Linear and Nonlinear Analysis
Volume 3, Number 2, 2017, 297–310

Yokohama Publishers
ISSN 2188-8167

ATTRACTIVE POINT AND WEAK CONVERGENCE THEOREMS FOR NORMALLY N-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

ATSUMASA KONDO* AND WATARU TAKAHASHI†

Abstract. In this paper, we introduce a new class of nonlinear mappings that contains generalized hybrid mappings [5], normally generalized hybrid mappings [17] and N-generalized hybrid mappings [9] as special cases. After proving an attractive point theorem that guarantees the existence of attractive points, we establish both Baillon's type and Mann's type weak convergence theorems of finding attractive points which are demonstrated without assuming that the domain of the mapping is closed. For the Baillon's type theorem, even convexity is dispensable. The results in this paper simultaneously extend many existing results in the literature.

1. Introduction

Throughout this paper, we denote a real Hilbert space by H and its inner product and norm by $\langle \cdot,\ \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty subset of H and let T be a mapping from C to H. The set of fixed points and attractive points of T are denoted by

$$\begin{aligned} F(T) &= \{u \in C : Tu = u\} \text{ and} \\ A(T) &= \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\}, \end{aligned}$$

respectively. The concept of attractive points was introduced by Takahashi and Takeuchi [15]. A mapping $T : C \to H$ is called:
(i) *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle x - y,\ Tx - Ty \rangle$ for all $x, y \in C$,
(ii) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.
It is well-known that the class of firmly nonexpansive mappings plays an important role in optimization theory in Hilbert spaces, and the class of nonexpansive mappings contains firmly nonexpansive mappings. It is easy to verify that a fixed point of a nonexpansive mapping is an attractive point. Baillon [3] proved the following mean convergence theorem for nonexpansive mappings in a Hilbert space.

2010 *Mathematics Subject Classification.* 47H10, 47H05.
Key words and phrases. Attractive point, fixed point, normally N-generalized hybrid mappings, weak convergence.
*The first author is supported by the Ryousui Gakujutsu Foundation of Shiga University.
†The second author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

Theorem 1.1. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, the sequence $\left\{S_n z \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ weakly converges to a fixed point of T.*

The following theorem of Mann's type iteration [8] was proved by Reich [10].

Theorem 1.2. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$. Let $\{x_n\}$ be a sequence in C defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n \quad (n = 1, 2, \cdots),$$

where $x_1 \in C$ is given. Then, $\{x_n\}$ weakly converges to a fixed point of T.

To extend the above theorems, various classes of nonlinear mappings have been proposed. A mapping $T : C \to C$ is called

(iii) *nonspreading* [6] if $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2$ for all $x, y \in C$,

(iv) *hybrid* [14] if $3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|x - Ty\|^2$ for all $x, y \in C$,

(v) *generalized hybrid* [5] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \text{ for all } x, y \in C.$$

The classes of nonexpansive, nonspreading and hybrid mappings are all generalizations of firmly nonexpansive mappings. The class of generalized hybrid mappings contains all mappings (i)–(iv). For mappings (ii)–(v), fixed and attractive point approximation methods have been extensively studied. Takahashi and Yao [19] studied nonexpansive, nonspreading and hybrid mappings. They proved fixed point theorems and demonstrated Baillon's type weak convergence theorems. Hojo and Takahashi [4] proved weak and strong convergence theorems of finding a fixed point for a generalized hybrid mapping in a Hilbert space. Takahashi and Yao [18] studied generalized hybrid mappings in a Banach space and proved a Mann's type weak convergence theorem.

Takahashi, Wong and Yao [17] and Maruyama, Takahashi and Yao [9] introduced new types of nonlinear mappings, which are more general than mappings (i)–(v). A mapping $T : C \to C$ is called

(vi) *normally generalized hybrid* [17] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(a) $\alpha + \beta + \gamma + \delta \geq 0$,

(b) $\alpha + \beta > 0$, or $\alpha + \gamma > 0$,

(c) $\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \leq 0$ for all $x, y \in C$,

(vii) *N-generalized hybrid* [9] if there exist $(\alpha_n, \beta_n)_{n=0}^{N} \in \mathbb{R}^{2(N+1)}$ such that

(a) $\sum_{n=0}^{N} \alpha_n = \sum_{n=0}^{N} \beta_n = 1$,

(b) $\sum_{n=0}^{N} \alpha_n \|T^n x - Ty\|^2 \leq \sum_{n=0}^{N} \beta_n \|T^n x - y\|^2$ for all $x, y \in C$.

The classes of normally generalized hybrid mappings and N-generalized hybrid mappings contain generalized hybrid mappings. For methods of fixed/attractive point approximation, see [20], [1] and [2], in addition to [17] and [9].

In this paper, we propose a new class of nonlinear mappings that contains all the mappings (i)–(vii) as special cases and establish three types of results. First, an attractive point theorem is proved (Theorem 3.3 in Section 3). Second, a weak convergence theorem of Baillon's type is established: an averaged sequence converges weakly to an attractive point (Theorem 4.2 in Section 4). Finally, we consider a Mann's type weak convergence theorem of finding an attractive point (Theorem 5.2 in Section 5) that shows how to approximate attractive points. While the Mann's type weak convergence theorem is proved without assuming that the domain of a mapping is closed, the Baillon's type theorem does not require even convexity. The same types of results regarding fixed points are also established as Theorems 3.5, 4.3 and 5.3.

2. Preliminaries and lemmas

In this section, we briefly present background information and results. For more details, see Takahashi [12], [13] and earlier studies. We know that

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,\ y\rangle + \|y\|^2 \tag{2.1}$$

for all $x, y \in H$. Let C be a nonempty subset of H. For $T : C \to H$ and $u \in H$, it is easy to verify that $u \in A(T)$ if and only if

$$\|Ty-y\|^2 + 2\langle Ty-y,\ y-u\rangle \leq 0 \tag{2.2}$$

for all $y \in C$. Indeed, for $u \in H$,

$$\begin{aligned}
&u \in A(T)\\
&\Longleftrightarrow \|Ty-u\|^2 \leq \|y-u\|^2 \text{ for all } y \in C\\
&\Longleftrightarrow \|Ty-y\|^2 + 2\langle Ty-y,\ y-u\rangle + \|y-u\|^2 \leq \|y-u\|^2 \text{ for all } y \in C\\
&\Longleftrightarrow \|Ty-y\|^2 + 2\langle Ty-y,\ y-u\rangle \leq 0 \text{ for all } y \in C.
\end{aligned}$$

A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx-u\| \leq \|x-u\|,$$

for all $x \in C$ and $u \in F(T)$. It is well-known that all types of mappings (i)–(vii) with $F(T) \neq \emptyset$ are quasi-nonexpansive. Furthermore, we know that the set of fixed points $F(T)$ of a quasi-nonexpansive mapping is closed and convex (see [5]), which plays important roles in the existing literature.

Strong and weak convergence of a sequence $\{x_n\}$ in H to a point $x \in H$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively, where weak convergence $x_n \rightharpoonup x$ means that $\langle x_n,\ y\rangle \to \langle x,\ y\rangle$ for all $y \in H$. It is well-known that a weakly convergent sequence in a Hilbert space H is bounded. Furthermore, if $x_n \to x$ and $y_n \rightharpoonup y$, then $\langle x_n,\ y_n\rangle \to \langle x,\ y\rangle$. We also know that a Hilbert space satisfies *Opial's condition*, that is, if a sequence $\{x_n\}$ in H satisfies $x_n \rightharpoonup u$ and $v \neq u$, then

$$\liminf_{n\to\infty} \|x_n-u\| < \liminf_{n\to\infty} \|x_n-v\|.$$

We denote the set of natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let l^∞ be the Banach space of bounded real sequences, that is,

$$l^\infty = \left\{\{x_n\} : x_n \in \mathbb{R}\ (\forall n \in \mathbb{N}) \text{ and } \sup_{n\in\mathbb{N}} |x_n| < \infty\right\},$$

where the norm of l^∞ is the supremum norm. Let $(l^\infty)^*$ be the dual space of l^∞, and $\mu \in (l^\infty)^*$. For simplicity, we often denote $\mu(\{x_n\})$ by $\mu_n x_n$ if no ambiguity arises.

A linear continuous functional $\mu \in (l^\infty)^*$ is called a *mean* if $\mu(\{1,1,1,\cdots\}) = \|\mu\| = 1$. When a mean additionally satisfies $\mu_n(x_n) = \mu_n(x_{n+1})$, it is called a *Banach limit* on l^∞. It is well-known that the Banach limit exists, which is demonstrated using the Hahn–Banach theorem, and for $\{x_n\} \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n x_n \le \limsup_{n\to\infty} x_n. \tag{2.3}$$

As a direct consequence from (2.3), if $x_n \to a \in \mathbb{R}$, then $\mu_n x_n = a$; see [12].

Let C be a nonempty, closed and convex subset of H. We know that for any $x \in H$, there exists a unique nearest point $u \in C$, that is, $\|x-u\| = \inf_{z\in C}\|x-z\|$. This correspondence is called the *metric projection*, and is denoted by P_C, that is, $P_C x = u$. The metric projection is an example of firmly nonexpansive mappings, and thus, nonexpansive mappings. We know that if P_C is the metric projection of H onto C, then

$$\langle x - P_C x,\ P_C x - z\rangle \ge 0$$

for all $x \in H$ and $z \in C$.

The following lemmas are used in the proofs of the main theorems of this paper.

Lemma 2.1 ([7, 11])**.** *Let μ be a mean on l^∞. Then, for any bounded sequence $\{x_n\}$ in H, there is a unique element $u \in \overline{co}\{x_n\}$ such that*

$$\mu_n \langle x_n,\ v\rangle = \langle u,\ v\rangle$$

for all $v \in H$, where $\overline{co}\{x_n\}$ is the closure of the convex hull of $\{x_n : n \in \mathbb{N}\}$.

Lemma 2.2 ([15])**.** *Let C be a nonempty, closed and convex subset of H, let T be a mapping from C to itself and let P_C be the metric projection of H onto C. Assume that $A(T) \ne \emptyset$. Then, if $u \in A(T)$, then $P_C u \in F(T)$. Thus, if $A(T) \ne \emptyset$, then $F(T) \ne \emptyset$.*

Lemma 2.3 ([15])**.** *Let C be a nonempty subset of H and let T be a mapping from C to H. Then, $A(T)$ is a closed and convex subset of H.*

From this lemma, if $A(T) \ne \emptyset$, then the metric projection $P_{A(T)}$ of H onto $A(T)$ is well-defined.

Lemma 2.4 ([16])**.** *Let A be a nonempty, closed and convex subset of H, let P_A be te metric projection of H onto A and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - q\| \le \|x_n - q\|$ for all $q \in A$ and $n \in \mathbb{N}$, then $\{P_A x_n\}$ is a convergent sequence in A.*

Lemma 2.5 ([15])**.** *Let C be a nonempty subset of H and let T be a mapping from C to H. Then, $A(T) \cap C \subset F(T)$.*

Proof. Let $u \in A(T) \cap C$. We will prove that $u = Tu$. Since $u \in A(T)$, it holds that

$$\|Tx - u\| \le \|x - u\| \tag{2.4}$$

for all $x \in C$. Since $u \in C$, substituting $x = u$ into (2.4), we obtain the desired result. □

Lemma 2.6 ([9]). *Let x, y, z be elements of a Hilbert space H and let a, b, c be real numbers such that $a + b + c = 1$. Then,*

$$\|ax + by + cz\|^2$$
$$= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2.$$

Additionally, if $a, b, c \in [0, 1]$, then

$$\|ax + by + cz\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2.$$

Lemma 2.7. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let $u \in H$. Then, $x_n \rightharpoonup u$ is equivalent to the following condition: for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$, it holds $w = u$.*

3. Attractive and fixed point theorems

In this section, we introduce a broad class of nonlinear mappings that contains all the mappings (i)–(vii) mentioned in Introduction and then prove the existence of attractive and fixed points.

Definition 3.1. A mapping $T : C \to C$ is called *normally N-generalized hybrid* if there exist real numbers $(\alpha_n, \beta_n)_{n=0}^N \in \mathbb{R}^{2(N+1)}$ such that

(a) $\sum_{n=0}^N (\alpha_n + \beta_n) \geq 0$;
(b) $\sum_{n=0}^N \alpha_n > 0$;
(c) $\sum_{n=0}^N \alpha_n \|T^n x - Ty\|^2 + \sum_{n=0}^N \beta_n \|T^n x - y\|^2 \leq 0$ for all $x, y \in C$.

We also call such a mapping $(\alpha_n, \beta_n)_{n=0}^N$-normally N-generalized hybrid.

The class of normally N-generalized hybrid mappings contains all the mappings (i)–(vii) mentioned in Introduction. In fact, if $\sum_{n=0}^N \alpha_n = 1$ and $\sum_{n=0}^N \beta_n = -1$, it is an N-generalized hybrid mapping, and if $N = 1$, it is a normally generalized hybrid mapping. The class of normally generalized hybrid mappings contains the mappings (i)–(v). In what follows, we consider the case of $N = 2$ because the generalization for the case in which T is a normally N-generalized hybrid mapping is straightforward. A normally 2-generalized hybrid mapping that has a fixed point is quasi-nonexpansive.

Proposition 3.2. *Let C be a nonempty subset of H and let $T : C \to C$ be a $(\alpha_n, \beta_n)_{n=0}^2$-normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Then, T is quasi-nonexpansive.*

Proof. Let $y \in C$ and $u \in F(T)$. We will prove that $\|Ty - u\| \leq \|y - u\|$. Since T is $(\alpha_n, \beta_n)_{n=0}^2$-normally 2-generalized hybrid,

$$\alpha_2 \left\|T^2 u - Ty\right\|^2 + \alpha_1 \|Tu - Ty\|^2 + \alpha_0 \|u - Ty\|^2$$
$$+ \beta_2 \left\|T^2 u - y\right\|^2 + \beta_1 \|Tu - y\|^2 + \beta_0 \|u - y\|^2 \leq 0.$$

Using $u = Tu$, we have

$$(\alpha_2 + \alpha_1 + \alpha_0)\|u - Ty\|^2 + (\beta_2 + \beta_1 + \beta_0)\|u - y\|^2 \leq 0.$$

Since $\sum_{n=0}^{2}(\alpha_n+\beta_n)\geq 0$, it holds that

$$\begin{aligned}(\alpha_2+\alpha_1+\alpha_0)\|u-Ty\|^2 &\leq -(\beta_2+\beta_1+\beta_0)\|u-y\|^2\\ &\leq (\alpha_2+\alpha_1+\alpha_0)\|u-y\|^2.\end{aligned}$$

Since $\sum_{n=0}^{2}\alpha_n>0$, we obtain the desired result. □

The following theorem asserts that a normally 2-generalized hybrid mapping has an attractive point in H.

Theorem 3.3. *Let C be a nonempty subset of H and let $T: C\to C$ be a $(\alpha_n,\beta_n)_{n=0}^{2}$-normally 2-generalized hybrid mapping. Assume that there exists $z\in C$ such that $\{T^n z\}$ is a bounded sequence in C. Then, $A(T)\neq\emptyset$.*

Proof. Let $\mu\in(l^\infty)^*$ be a Banach limit. From Lemma 2.1, for the bounded sequence $\{T^n z\}$, there is a unique element $u\in H$ such that

$$\mu_n\langle T^n z,\ v\rangle=\langle u,\ v\rangle \tag{3.1}$$

for all $v\in H$. Let $y\in C$. We will show that

$$\|Ty-y\|^2+2\langle Ty-y,\ y-u\rangle\leq 0,$$

which means that $u\in A(T)$ from (2.2).

Since T is a normally 2-generalized hybrid mapping,

$$\begin{aligned}\alpha_2\left\|T^{n+2}z-Ty\right\|^2+\alpha_1\left\|T^{n+1}z-Ty\right\|^2+\alpha_0\|T^n z-Ty\|^2&\\ +\beta_2\left\|T^{n+2}z-y\right\|^2+\beta_1\left\|T^{n+1}z-y\right\|^2+\beta_0\|T^n z-y\|^2&\leq 0.\end{aligned}$$

Since $\{T^n z\}$ is bounded, we can apply the Banach limit μ to both sides of the inequality. Then, we obtain

$$(\alpha_2+\alpha_1+\alpha_0)\mu_n\|T^n z-Ty\|^2+(\beta_2+\beta_1+\beta_0)\mu_n\|T^n z-y\|^2\leq 0.$$

Thus, from (2.1),

$$\begin{aligned}(\alpha_2+\alpha_1+\alpha_0)\mu_n\left[\|T^n z-y\|^2+2\langle T^n z-y,\ y-Ty\rangle+\|y-Ty\|^2\right]&\\ +(\beta_2+\beta_1+\beta_0)\mu_n\|T^n z-y\|^2&\leq 0.\end{aligned}$$

Since $\alpha_0+\alpha_1+\alpha_2+\beta_0+\beta_1+\beta_2\geq 0$, the following holds

$$(\alpha_2+\alpha_1+\alpha_0)\mu_n\left[2\langle T^n z-y,\ y-Ty\rangle+\|y-Ty\|^2\right]\leq 0.$$

Since $\alpha_0+\alpha_1+\alpha_2>0$, we have from (3.1) that

$$2\langle u-y,\ y-Ty\rangle+\|y-Ty\|^2\leq 0.$$

This completes the proof. □

We can easily obtain the following corollary from Theorem 3.3.

Corollary 3.4. *Let C be a nonempty subset of H and let $T: C\to C$ be a normally 2-generalized hybrid mapping. Then, the following three statements are equivalent:*

(I) *for any $x\in C$, $\{T^n x\}$ is a bounded sequence in C,*

(II) *there exists $z\in C$ such that $\{T^n z\}$ is a bounded sequence in C,*

(III) $A(T) \neq \emptyset$.

Proof. (I) $\Longrightarrow$ (II) obviously holds and (II) $\Longrightarrow$ (III) was already established as the previous theorem.

We will prove (III) $\Longrightarrow$ (I). Let $x \in C$, $u \in A(T)$ and $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|T^n x\| &\leq \|T^n x - u\| + \|u\| \\ &\leq \left\|T^{n-1}x - u\right\| + \|u\| \\ &\quad \cdots\cdots \\ &\leq \|x - u\| + \|u\| . \end{aligned}$$

This shows that the sequence $\{T^n x\}$ is bounded. □

Adding that C is closed and convex in Theorem 3.3, we can obtain the following fixed point theorem:

Theorem 3.5. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Assume that there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. Then, $F(T) \neq \emptyset$.*

Proof. From the previous theorem, we have $A(T) \neq \emptyset$. Then, from Lemma 2.2, we obtain the desired result. □

The following corollary is directly derived from Corollary 3.4 and Theorem 3.5.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Then, the following four statements are equivalent:*

(I) *for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C,*
(II) *there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C,*
(III) $A(T) \neq \emptyset$,
(IV) $F(T) \neq \emptyset$.

4. Weak convergence theorem of Baillon's type

In this section, we establish a mean convergence theorem of Baillon's type without relying on either the convexity or closedness of C for normally 2-generalized mappings in a Hilbert space. The basic technique of the proof was developed by Takahashi [11]. We start with proving the following lemma.

Lemma 4.1. *Let C be a nonempty subset of H and let $T : C \to C$ be a $(\alpha_n, \beta_n)_{n=0}^{2}$-normally 2-generalized hybrid mapping. Assume that there is an element $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. Define $S_n z \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k z$ and assume $S_{n_i} z \rightharpoonup u$, where $\{S_{n_i} z\}$ is a subsequence of $\{S_n z\}$. Then $u \in A(T)$.*

Proof. Let $y \in C$. We will prove that $\|Ty - y\|^2 + 2\langle Ty - y,\ y - u\rangle \leq 0$. Since T is a normally 2-generalized hybrid mapping, we have

$$\begin{aligned} \alpha_2 \left\|T^{k+2}z - Ty\right\|^2 + \alpha_1 \left\|T^{k+1}z - Ty\right\|^2 + \alpha_0 \left\|T^k z - Ty\right\|^2 \\ + \beta_2 \left\|T^{k+2}z - y\right\|^2 + \beta_1 \left\|T^{k+1}z - y\right\|^2 + \beta_0 \left\|T^k z - y\right\|^2 \leq 0 \end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$. We have from (2.1) that

$$\begin{aligned}
&\alpha_2 \left[\left\| T^{k+2}z - y \right\|^2 + 2 \left\langle T^{k+2}z - y,\ y - Ty \right\rangle + \|y - Ty\|^2 \right] \\
&\quad + \alpha_1 \left[\left\| T^{k+1}z - y \right\|^2 + 2 \left\langle T^{k+1}z - y,\ y - Ty \right\rangle + \|y - Ty\|^2 \right] \\
&\quad + \alpha_0 \left[\left\| T^{k}z - y \right\|^2 + 2 \left\langle T^{k}z - y,\ y - Ty \right\rangle + \|y - Ty\|^2 \right] \\
&\quad + \beta_2 \left\| T^{k+2}z - y \right\|^2 + \beta_1 \left\| T^{k+1}z - y \right\|^2 + \beta_0 \left\| T^{k}z - y \right\|^2 \leq 0.
\end{aligned}$$

Thus, it holds that

$$\begin{aligned}
&(\alpha_2 + \alpha_1 + \alpha_0) \|y - Ty\|^2 + (\alpha_2 + \beta_2) \left\| T^{k+2}z - y \right\|^2 \\
&\quad + (\alpha_1 + \beta_1) \left\| T^{k+1}z - y \right\|^2 + (\alpha_0 + \beta_0) \left\| T^{k}z - y \right\|^2 \\
&\quad + 2 \left\langle \alpha_2 T^{k+2}z + \alpha_1 T^{k+1}z + \alpha_0 T^k z - (\alpha_2 + \alpha_1 + \alpha_0) y,\ y - Ty \right\rangle \leq 0.
\end{aligned}$$

Using the condition $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$, we obtain

$$\begin{aligned}
&(\alpha_2 + \alpha_1 + \alpha_0) \|y - Ty\|^2 + (\alpha_2 + \beta_2) \left\| T^{k+2}z - y \right\|^2 \\
&\quad + (\alpha_1 + \beta_1) \left\| T^{k+1}z - y \right\|^2 - [(\alpha_2 + \beta_2) + (\alpha_1 + \beta_1)] \left\| T^{k}z - y \right\|^2 \\
&\quad + 2\langle \alpha_2 T^{k+2}z + \alpha_1 T^{k+1}z + (\alpha_2 + \alpha_1 + \alpha_0) T^k z - (\alpha_2 + \alpha_1) T^k z \\
&\qquad - (\alpha_2 + \alpha_1 + \alpha_0) y, \quad y - Ty \rangle \leq 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\alpha_2 + \alpha_1 + \alpha_0) \|y - Ty\|^2 + (\alpha_2 + \beta_2) \left(\left\| T^{k+2}z - y \right\|^2 - \left\| T^{k}z - y \right\|^2 \right) \\
&\quad + (\alpha_1 + \beta_1) \left(\left\| T^{k+1}z - y \right\|^2 - \left\| T^{k}z - y \right\|^2 \right) \\
&\quad + 2\langle \alpha_2 \left(T^{k+2}z - T^k z \right) + \alpha_1 \left(T^{k+1}z - T^k z \right) + (\alpha_2 + \alpha_1 + \alpha_0) T^k z \\
&\qquad - (\alpha_2 + \alpha_1 + \alpha_0) y, \quad y - Ty \rangle \leq 0.
\end{aligned}$$

Summing these inequalities with respect to k from 0 to $n-1$, we have

$$\begin{aligned}
&(\alpha_2+\alpha_1+\alpha_0)\, n \|y-Ty\|^2 \\
&\qquad + (\alpha_2+\beta_2)\left(\left\|T^{n+1}z-y\right\|^2 + \|T^n z-y\|^2 - \|Tz-y\|^2 - \|z-y\|^2\right) \\
&\qquad\qquad + (\alpha_1+\beta_1)\left(\|T^n z-y\|^2 - \|z-y\|^2\right) \\
&\quad + 2\langle \alpha_2\left(T^{n+1}z+T^n z-Tz-z\right) + \alpha_1\left(T^n z-z\right) + (\alpha_2+\alpha_1+\alpha_0)\sum_{k=0}^{n-1} T^k z \\
&\qquad\qquad - (\alpha_2+\alpha_1+\alpha_0)\, ny, \quad y-Ty\rangle \leq 0
\end{aligned}$$

for all $n \in \mathbb{N}$. Dividing by n, we have

$$\begin{aligned}
&(\alpha_2+\alpha_1+\alpha_0) \|y-Ty\|^2 \\
&\qquad + \frac{\alpha_2+\beta_2}{n}\left(\left\|T^{n+1}z-y\right\|^2 + \|T^n z-y\|^2 - \|Tz-y\|^2 - \|z-y\|^2\right) \\
&\qquad\qquad + \frac{\alpha_1+\beta_1}{n}\left(\|T^n z-y\|^2 - \|z-y\|^2\right) \\
&\quad + 2\langle \frac{\alpha_2}{n}\left(T^{n+1}z+T^n z-Tz-z\right) + \frac{\alpha_1}{n}\left(T^n z-z\right) + (\alpha_2+\alpha_1+\alpha_0) S_n z \\
&\qquad\qquad - (\alpha_2+\alpha_1+\alpha_0)\, y, \quad y-Ty\rangle \leq 0.
\end{aligned}$$

Replacing n by n_i and taking the limit as $i \to \infty$, we obtain

$$(\alpha_2+\alpha_1+\alpha_0)\|y-Ty\|^2 + 2\left\langle (\alpha_2+\alpha_1+\alpha_0)\, u - (\alpha_2+\alpha_1+\alpha_0)\, y,\ y-Ty\right\rangle \leq 0.$$

Since $\alpha_2+\alpha_1+\alpha_0 > 0$, we obtain that $\|y-Ty\|^2 + 2\langle u-y,\ y-Ty\rangle \leq 0$ for all $y \in C$. This means $u \in A(T)$. □

Theorem 4.2. *Let C be a nonempty subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$. Let $P_{A(T)}$ be the metric projection from H onto $A(T)$. Then, for any $x \in C$, the sequence $\left\{S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ converges weakly to $u \in A(T)$, where $u = \lim_{n\to\infty} P_{A(T)} T^n x$.*

Proof. Note from Lemma 2.3 that the metric projection $P_{A(T)}$ is well-defined. Let $x \in C$ and define $S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x \in H$. Since $A(T) \neq \emptyset$, from Corollary 3.4, $\{T^n x\}$ is bounded. It is obvious that

$$\left\|T^{n+1}x - q\right\| \leq \|T^n x - q\| \tag{4.1}$$

for all $q \in A(T)$ and $n \in \mathbb{N}$. Thus, from Lemma 2.4, the sequence $\left\{P_{A(T)} T^n x\right\}$ is convergent in $A(T)$.

Let $v \equiv \lim_{n\to\infty} P_{A(T)} T^n x \in A(T)$. To prove $S_n x \rightharpoonup v$, we will show that if $\{S_{n_i} x\}$ is a subsequence of $\{S_n x\}$ such that $S_{n_i} x \rightharpoonup u$, then $u = v$. From Lemma 4.1, we have that $u \in A(T)$. Note that the real sequence $\left\{\left\|T^n x - P_{A(T)} T^n x\right\|\right\}$ is monotone decreasing. Indeed, from $P_{A(T)} T^n x \in A(T)$ and (4.1),

$$\begin{aligned}
\left\|T^{n+1}x - P_{A(T)} T^{n+1} x\right\| &\leq \left\|T^{n+1}x - P_{A(T)} T^n x\right\| \\
&\leq \left\|T^n x - P_{A(T)} T^n x\right\|
\end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $P_{A(T)}$ is the metric projection from H onto $A(T)$ and $u \in A(T)$,

$$\left\langle T^k x - P_{A(T)} T^k x,\ P_{A(T)} T^k x - u \right\rangle \geq 0$$

for all $k \in \mathbb{N} \cup \{0\}$. Thus,

$$\left\langle T^k x - P_{A(T)} T^k x,\ P_{A(T)} T^k x - v + v - u \right\rangle \geq 0.$$

Using (4.2), we have

$$\begin{aligned} \left\langle T^k x - P_{A(T)} T^k x,\ -(v-u) \right\rangle &\leq \left\langle T^k x - P_{A(T)} T^k x,\ P_{A(T)} T^k x - v \right\rangle \\ &\leq \left\| T^k x - P_{A(T)} T^k x \right\| \left\| P_{A(T)} T^k x - v \right\| \\ &\leq \left\| x - P_{A(T)} x \right\| \left\| P_{A(T)} T^k x - v \right\| \end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$. Summing these inequalities with respect to k from 0 to $n-1$, we obtain

$$\left\langle \sum_{k=0}^{n-1} T^k x - \sum_{k=0}^{n-1} P_{A(T)} T^k x,\ -(v-u) \right\rangle \leq \left\| x - P_{A(T)} x \right\| \cdot \sum_{k=0}^{n-1} \left\| P_{A(T)} T^k x - v \right\|.$$

Dividing by n, we have

$$\left\langle S_n x - \frac{1}{n} \sum_{k=0}^{n-1} P_{A(T)} T^k x,\ -(v-u) \right\rangle \leq \left\| x - P_{A(T)} x \right\| \cdot \frac{1}{n} \sum_{k=0}^{n-1} \left\| P_{A(T)} T^k x - v \right\|.$$

Replacing n by n_{ij} and taking the limit as $j \to \infty$, we obtain that

$$\langle u - v,\ -(v-u) \rangle \leq 0.$$

This implies that $u = v$. □

Adding that C is closed and convex in Theorem 4.2, we obtain a Baillon's type weak convergence theorem of finding a fixed point for a normally 2-generalized hybrid mapping in Hilbert spaces.

Theorem 4.3. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, the sequence $\left\{ S_n x \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^k x \right\}$ converges weakly to $u \in F(T)$.*

Proof. Since $F(T) \neq \emptyset$ is assumed, from Corollary 3.6, we have $A(T) \neq \emptyset$. Furthermore, from Lemma 2.3, $A(T)$ is a closed and convex subset of H. Thus, the metric projection $P_{A(T)}$ and is well-defined. Let $x \in C$ and define $S_n x \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^k x \in C$. From Theorem 4.2, $S_n x$ converges weakly to $u = \lim_{n \to \infty} P_{A(T)} T^n x \in A(T)$.

To complete the proof, we will show that $u \in F(T)$. Since C is convex, $\{S_n x\}$ is a sequence in C. Since C is weakly closed in H and $S_n x \rightharpoonup u$, we have $u \in C$. Since $u \in A(T) \cap C$, from Lemma 2.5, we obtain $u \in F(T)$. □

5. Weak convergence theorem of Mann's type

In this section, we present a Mann's type weak convergence theorem for normally 2-generalized hybrid mappings to their attractive points in Hilbert spaces. At the outset, an additional lemma is prepared.

Lemma 5.1. *Let C be a nonempty subset of H and let $T : C \to C$ be a $(\alpha_n, \beta_n)_{n=0}^{2}$-normally 2-generalized hybrid mapping. If a sequence $\{x_n\}$ in C satisfies $Tx_n - x_n \to 0$, $T^2x_n - x_n \to 0$ and $x_n \rightharpoonup u$, then $u \in A(T)$.*

Proof. Let $y \in C$. We will show that

$$\|Ty - y\|^2 + 2\langle Ty - y,\ y - u\rangle \le 0,$$

which means that $u \in A(T)$. Since T is $(\alpha_n, \beta_n)_{n=0}^{2}$-normally 2-generalized hybrid,

$$\begin{aligned}\alpha_2 \left\|T^2x_n - Ty\right\|^2 + \alpha_1 \|Tx_n - Ty\|^2 + \alpha_0 \|x_n - Ty\|^2 \\ + \beta_2 \left\|T^2x_n - y\right\|^2 + \beta_1 \|Tx_n - y\|^2 + \beta_0 \|x_n - y\|^2 \le 0.\end{aligned}$$

Using (2.1), we have

$$\begin{aligned}&\alpha_2 \left[\left\|T^2x_n - x_n\right\|^2 + 2\left\langle T^2x_n - x_n,\ x_n - Ty\right\rangle + \|x_n - Ty\|^2\right] \\ &+ \alpha_1 \left[\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n,\ x_n - Ty\rangle + \|x_n - Ty\|^2\right] + \alpha_0 \|x_n - Ty\|^2 \\ &+ \beta_2 \left[\left\|T^2x_n - x_n\right\|^2 + 2\left\langle T^2x_n - x_n,\ x_n - y\right\rangle + \|x_n - y\|^2\right] \\ &+ \beta_1 \left[\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n,\ x_n - y\rangle + \|x_n - y\|^2\right] + \beta_0 \|x_n - y\|^2 \le 0.\end{aligned}$$

Since $\{x_n\}, \{Tx_n\}$ and $\{T^2x_n\}$ are bounded sequences, we can apply a Banach limit $\mu \in (l^\infty)^*$ to both sides of the inequality. From $Tx_n - x_n \to 0$ and $T^2x_n - x_n \to 0$, we have $\|T^2x_n - x_n\|^2 \to 0$, $\|Tx_n - x_n\|^2 \to 0$, $\langle T^2x_n - x_n, x_n - Ty\rangle \to 0$ and $\langle Tx_n - x_n,\ x_n - y\rangle \to 0$. Then, we obtain

$$(\alpha_2 + \alpha_1 + \alpha_0)\,\mu_n \|x_n - Ty\|^2 + (\beta_2 + \beta_1 + \beta_0)\,\mu_n \|x_n - y\|^2 \le 0.$$

Since $\alpha_0 + \alpha_1 + \alpha_2 + \beta_0 + \beta_1 + \beta_2 \ge 0$,

$$\begin{aligned}(\alpha_2 + \alpha_1 + \alpha_0)\,\mu_n \|x_n - Ty\|^2 &\le -(\beta_2 + \beta_1 + \beta_0)\,\mu_n \|x_n - y\|^2 \\ &\le (\alpha_2 + \alpha_1 + \alpha_0)\,\mu_n \|x_n - y\|^2.\end{aligned}$$

It holds from $\alpha_2 + \alpha_1 + \alpha_0 > 0$ that

$$\mu_n \|x_n - Ty\|^2 \le \mu_n \|x_n - y\|^2.$$

We obtain

$$\mu_n \left[\|x_n - y\|^2 + 2\langle x_n - y,\ y - Ty\rangle + \|y - Ty\|^2\right] \le \mu_n \|x_n - y\|^2,$$

and thus,

$$\mu_n \left[2\langle x_n - y,\ y - Ty\rangle + \|y - Ty\|^2\right] \le 0.$$

Since $x_n \rightharpoonup u$, we obtain $2\langle u - y,\ y - Ty\rangle + \|y - Ty\|^2 \le 0$. This completes the proof. □

Now, we can prove a Mann's type weak convergence theorem for a normally 2-generalized hybrid mapping in Hilbert spaces.

Theorem 5.2. *Let C be a nonempty and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$. Let $P_{A(T)}$ be the metric projection from H onto $A(T)$. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be real sequences in the interval $(0,1)$ such that $a_n + b_n + c_n = 1$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$ for any $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to an element u of $A(T)$, where $u = \lim_{n\to\infty} P_{A(T)} x_n$.

Proof. First, note that the real sequence $\{\|x_n - q\|\}$ is monotone decreasing for all $q \in A(T)$. Indeed, from Lemma 2.6 and Proposition 3.2,

$$\begin{aligned} \|x_{n+1} - q\| &= \left\|a_n (x_n - q) + b_n (T x_n - q) + c_n \left(T^2 x_n - q\right)\right\| \\ &\leq a_n \|x_n - q\| + b_n \|T x_n - q\| + c_n \left\|T^2 x_n - q\right\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\| \end{aligned} \tag{5.1}$$

for all $q \in A(T)$ and $n \in \mathbb{N}$. As direct consequences, the sequence $\{x_n\}$ is bounded, and $\{\|x_n - q\|\}$ is convergent for all $q \in A(T)$. Furthermore, from Lemma 2.4, $\left\{P_{A(T)} x_n\right\}$ is convergent in $A(T)$.

Next, we verify that $x_n - T x_n \to 0$ and $T^2 x_n - x_n \to 0$. We obtain from Lemma 2.6 and Proposition 3.2 that

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \left\|a_n (x_n - q) + b_n (T x_n - q) + c_n \left(T^2 x_n - q\right)\right\|^2 \\ &= a_n \|x_n - q\|^2 + b_n \|T x_n - q\|^2 + c_n \left\|T^2 x_n - q\right\|^2 \\ &\quad - a_n b_n \|x_n - T x_n\|^2 - b_n c_n \left\|T x_n - T^2 x_n\right\|^2 - c_n a_n \left\|T^2 x_n - x_n\right\|^2 \\ &\leq \|x_n - q\|^2 - a_n b_n \|x_n - T x_n\|^2 - b_n c_n \left\|T x_n - T^2 x_n\right\|^2 \\ &\quad - c_n a_n \left\|T^2 x_n - x_n\right\|^2 \end{aligned}$$

for all $q \in A(T)$ and $n \in \mathbb{N}$. Thus,

$$\begin{aligned} & a_n b_n \|x_n - T x_n\|^2 + b_n c_n \left\|T x_n - T^2 x_n\right\|^2 + c_n a_n \left\|T^2 x_n - x_n\right\|^2 \\ \leq\ & \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \end{aligned} \tag{5.2}$$

for all $q \in A(T)$ and $n \in \mathbb{N}$. Since $\{\|x_n - q\|\}$ is convergent for any $q \in A(T)$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$, we have from (5.2) that

$$x_n - T x_n \to 0 \text{ and } T^2 x_n - x_n \to 0. \tag{5.3}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $u \in H$ such that $x_{n_i} \rightharpoonup u$. From Lemma 5.1 and (5.3), we have $u \in A(T)$. We will prove that $x_n \rightharpoonup u$. For that aim, we demonstrate

$$[x_{n_k} \rightharpoonup u_1 \text{ and } x_{n_l} \rightharpoonup u_2] \Longrightarrow u_1 = u_2, \tag{5.4}$$

where $\{x_{n_k}\}$ and $\{x_{n_l}\}$ are subsequences of $\{x_n\}$. Suppose by way of contradiction that $u_1 \neq u_2$. From Lemma 5.1, (5.3) and the assumptions $x_{n_k} \rightharpoonup u_1$ and $x_{n_l} \rightharpoonup u_2$, we have $u_1, u_2 \in A(T)$, and thus, two real sequences $\{\|x_n - u_1\|\}$ and $\{\|x_n - u_2\|\}$ are convergent. From Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - u_1\| &= \lim_{k\to\infty} \|x_{n_k} - u_1\| \\ &< \lim_{k\to\infty} \|x_{n_k} - u_2\| \\ &= \lim_{n\to\infty} \|x_n - u_2\| \\ &= \lim_{l\to\infty} \|x_{n_l} - u_2\| \\ &< \lim_{l\to\infty} \|x_{n_l} - u_1\| \\ &= \lim_{n\to\infty} \|x_n - u_1\| . \end{aligned}$$

This is a contradiction. Thus, $u_1 = u_2$. To conclude $x_n \rightharpoonup u$, from Lemma 2.7, it is sufficient to show that if $x_{n_j} \rightharpoonup w$, then $w = u$, where $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$. Assume that $x_{n_j} \rightharpoonup w$. Since $x_{n_i} \rightharpoonup u$, from (5.4), we get that $w = u$. Therefore, we obtain $x_n \rightharpoonup u$.

To complete the proof, we will show that $u = \lim_{n\to\infty} P_{A(T)}x_n$. Let $v \equiv \lim_{n\to\infty} P_{A(T)}x_n$. Since $u \in A(T)$, it holds that

$$\left\langle x_n - P_{A(T)}x_n,\ P_{A(T)}x_n - u \right\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup u$ and $P_{A(T)}x_n \to v$, we obtain $\langle u - v,\ v - u\rangle \geq 0$. This means that $u = v$. □

Adding that C is closed in Theorem 5.2, we obtain a Mann's type weak convergence theorem of finding a fixed point for a normally 2-generalized hybrid mapping in Hilbert spaces.

Theorem 5.3. *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be real sequences in the interval $(0, 1)$ such that $a_n + b_n + c_n = 1$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to an element u of $F(T)$.

Proof. From Theorem 5.2, we have that the sequence $\{x_n\}$ in C converges weakly to $u = \lim_{n\to\infty} P_{A(T)}x_n \in A(T)$. Since C is closed and convex in H and $x_n \rightharpoonup u$, we have $u \in C$. Since $u \in A(T) \cap C$, from Lemma 2.5, we obtain $u \in F(T)$. □

References

[1] S. Alizadeh and F. Moradlou, *Weak and strong convergence theorems for m-generalized hybrid mappings in Hilbert spaces*, Topol. Methods Nonlinear Anal. **46** (2015), 315–328.

[2] S. Alizadeh and F. Moradlou, *Weak convergence theorems for 2-generalized hybrid mappings and equilibrium problems*, Commun. Korean Math. Soc. **31** (2016), 765–777.

[3] J. B. Baillon, *Un Théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Ser. A–B **280** (1975), 1511–1514.

[4] M. Hojo and W. Takahashi, *Weak and strong convergence theorems for generalized hybrid mappings in Hilbert spaces*, Sci. Mathe. Jpn. **73** (2011), 31–40.
[5] P. Kocourek, W. Takahashi, J. C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.
[6] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.
[7] L.-J. Lin, and W. Takahashi, *Attractive point theorem and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **16** (2012), 1763–1779.
[8] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[9] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185–197.
[10] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.
[11] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.
[12] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishes, Yokohama, 2000.
[13] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishes, Yokohama, 2009.
[14] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[15] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.
[16] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.
[17] W. Takahashi, N. C. Wong, J. C. Yao, *Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13** (2012), 745–757.
[18] W. Takahashi and J. C. Yao, *Weak convergence theorems for generalized hybrid mappings in Banach spaces*, J. Nonlinear Anal. Optim. **2** (2011), 155–166.
[19] W. Takahashi and J. C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.
[20] I. Termwuttipong, T. Pongsriiam and M. Yao, *Nonlinear Ergodic Theorem without Convexity for 2- Generalized Hybrid Mappings in a Hilbert Space*, Nonlinear Analysis and Convex Analysis, 1–11, Yokohama Publishes, Yokohama, 2010.

Manuscript received 13 April 2017
revised 8 May 2017

Atsumasa Kondo
Department of Economics, Shiga University, Banba 1-1-1, Hikone, Shiga 522-0069, Japan
E-mail address: `a-kondo@biwako.shiga-u.ac.jp`

Wataru Takahashi
Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan; Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: `wataru@is.titech.ac.jp; wataru@a00.itscom.net`