned} \inf\{(f+g)(y) \mid \langle v, y\rangle \geq t\} &\geq \inf\{f(y) \mid \langle v, y\rangle \geq t\} + \inf\{g(y) \mid \langle v, y\rangle \geq t\} \\ &\geq f(x) + g(x). \end{aligned}$$

This shows that $(v,t) \in \partial^Q(f+g)(x)$. □

Theorem 4.3. *Let f and g be real-valued usc quasiconvex functions. For each $x \in \mathbb{R}^n$, assume that at least one of the following conditions is satisfied:*

(i) $L(f,<,f(x)) \subset L(g,<,g(x))$,
(ii) $L(g,<,g(x)) \subset L(f,<,f(x))$,
(iii) $\partial^Q f(x) \subset \partial^Q g(x)$,
(iv) $\partial^Q g(x) \subset \partial^Q f(x)$.

Then, $f+g$ is quasiconvex.

In addition, for each $x \in \mathbb{R}^n$, (i) $\Longrightarrow$ (iv), (ii) $\Longrightarrow$ (iii).

Proof. Let $x \in \mathbb{R}^n$. At first, we show that (i) $\Longrightarrow$ (iv). Assume that (i) holds and $(v,t) \in \partial^Q g(x)$. Then

$$L(v,\geq,t) \subset L(g,\geq,g(x)) \subset L(f,\geq,f(x)).$$

This shows that $(v,t) \in \partial^Q f(x)$. Similarly, we can prove that (ii) $\Longrightarrow$ (iii).

Let $x \in \mathbb{R}^n$. By Theorem 3.1, $\partial^Q f(x) \neq \emptyset$ and $\partial^Q g(x) \neq \emptyset$ since f and g are usc quasiconvex. If (iii) holds, then

$$\partial^Q f(x) \cap \partial^Q g(x) = \partial^Q f(x) \neq \emptyset.$$

If (iv) holds, then

$$\partial^Q f(x) \cap \partial^Q g(x) = \partial^Q g(x) \neq \emptyset.$$

Hence, if at least one of the conditions (i), (ii), (iii), and (iv) is satisfied, then $\partial^Q f(x) \cap \partial^Q g(x) \neq \emptyset$. By Lemma 4.2, $\partial^Q(f+g)(x) \neq \emptyset$. Hence by Theorem 4.1, $f+g$ is quasiconvex. □

Next, we show a set of quasiconvex functions which is closed under addition and multiplication by positive scalars.

Theorem 4.4. *Let f be a real-valued usc quasiconvex function on $\mathbb{R}^n$, and $\mathcal{F}_f$ as follows:*

$$\mathcal{F}_f = \{g : \mathbb{R}^n \to \mathbb{R},\ usc \mid \partial^{GP} f(x) \subset \partial^{GP} g(x), \forall x \in \mathbb{R}^n\}.$$

Then, the following statements holds:

(i) *$f \in \mathcal{F}_f$,*
(ii) *a constant function on $\mathbb{R}^n$ is an element of $\mathcal{F}_f$,*
(iii) *for each $g \in \mathcal{F}_f$, g is quasiconvex,*

(iv) *for each* $g_1, \dots, g_m \in \mathcal{F}_f$, *and* $\alpha_1, \dots, \alpha_m > 0$, $\sum_{i=1}^m \alpha_i g_i \in \mathcal{F}_f$,
(v) *for each* $g_1, \dots, g_m \in \mathcal{F}_f$, $\max_{i=1,\dots,m} g_i \in \mathcal{F}_f$.

Proof. It is clear that (i) holds. For each constant function α and $x \in \mathbb{R}^n$, x is a global minimizer of α in $\mathbb{R}^n$. Hence, by Theorem 2.1, Greenberg-Pierskalla subdifferential of α at x is the whole space $\mathbb{R}^n$. This means that (ii) holds.

(iii) Let $g \in \mathcal{F}_f$ and $x \in \mathbb{R}^n$. By Theorem 3.1, there exists $v_0 \in \mathbb{R}^n$ such that $(v_0, \langle v_0, x\rangle) \in \partial^Q f(x)$ since f is usc quasiconvex. By Theorem 2.1,

$$v_0 \in \partial^{GP} f(x) \subset \partial^{GP} g(x) = \{v \in \mathbb{R}^n \mid (v, \langle v, x\rangle) \in \partial^Q g(x)\}.$$

By Theorem 3.1, g is quasiconvex.

(iv) Let $g_1, \dots, g_m \in \mathcal{F}_f$, $\alpha_1, \dots, \alpha_m > 0$, and $x \in \mathbb{R}^n$. By Lemma 4.2 and Theorem 2.1,

$$\begin{aligned} \partial^Q \left(\sum_{i=1}^m \alpha_i g_i\right)(x) &\supset \partial^Q (\alpha_1 g_1)(x) \cap \cdots \cap \partial^Q (\alpha_m g_m)(x) \\ &= \partial^Q g_1(x) \cap \cdots \cap \partial^Q g_m(x) \\ &\supset \partial^Q f(x). \end{aligned}$$

We can easily show that $\partial^{GP} f(x) \subset \partial^{GP} \left(\sum_{i=1}^m \alpha_i g_i\right)(x)$ by Theorem 2.1. Hence $\sum_{i=1}^m \alpha_i g_i \in \mathcal{F}_f$.

(v) Let $g_1, \dots, g_m \in \mathcal{F}_f$, and $x \in \mathbb{R}^n$. Then, there exists $i_0 \in \{1, \dots, m\}$ such that $\max_{i=1,\dots,m} g_i(x) = g_{i_0}(x)$. For each $v \in \partial^{GP} g_{i_0}(x)$,

$$\begin{aligned} \max_{i=1,\dots,m} g_i(x) &= g_{i_0}(x) \\ &\le \inf\{g_{i_0}(y) \mid \langle v, y\rangle \ge \langle v, x\rangle\} \\ &\le \inf\left\{\max_{i=1,\dots,m} g_i(y) \;\middle|\; \langle v, y\rangle \ge \langle v, x\rangle\right\}. \end{aligned}$$

This shows that $v \in \partial^{GP}\left(\max_{i=1,\dots,m} g_i\right)(x)$. Hence,

$$\partial^{GP} f(x) \subset \partial^{GP} g_{i_0}(x) \subset \partial^{GP}\left(\max_{i=1,\dots,m} g_i\right)(x).$$

This completes the proof. □

5. Applications to duality theorems

In this section, we investigate Lagrange duality and surrogate duality for quasiconvex programming as applications of our results. In addition, we show an example of $\mathcal{F}_f$, a set of quasiconvex functions which is closed under addition and multiplication by positive scalars.

Let f be a real-valued usc quasiconvex function, I an index set, f_0, $g_i \in \mathcal{F}_f$, and assume that $A = \{x \in \mathbb{R}^n \mid g_i(x) \le 0, \forall i \in I\} \neq \emptyset$. Consider the following quasiconvex programming problem (P):

$$(P)\begin{cases} \text{minimize } f_0(x), \\ \text{subject to } g_i(x) \le 0, \forall i \in I. \end{cases}$$

In convex and quasiconvex programming, two types of duality theorems have been studied mainly. The following equation is called Lagrange duality:

$$\inf_{x \in A} f_0(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in \mathbb{R}^n} \left\{ f_0(x) + \sum_{i \in I} \lambda_i g_i(x) \right\},$$

where $\mathbb{R}_+^{(I)} = \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\} \text{ is finite}\}$. The following equation is called surrogate duality:

$$\inf_{x \in A} f_0(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf \left\{ f_0(x) \,\middle|\, \sum_{i \in I} \lambda_i g_i(x) \leq 0 \right\}.$$

Sum of constraint functions and the objective function appears in both of these dualities. In convex programming, Lagrange duality have been studied mainly. Since the set of all convex functions is closed under addition and multiplication by positive scalars, Lagrange dual problem is also a convex programming problem. In quasiconvex programming, surrogate duality have been studied mainly. However, since the set of all quasiconvex functions is not closed under addition, we assume that constraint functions are convex, see [14, 29, 32, 34]. Then, surrogate dual problem is also a quasiconvex programming problem.

By using our results in this paper, the Lagrange dual problem and surrogate dual problem of (P) are also quasiconvex programming since $\mathcal{F}_f$ is closed under addition and multiplication by positive scalars. We can solve the problem (P) by using Lagrange and surrogate duality theorems.

Finally, we introduce an example of $\mathcal{F}_f$.

Example 5.1. Let f be the following real-valued continuous function on $\mathbb{R}^n$:

$$f(x) = \|x\|.$$

Then, $\mathcal{F}_f$ is the following set:

$$\mathcal{F}_f = \left\{ g : \mathbb{R}^n \to \mathbb{R}, \text{ usc} \,\middle|\, \begin{array}{l} \forall x \in \mathbb{R}^n, L(g, <, g(x)) \subset \{y \in \mathbb{R}^n \mid \langle x, y \rangle < \langle x, x \rangle\} \\ g(0) = \min_{x \in \mathbb{R}^n} g(x) \end{array} \right\}.$$

Let $g \in \mathcal{F}_f$. Since $0 \in \mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$, $0 \in \partial^{GP} f(0) \subset \partial^{GP} g(0)$ by Theorem 2.1. This shows that 0 is a global minimizer of g in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. We show that $x \in \partial^{GP} f(x)$. Actually, if $x = 0$, then $\partial^{GP} f(x) = \mathbb{R}^n$. Furthermore, if $x \neq 0$, then we can check that $\partial^{GP} f(x) = \{\lambda x \mid \lambda > 0\}$. Since $g \in \mathcal{F}_f$, $x \in \partial^{GP} f(x) \subset \partial^{GP} g(x)$. By the definition of Greenberg-Pierskalla subdifferential,

$$\inf\{g(y) \mid \langle x, y \rangle \geq \langle x, x \rangle\} \geq g(x),$$

that is,

$$L(g, <, g(x)) \subset \{y \in \mathbb{R}^n \mid \langle x, y \rangle < \langle x, x \rangle\}.$$

Let g be a real-valued usc function on $\mathbb{R}^n$. Assume that $L(g, <, g(x)) \subset \{y \in \mathbb{R}^n \mid \langle x, y \rangle < \langle x, x \rangle\}$ for each $x \in \mathbb{R}^n$ and $g(0) = \min_{x \in \mathbb{R}^n} g(x)$. Let $x \in \mathbb{R}^n$. If $x = 0$, then it is clear that $\partial^{GP} f(x) = \partial^{GP} g(x) = \mathbb{R}^n$. Assume that $x \neq 0$ and let $v \in \partial^{GP} f(x)$. Then, there exists $\lambda > 0$ such that $v = \lambda x$. By the assumption,

$$L(g, <, g(x)) \subset \{y \in \mathbb{R}^n \mid \langle x, y \rangle < \langle x, x \rangle\} = \{y \in \mathbb{R}^n \mid \langle \lambda x, y \rangle < \langle \lambda x, x \rangle\},$$

that is, $v = \lambda x \in \partial^{GP} g(x)$. This shows that $g \in \mathcal{F}_f$.

Additionally, we show the following inclusion:

$$\mathcal{F}_f \supset \{g : \mathbb{R}^n \to \mathbb{R} \mid \forall \alpha \in \mathbb{R}, \exists r \in [0, +\infty] \text{ s.t. } L(g, <, \alpha) = B(0, r)\},$$

where $B(0, r) = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ for each $r > 0$, $B(0, 0) = \emptyset$, and $B(0, \infty) = \mathbb{R}^n$.

Let g be a real-valued function, and assume that for each $\alpha \in \mathbb{R}$, there exists $r \in [0, +\infty]$ such that $L(g, <, \alpha) = B(0, r)$. It is clear that g is usc quasiconvex. Let $x \in \mathbb{R}^n$, then there exists $r_x \in [0, +\infty]$ such that $L(g, <, g(x)) = B(0, r_x)$. Since $x \notin L(g, <, g(x))$, $\|x\| \geq r_x$. Hence,

$$L(g, <, g(x)) = B(0, r_x) \subset B(0, \|x\|) = L(f, <, f(x)).$$

By Theorem 4.3, $\partial^{GP} f(x) \subset \partial^{GP} g(x)$, that is, $g \in \mathcal{F}_f$.

6. Conclusion

In this paper, we study quasiconvexity of sum of quasiconvex functions. We show characterizations of quasiconvexity via usc functions in terms of Q-subdifferential and Q-conjugate in Theorem 3.1. By using these characterizations, we show sufficient conditions for quasiconvexity of sum of quasiconvex functions in Theorem 4.3. In Theorem 4.4, we introduce $\mathcal{F}_f$, a set of quasiconvex functions which is closed under addition and multiplication by positive scalars. In addition, we show an example of $\mathcal{F}_f$ in Section 5. Finally, as applications, we study Lagrange duality and surrogate duality for quasiconvex programming by using our results.

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References

[1] M. Avriel, W. E. Diewert, S. Schaible and I. Zang, *Generalized concavity*, Math. Concepts Methods Sci. Engrg. Plenum Press, New York, 1988.

[2] J. P. Crouzeix and J. A. Ferland, *Criteria for quasiconvexity and pseudoconvexity: relationships and comparisons*, Math. Programming. **23** (1982), 193–205.

[3] J. P. Crouzeix and P. O. Lindberg, *Additively decomposed quasiconvex functions*, Math. Programming. **35** (1986), 42–57.

[4] A. Daniilidis, N. Hadjisavvas and J. E. Martínez-Legaz, *An appropriate subdifferential for quasiconvex functions*, SIAM J. Optim. **12** (2001), 407–420.

[5] G. Debreu and T. C. Koopmans, *Additively decomposed quasiconvex functions*, Math. Programming. **24** (1982), 1–38.

[6] D. H. Fang, X. F. Luo and X. Y. Wang, *Strong and Total Lagrange Dualities for Quasiconvex Programming*, J. Appl. Math. Art. ID 453912, (2014), 8 pp. DOI:10.1155/2014/453912.

[7] H. J. Greenberg and W. P. Pierskalla, *Quasi-conjugate functions and surrogate duality*, Cah. Cent. Étud. Rech. Opér. **15** (1973), 437–448.

[8] J. M. Gutiérrez Díez, *Infragradientes y Direcciones de Decrecimiento*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid. **78** (1984), 523–532.

[9] Y. Hu, X. Yang and C. K. Sim, *Inexact subgradient methods for quasi-convex optimization problems*, European J. Oper. Res. **240** (2015), 315–327.

[10] V. I. Ivanov, *First order characterizations of pseudoconvex functions*, Serdica Math. J. **27** (2001), 203–218.

[11] V. I. Ivanov, *Characterizations of pseudoconvex functions and semistrictly quasiconvex ones*, J. Global Optim. **57** (2013), 677–693.

[12] V. Jeyakumar, W. Oettli and M. Natividad, *A solvability theorem for a class of quasiconvex mappings with applications to optimization*, J. Math. Anal. Appl. **179** (1993), 537–546.

[13] N. T. H. Linh and J. P. Penot, *Optimality conditions for quasiconvex programs*, SIAM J. Optim. **17** (2006), 500–510.

[14] D. G. Luenberger, *Quasi-convex programming*, SIAM J. Appl. Math. **16** (1968), 1090–1095.

[15] J. E. Martínez-Legaz, *A generalized concept of conjugation*, Lecture Notes in Pure and Appl. Math. **86** (1983), 45–59.

[16] J. E. Martínez-Legaz, *A new approach to symmetric quasiconvex conjugacy*, Lecture Notes in Econom. and Math. Systems. **226** (1984), 42–48.

[17] J. E. Martínez-Legaz, *Quasiconvex duality theory by generalized conjugation methods*, Optimization. **19** (1988), 603–652.

[18] J. E. Martínez-Legaz and P. H. Sach, *A New Subdifferential in Quasiconvex Analysis*, J. Convex Anal. **6** (1999), 1–11.

[19] J. J. Moreau, *Inf-convolution, sous-additivité, convexité des fonctions numériques*, J. Math. Pures Appl. **49** (1970), 109–154.

[20] J. P. Penot, *What is quasiconvex analysis?*, Optimization. **47** (2000), 35–110.

[21] J. P. Penot, *Projective dualities for quasiconvex problems*, J. Global Optim. **62** (2015), 411–430.

[22] J. P. Penot and M. Volle, *On quasi-convex duality*, Math. Oper. Res. **15** (1990), 597–625.

[23] F. Plastria, *Lower subdifferentiable functions and their minimization by cutting planes*, J. Optim. Theory Appl. **46** (1985), 37–53.

[24] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, NJ, 1970.

[25] S. Suzuki and D. Kuroiwa, *Set containment characterization for quasiconvex programming*, J. Global Optim. **45** (2009), 551–563.

[26] S. Suzuki and D. Kuroiwa, *Optimality conditions and the basic constraint qualification for quasiconvex programming*, Nonlinear Anal. **74** (2011), 1279–1285.

[27] S. Suzuki and D. Kuroiwa, *Subdifferential calculus for a quasiconvex function with generator*, J. Math. Anal. Appl. **384** (2011), 677–682.

[28] S. Suzuki and D. Kuroiwa, *Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming*, Nonlinear Anal. **75** (2012), 2851–2858.

[29] S. Suzuki and D. Kuroiwa, *Necessary and sufficient constraint qualification for surrogate duality*, J. Optim. Theory Appl. **152** (2012), 366–367.

[30] S. Suzuki and D. Kuroiwa, *Some constraint qualifications for quasiconvex vector-valued systems*, J. Global Optim. **55** (2013), 539–548.

[31] S. Suzuki and D. Kuroiwa, *Characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential*, J. Global Optim. **62** (2015), 431–441.

[32] S. Suzuki and D. Kuroiwa, *A constraint qualification characterizing surrogate duality for quasiconvex programming*, Pac. J. Optim. **12** (2016), 87–100.

[33] S. Suzuki and D. Kuroiwa, *Generators and constraint qualifications for quasiconvex inequality systems*, J. Nonlinear Convex Anal. to appear.

[34] S. Suzuki, D. Kuroiwa and G. M. Lee, *Surrogate duality for robust optimization*, European J. Oper. Res. **231** (2013), 257–262.

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